LECTURES ON GEOMETRY OF SUPERSYMMETRY

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ABSTRACT. $\boxed{\text{updated by Oct 9 2017. To be continued.}}$ This is note for my course on supersymmetry in the fall 2017 at Tsinghua university. We discuss supersymmetric gauge theories in various dimensions, their geometric structures and dualities.

CONTENTS

We will be working with the symmetric monoidal category of $\mathbb{Z}_2$-graded algebras over a field k of characteristic 0. Let A be a k-algebra with $\mathbb{Z}_2$-graded decomposition

$$A = A^0 \oplus A^1.$$

We will write $|a_i| = i, a_i \in A^i$, for the grading. The monoidal structure is given by the graded tensor product $\hat{\otimes}_k$ defined as follows.

Definition 0.1. Let A, B be two $\mathbb{Z}_2$-graded algebras over k. We define the graded tensor product $A\hat{\otimes}_k B$ as the $\mathbb{Z}_2$-graded algebra whose underlying $\mathbb{Z}_2$-graded vector space is $A \otimes_k B$, with multiplication defined by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 a_2) \otimes (b_1 b_2).$$

1. CLIFFORD ALGEBRA

Definition 1.1. Let V be a vector space over the field k, and Q is a quadratic form on V. We always assume q be non-degenerate. Let $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ be the tensor algebra. The **Clifford algebra** $Cl(V, Q)$ is defined to be the quotient of $T(V)$ by the relation

$$x^2 = -Q(x), \quad x \in V.$$

The tensor product $\otimes$ on $T(V)$ induces a product on $Cl(V, Q)$ denoted by $\cdot$.

Equivalently, $Cl(V, Q)$ is the associative algebra freely generated by V and the relation

$$a \cdot b + b \cdot a = -2 \langle a, b \rangle \quad \forall a, b \in V,$$

where $\langle -, - \rangle$ is the inner product on V associated to q.

Let $T(V) = T^{even}(V) \oplus T^{odd}(V)$ be the decomposition into even and odd number of tensors of V in $T(V)$. It equips $Cl(V, Q)$ with the structure of $\mathbb{Z}_2$-graded algebra by

$$Cl(V, Q) = Cl^0(V, q) \oplus Cl^1(V, Q)$$

where $Cl^0(V, Q)$ and $Cl^1(V, Q)$ are the images of $T^{even}(V)$ and $T^{odd}(V)$ respectively.

Let $T^{\leq p}(V) = \bigoplus_{n \leq p} V^{\otimes n}$. Let $\mathcal{F}^p$ be its image in $Cl(V, Q)$. Then the filtration

$$\mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset Cl(V, Q)$$

equips $Cl(V, Q)$ with the structure of filtered algebra. Let $Gr_{\mathcal{F}}(Cl(V, Q))$ be the associated graded algebra.

Lemma 1.2. *There is a canonical isomorphism of algebras*

$$Gr_{\mathcal{F}}(Cl(V, Q)) \cong \wedge^* V$$

where $\wedge^* V$ is the exterior algebra.

In particular, there is an explicit isomorphism of vector spaces

$$\rho : \wedge^* V \to Cl(V, Q), \quad v_1 \wedge \cdots \wedge v_p \to \sum_{\sigma \in S_p} (-1)^{\sigma} v_{\sigma(1)} \cdots v_{\sigma(p)}.$$

Lemma 1.3. *Suppose $V = V_1 \oplus V_2$ and $Q = Q_1 + Q_2$, where Q_i is a quadratic form on V_i. Then there is a canonical isomorphism of $\mathbb{Z}_2$-graded k-algebras*

$$Cl(V, Q) \cong Cl(V_1, Q_1) \hat{\otimes}_k Cl(V_2, Q_2).$$

Example 1.4. Let V be a real vector space, and $Q = x_1^2 + \cdots x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$. The associated Clifford algebra will be denoted by $Cl_{p,q}$. Moreover $Cl_p \equiv Cl_{p,0}$ for simplicity. We have

