INTRODUCTION TO ALGEBRAIC TOPOLOGY

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ABSTRACT. $\boxed{\text{Updated by Mar 5 2018. To be continue.}}$ This is course note for Algebraic Topology in Spring 2018 at Tsinghua university.

Coure References:

(1) Hatcher: Algebraic Topology
(2) Bott and Tu: Differential forms in algebraic topology.
(3) May: A Concise Course in Algebraic Topology
(4) Spanier: Algebraic Topology.

CONTENTS

1. CATEGORY AND FUNCTOR

Category.

Definition 1.1. A **category** $\mathcal{C}$ consists of

(1) a class of *objects*: $\mathrm{Obj}(\mathcal{C})$
(2) *morphisms*: a set $\mathrm{Hom}_{\mathcal{C}}(A,B), \forall A, B \in \mathrm{Obj}(\mathcal{C})$. An element $f \in \mathrm{Hom}(A,B)$ will be denoted by

$$A \xrightarrow{f} B \quad \text{or} \quad f : A \to B.$$

(3) *composition*:

$$\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C), \forall A,B,C \in \mathrm{Obj}(\mathcal{C})$$
$$f \times g \to g \circ f$$

satisfying the following axioms

(1) *associativity*: $h \circ (g \circ f) = (h \circ g) \circ f$ for any $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$.
(2) *identity*: $\forall A \in \mathrm{Obj}(\mathcal{C}), \exists 1_A \in \mathrm{Hom}(A,A)$ called the identity element, such that

$$f \circ 1_A = f = 1_B \circ f, \quad \forall A \xrightarrow{f} B.$$

Definition 1.2. A morphism $f : A \to B$ is called an **equivalence/invertible** if $\exists g : B \to A$ such that

$$f \circ g = 1_B, \quad g \circ f = 1_A.$$

Two objects A, B are called **equivalent** if there exists an equivalence $f : A \to B$.

Definition 1.3. A category where all morphisms are equivalences is called a **groupoid**.

Definition 1.4. A **subcategory** $\mathcal{C}' \subset \mathcal{C}$ is a category such that

- $\mathrm{Obj}(\mathcal{C}') \subset \mathrm{Obj}(\mathcal{C})$
- $\mathrm{Hom}_{\mathcal{C}'}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B), \forall A, B \in \mathrm{Obj}(\mathcal{C}')$
- composition coincides.

$\mathcal{C}'$ is called a full subcategory of $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B), \forall A, B \in \mathrm{Obj}(\mathcal{C}')$.

Definition 1.5. Let $\sim$ be an equivalence relation defined on each $\mathrm{Hom}(A, B), A, B \in \mathrm{Obj}(\mathcal{C})$ satisfying

$$f_1 \sim f_2, g_1 \sim g_2 \Longrightarrow g_1 \circ f_1 \sim g_2 \circ f_2.$$

Then we define the quotient category $\mathcal{C}' = \mathcal{C}/\sim$ by

- $\mathrm{Obj}(\mathcal{C}') = \mathrm{Obj}(\mathcal{C}')$
- $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)/\sim, \forall A, B \in \mathrm{Obj}(\mathcal{C}')$

Example 1.6. We will frequently use the following categories.

- $\underline{\mathrm{Set}}$: the category of set.
- $\underline{\mathrm{Vect}}$: the category of vector spaces.
- $\underline{\mathrm{Group}}$: the category of groups.
- $\underline{\mathrm{Ab}}$: the category of abelian groups.
- $\underline{\mathrm{Ring}}$: the category of rings.

$\underline{\mathrm{Vect}} \subset \underline{\mathrm{Set}}$ is a subcategory, and $\underline{\mathrm{Ab}} \subset \underline{\mathrm{Group}}$ is a full subcategory.

The main object of our interest is the category of topological spaces $\underline{\mathrm{Top}}$

- objects of $\underline{\mathrm{Top}}$ are topological spaces.
- morphism $f : X \to Y$ is a continuous map.

Definition 1.7. Given $X, Y \in \underline{\mathrm{Top}}$, $f_0, f_1 : X \to Y$ are said to to homotopic, denoted by $f_0 \simeq f_1$, if

$$\exists F : X \times I \to Y, \quad \text{such that} \quad F|_{X\times 0} = f_0, F|_{X\times 1} = f_1. \quad I = [0, 1].$$

Homotopy defines an equivalence relation on $\underline{\mathrm{Top}}$. We denote its quotient category by

$$\underline{\mathrm{hTop}} = \underline{\mathrm{Top}} / \simeq .$$

We also denote

$$\mathrm{Hom}_{\underline{\mathrm{hTop}}}(X, Y) = [X, Y].$$

Definition 1.8. Two topological spaces X, Y are said to have the **same homotopy type** (or homotopy equivalent) if they are equivalent in $\underline{\mathrm{hTop}}$.