$$Cl_{p,q} \cong Cl_{p,0} \hat{\otimes}_{\mathbb{R}} Cl_{0,q}.$$

Example 1.5. Let V be a complex vector space of dimension n, then all non-degenerate quadratic forms are equivalent. The associated Clifford algebra will be denoted by $\mathbb{C}l_n$ and we have

$$\mathbb{C}l_n \cong \mathbb{C}l_1^{\hat{\otimes}_{\mathbb{C}}^n}.$$

Lemma 1.6. *Let $k(n)$ denote the $n \times n$ matrix algebra with entries in k. The we have algebra isomorphisms*

$$Cl_{1,0} \cong \mathbb{C}, \quad Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}, \quad Cl_{2,0} \cong \mathbb{H}, \quad Cl_{1,1} \cong Cl_{0,2} \cong \mathbb{R}(2)$$

$$\mathbb{C}l_1 \cong \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{C}l_2 \cong \mathbb{C}(2).$$

Here $\mathbb{H}$ are quaternions.

Proof. We prove $Cl_{1,1} \cong Cl_{0,2} \cong \mathbb{R}(2)$. As a vector space

$$Cl_{1,1} = \mathbb{R} \oplus \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}xy$$

with multiplication structure by $x^2 = 1, y^2 = -1, xy = -yx$. It is identified with $\mathbb{R}(2)$ by

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Similarly,

$$Cl_{0,2} = \mathbb{R} \oplus \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}xy$$

with multiplication structure by $x^2 = 1, y^2 = 1, xy = -yx$. It is identified with $\mathbb{R}(2)$ by

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

□

The Clifford algebras are easily classified with the help of the following proposition.

Proposition 1.7. *We have the following algebra isomorphisms*

(1) $Cl_{p+2,q} \cong Cl_{2,0} \otimes Cl_{q,p}, \quad Cl_{p,q+2} \cong Cl_{0,2} \otimes Cl_{q,p}, \quad Cl_{p+1,q+1} \cong Cl_{1,1} \otimes Cl_{p,q}.$
(2) *Bott periodicity (real case):* $Cl_{p+4,q} \cong Cl_{p,q+4}, \quad Cl(p+8,q) \cong Cl(p,q)(16).$
(3) *Bott periodicity (complex case):* $\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2 \cong \mathbb{C}l_n(2).$
(4) $Cl_{p,q} \cong Cl^0_{p+1,q}.$

Proof. (1)The isomorphism

$$Cl_{2,0} \otimes Cl_{q,p} \to Cl_{p+2,q}$$

is realized in generators

$$e_i \to e_i, \tilde{e}_\alpha \to e_{12}\tilde{e}_\alpha, \quad 1 \leq i \leq 2, 1 \leq \alpha \leq p+q, \quad e_{12} = e_1 e_2.$$

(4) The isomorphism

$$Cl_{p,q} \cong Cl^0_{p+1,q}$$

is realized in generators

$$e_i \to e_1 e_{i+1}, \quad 1 \leq i \leq p+q.$$

□

Combining Lemma 1.6 and Proposition 1.7, we find the following table

n	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
$Cl_{0,n}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$Cl_{n-1,1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$Cl_{1,n-1}$	$\mathbb{C}$	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$
$\mathbb{C}l_n$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$\mathbb{C}(16)$

TABLE 1. Clifford algebras

2. SPIN GROUP

Definition 2.1. We define the following operations on Clifford algebras

(1) **Reflection automorphism**: let $x = v_1 \cdots v_m$, then $\hat{x} = (-)^m v_1 \cdots v_m$.
(2) **Transpose anti-automorphism**: let $x = v_1 \cdots v_m$, then $x^t = v_m \cdots v_1$.
(3) **Conjugate anti-automorphism**: $x^* = \hat{x}^t$.

Definition 2.2. Let $Cl^\times(V,Q)$ be the multiplicative group of units in $Cl(V,Q)$. We define the twisted conjugation action of $Cl^\times(V,Q)$ on the Clifford algebra

$$\hat{Ad} : Cl^\times(V,Q) \to Gl(Cl(V,Q))$$

by

$$\hat{Ad}_x(y) = \hat{x} y x^{-1}.$$

Example 2.3. Let $x, y \in V$, then

$$\hat{Ad}_x(y) = y - 2\frac{\langle x,y\rangle}{\langle x,x\rangle} x$$

is the reflection along the hyperplane orthogonal to x.

Definition 2.4. The **Clifford group** $\Gamma(V,Q)$ is defined by

$$\Gamma(V,Q) = \{x \in Cl^\times(V,q) | \hat{Ad}_x : V \to V \text{ preserves } V\}.$$

Lemma 2.5. *The Clifford action $\Gamma(V,Q) : V \to V$ preserves the quadratic form.*

Proof. In fact, $\forall v \in V$,

$$\left\langle \hat{Ad}_x v, \hat{Ad}_x v \right\rangle = \hat{x} v x^{-1} \widehat{\hat{x} v x^{-1}} = \langle v, v\rangle .$$

□

Corollary 2.6. *There is an exact sequence of groups*

$$0 \to k^\times \to \Gamma(V,Q) \stackrel{\hat{Ad}}{\to} O(V,Q) \to 0.$$

Here $O(V,Q)$ is the orthogonal group of q-preserving linear automorphisms of V.

Proof. $\hat{Ad}$ is defined by the previous lemma. Let $x \in \Gamma(V,Q) \cap \ker(\hat{Ad})$. Then

$$vx = \hat{x}v, \quad \forall v \in V.$$

Using the filtration $\mathcal{F}^\bullet$ on $Cl(V,Q)$, it is enough to show that given $x \in \wedge^p V$, $p > 0$, if

$$\iota_{\langle v,-\rangle} x = 0 \in \wedge^{p-1} V \text{ for any } v \in V,$$

then $x = 0$. Here $\langle v, -\rangle$ is viewed as an element of V^* and ι is the natural contraction. But this is obvious.

On the other hand, Cartan-Dieudonné theorem states that every element of $O(V,Q)$ is a composition of at most $\dim_k V$ reflections ($char(k) \neq 2$), hence a Clifford action. □

Definition 2.7. The **spin norm** $N : \Gamma(V,Q) \to k^\times$ is a group homomorphism defined by
$$N(x) = x\hat{x}^t.$$

The reason that $N(x) \in k^\times$ comes from the observation that $x\hat{x}^t \in \Gamma(V,Q) \cap \ker(\hat{A}d)$. In fact,
$$\hat{A}d_{x\hat{x}^t} v = \hat{x}\left(x^t v(\hat{x}^t)^{-1}\right)x^{-1} = \hat{x}\left(x^t v(\hat{x}^t)^{-1}\right)^t x^{-1} = v.$$

Definition 2.8. The **Pin group** and **Spin group** are defined by
$$\mathrm{Pin}(V,Q) = \{v_1 \cdots v_r \in Cl^\times(V,Q) | v_i \in V, Q(v_i) = \pm 1\}.$$
$$\mathrm{Spin}(V,Q) = \mathrm{Pin}(V,Q) \cap Cl^0(V,Q).$$

In the case when k is a spin field (i.e. $k^\times = (k^\times)^2 \cup -(k^\times)^2$), we still have a surjection
$$\hat{A}d : \mathrm{Pin}(V,Q) \to O(V,Q)$$
and the spin norm is
$$N : \mathrm{Pin}(V,Q) \to \pm 1.$$

Proposition 2.9. *In the real case $k = \mathbb{R}$, we have exact sequences*
$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(p,q) \to SO(p,q) \to 1$$
$$0 \to \mathbb{Z}_2 \to \mathrm{Pin}(p,q) \to O(p,q) \to 1.$$
Here $SO(p,q) = \{A \in O(p,q) | \det A = 1\}$.

Proof. Let $x \in \mathrm{Pin}(V,Q) \cap \mathbb{R}^\times$, then
$$N(x) = x^2 = \pm 1 \Longrightarrow x = \pm 1.$$
This implies the second exact sequence. The first exact sequence follows from the fact that each reflection has determinant -1. □

Proposition 2.10. $\mathrm{Spin}(p,q) \cong \mathrm{Spin}(q,p)$.