There is also a relative version as follows.

Definition 1.9. Let $A \subset X \in \underline{\mathrm{Top}}$, $f_0, f_1 : X \to Y$ such that $f_0|_A = f_1|_A : A \to Y$. We say f_0 is homotopic to f_1 relative to A, denoted by

$$f_0 \simeq f_1 \,\mathrm{rel}\, A$$

if there exists $F : X \times I \to Y$ such that

$$F|_{X\times 0} = f_0, \quad F|_{X\times 1} = f_1, \quad F|_{A\times t} = f_0|_A, \forall t \in I.$$

Functor.

Definition 1.10. Let $\mathcal{C}, \mathcal{D}$ be two categories. A **covariant functor** (or **contravariant functor**) $F : \mathcal{C} \to \mathcal{D}$ consists of

- $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, $A \to F(A)$
- $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)), \forall A, B \in \mathrm{Obj}(\mathcal{C})$. We denote by

$$A \xrightarrow{f} B \Longrightarrow F(A) \xrightarrow{F(f)} F(B)$$

(or $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)), \forall A, B \in \mathrm{Obj}(\mathcal{C})$, denoted by $A \xrightarrow{f} B \Longrightarrow F(B) \xrightarrow{F(f)} F(A)$)

satisfying

- $F(g \circ f) = F(g) \circ F(f)$ (or $F(g \circ f) = F(f) \circ F(g)$) for any $A \xrightarrow{f} B \xrightarrow{g} C$
- $F(1_A) = 1_{F(A)}, \forall A \in \mathrm{Obj}(\mathcal{C})$.

F is called **faithful** (or **full**) if $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective (or surjective) $\forall A, B \in \mathrm{Obj}(\mathcal{C})$.

Example 1.11. $\forall X \in \mathrm{Obj}(\mathcal{C})$,

$$\mathrm{Hom}(X, -) : \mathcal{C} \to \underline{\mathrm{Set}}, \quad A \to \mathrm{Hom}(X, A)$$

defines a covariant functor. Similarly $\mathrm{Hom}(-, X)$ defines a contravariant functor. A functor $F : \mathcal{C} \to \underline{\mathrm{Set}}$ of such type is called *representable*.

Example 1.12. Let G be an abelian group. Given $X \in \underline{\mathrm{Top}}$, we will study its n-th cohomology $\mathrm{H}^n(X; G)$. It defines a functor

$$\mathrm{H}^n(-; G) : \underline{\mathrm{hTop}} \to \underline{\mathrm{Set}}, \quad X \to \mathrm{H}^n(X; G)$$

We will see that this functor is representable by the Eilenberg-Maclane space if we work with the subcategory of CW-complexes.

Example 1.13. We define a contravariant functor

$$\mathrm{Fun} : \underline{\mathrm{Top}} \to \underline{\mathrm{Ring}}, \quad X \to \mathrm{Fun}(X) = \mathrm{Hom}(X, \mathbb{R})$$

$\mathrm{F}(X)$ are continuous real functions on X. A classical theorem of Gelfand-Kolmogoroff says that two compact Hausdorff spaces X, Y are homeomorphic if and only if $\mathrm{Fun}(X), \mathrm{Fun}(Y)$ are ring isomorphic.

Proposition 1.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. $f : A \to B$ is an equivalence. Then $F(f) : F(A) \to F(B)$ is also an equivalence.*

Natural transformation.

Definition 1.15. Let $\mathcal{C}, \mathcal{D}$ be two categories. $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** $\tau : F \to G$ consists of morphisms

$$\tau = \{\tau_A : F(A) \to G(A) | \forall A \in \mathrm{Obj}(\mathcal{C})\}$$

such that the following diagram commutes for any $A, B \in \mathrm{Obj}(\mathcal{C})$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\tau_A"] & F(B) \arrow[d, "\tau_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

τ is called **natural equivalence** if τ_A is an equivalence for any $A \in \mathrm{Obj}(\mathcal{C})$. We write $F \simeq G$.

Definition 1.16. Two categories $\mathcal{C}, \mathcal{D}$ are called **isomorphic** if $\exists F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = 1_{\mathcal{D}}, G \circ F = 1_{\mathcal{C}}$. They are called **equivalent** if $\exists F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \simeq 1_{\mathcal{D}}, G \circ F \simeq 1_{\mathcal{C}}$

Proposition 1.17. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Then F is fully faithful.*

2. Fundamental Groupoid

Path connected component.

Definition 2.1. Let $X \in \underline{\text{Top}}$. A map $\gamma : I \to X$ is called a path from $\gamma(0)$ to $\gamma(1)$. We denote γ^{-1} be the path from $\gamma(1)$ to $\gamma(0)$ defined by $\gamma^{-1}(t) = \gamma(1-t)$. We denote $i_{x_0} : I \to X$ be the constant map to $x_0 \in X$.

Let us introduce an equivalence relation on X by

$$x_0 \sim x_1 \Longleftrightarrow \exists \text{ a path from } x_0 \text{ to } x_1.$$

We denote the quotient space

$$\pi_0(X) = X/\sim$$

which is the set of path connected components of X.

Proposition 2.2. $\pi_0 : \underline{\text{hTop}} \to \underline{\text{Set}}$ *defines a covariant functor.*

As a consequence, $\pi_0(X) \cong \pi_0(Y)$ if X, Y are homotopy equivalent.

Path category/fundamental groupoid.

Definition 2.3. Let $\gamma : I \to X$ be a path. We define the path class of γ

$$[\gamma] = \{\tilde{\gamma} : I \to X | \tilde{\gamma} \simeq \gamma \operatorname{rel} \partial I = \{0, 1\}\}$$

Definition 2.4. Let $\gamma_1, \gamma_2 : I \to X$ such that $\gamma_1(1) = \gamma_2(0)$. We define

$$\gamma_2 \star \gamma_1 : I \to X$$

by

$$\gamma_2 \star \gamma_1(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

$\star$ is not associative for strict paths. However, $\star$ defines an associative composition on path classes.

Theorem 2.5. *Let $X \in \underline{\text{Top}}$. We define a category $\Pi_1(X)$ as follows:*

- $\text{Obj}(\Pi_1(X)) = X$.
- $\text{Hom}_{\Pi_1(X)}(x_0, x_1)$=*path classes from x_0 to x_1.*
- $1_{x_0} = i_{x_0}$.

Then $\Pi_1(X)$ defines a category which is in fact a groupoid. The inverse of $[\gamma]$ is given by $[\gamma^{-1}]$. $\Pi_1(X)$ is called the ***fundamental groupoid*** *of X.*

Let $\mathcal{C}$ be a groupoid. Let $A \in \text{Obj}(\mathcal{C})$, then

$$\text{Aut}_{\mathcal{C}}(A) := \text{Hom}_{\mathcal{C}}(A, A)$$

forms a group. For any $f : A \to B$, it induces a group isomorphism

$$Ad_f : \text{Aut}_{\mathcal{C}}(A) \to \text{Aut}_{\mathcal{C}}(B)$$

$$g \to f \circ g \circ f^{-1}.$$

This naturally defines a functor

$$\begin{aligned} \mathcal{C} &\to \underline{\text{Group}} \\ A &\to \text{Aut}_{\mathcal{C}}(A) \\ f &\to Ad_f \end{aligned}$$

Specialize this to topological spaces, we find a functor

$$\boxed{\Pi_1(X) \to \underline{\text{Group}}}.$$

Definition 2.6. Let $x_0 \in X$, the group

$$\pi_1(X, x_0) := \text{Aut}_{\Pi_1(X)}(x_0)$$

is called the fundamental group of the pointed space (X, x_0).

Theorem 2.7. *Let X be path connected. Then for $x_0, x_1 \in X$, the have group isomorphism*

$$\pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Let $f : X \to Y$ be a continuous map. It defines a functor

$$\begin{aligned} \Pi_1(f) : \Pi_1(X) &\to \Pi_1(Y) \\ x &\to f(x) \\ [\gamma] &\to [f \circ \gamma]. \end{aligned}$$

Then Π_1 defines a functor

$$\boxed{\Pi_1 : \underline{\text{Top}} \to \underline{\text{Groupoid}}}, \quad X \to \Pi_1(X)$$

from the category $\underline{\text{Top}}$ to the category $\underline{\text{Groupoid}}$ of groupoids. Here morphisms in $\underline{\text{Groupoid}}$ are given by natural transformations.

Proposition 2.8. *Let $f, g : X \to Y$ be maps which are homotopic by $F : X \times I \to Y$. Let us define path classes*

$$\tau_{x_0} = [F|_{x_0 \times I}] \in \text{Hom}_{\Pi_1(Y)}(f(x_0), g(x_0)).$$

Then τ defines a natural transformation

$$\tau : \Pi_1(f) \Longrightarrow \Pi_1(g).$$

This proposition can be summarized by the following diagrams

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow F"] & Y \end{tikzcd} \Longrightarrow \begin{tikzcd} \Pi_1(X) \arrow[r, bend left, "\Pi_1(f)"] \arrow[r, bend right, "\Pi_1(g)"'] \arrow[r, phantom, "\Downarrow \tau"] & \Pi_1(Y) \end{tikzcd}$$

The following theorem is a formal consequence of the above proposition

Theorem 2.9. *If $f : X \to Y$ is a homotopy equivalence. Then*

$$\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$$

is an equivalence of categories. In particular, it induces group isomorphisms

$$\pi_1(X, x_0) \cong \pi_1(Y, f(x_0)),$$