Proof. We consider the map of $Cl_{p,q} \otimes_{\mathbb{R}} \mathbb{C}$ on generators
$$e_i \to \sqrt{-1}e_i.$$
This is well-defined on $Cl^0_{p,q}$ since it contains even number of products of $\sqrt{-1}$'s. □

Example 2.11. We can identify some spin groups in low dimensions
$$\mathrm{Spin}(2) \cong U(1), \qquad \mathrm{Spin}(3) \cong SU(2)$$
$$\mathrm{Spin}(4) = SU(2) \times SU(2), \qquad \mathrm{Spin}(3,1) \cong SL^*(2,\mathbb{C})$$
$$\mathrm{Spin}(6) \cong SU(4), \qquad \mathrm{Spin}(5,1) \cong SL^*(2,\mathbb{H}).$$
Here $SL^*(2,k) = \{A \in GL(2,k) | \det g = \pm 1\}$. This can be seen as follows.

- $\mathrm{Spin}(2)$. In this case we have $Cl_2 \cong \mathbb{H} = \mathbb{R} \oplus \mathbb{R}\,\mathrm{i} \oplus \mathbb{R}\,\mathrm{j} \oplus \mathbb{R}\,\mathrm{k}$.
$$\mathrm{Spin}(2) = \{a + b\,\mathrm{k}\,|a^2 + b^2 = 1, \quad a,b \in \mathbb{R}\}.$$
Given $\phi = e^{\mathrm{k}\,\theta} \in \mathrm{Spin}(2)$, $z = x\,\mathrm{i} + y\,\mathrm{j} \in \mathbb{R}^2$, we have $\hat{A}d_\phi : z \to e^{2\,\mathrm{k}\,\theta} z$.

- Spin(3). Let e_1, e_2, e_3 be orthonormal basis of $\mathbb{R}^3$. $Cl_3^0 \cong Cl_2 \cong \mathbb{H}$. Explicitly,
$$\mathbb{H} \to Cl_3^0, \quad \mathrm{i} \to e_2e_3, \quad \mathrm{j} \to e_3e_1, \quad \mathrm{k} \to e_1e_2.$$
Observe that the spin norm on Cl_3^0 can be identified with the norm on $\mathbb{H}$. It follows that
$$\mathrm{Spin}(3) \cong \{a \in \mathbb{H} | |a|^2 = 1\}.$$
Consider a two-dim representation of $\mathbb{H}$ by
$$\mathrm{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathrm{j} = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \mathrm{k} = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$$
It identifies
$$\mathbb{H} \cong \{A \in M_2(\mathbb{C}) | \bar{A} = \gamma A \gamma^{-1}\}, \quad \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
The norm of $\mathbb{H}$ can be identified with the determinant
$$t^2 + x^2 + y^2 + z^2 = \det(t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$
This gives another isomorphism
$$SU(2) \cong \{a \in \mathbb{H} | |a|^2 = 1\}.$$
It follows that $\mathrm{Spin}_3 \cong SU(2)$. If we identify $\mathbb{R}^3$ as the imaginary part of $\mathbb{H}$, then the homomorphism $\mathrm{Spin}(3) \to SO(3)$ is realized by
$$\hat{Ad}_a : v \to ava^\dagger, \quad a \in \mathrm{Spin}(3) \subset H, \quad v \in \mathrm{Im}(\mathbb{H}).$$
- Spin(4). $SU(2)$ acts on $\mathbb{H} \cong \mathbb{R}^4$ from both sides, which gives the following map
$$SU(2)_L \times SU(2)_R \to GL(4, \mathbb{R}), \quad A_L \times A_R : q \to A_L q A_R^{-1}.$$
Since the norm is given by the derminant, it actually maps to $SO(4)$. It follows that $Spin_4$ can be identified with two copies of $SU(2)$
$$Spin(4) \cong SU(2) \times SU(2).$$
- Spin(3,1). We consider $\mathbb{R}^{3,1}$ with metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$. We define the Pauli matrices
$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
Let $x = \{x^0, x^1, x^2, x^3\}$ be coordinates on $\mathbb{R}^{3,1}$. The Pauli representation
$$x \to A(x) = \sum_{i=0}^{3} x^i \sigma_i$$
identifies $\mathbb{R}^{3,1}$ with the space of 2×2 Hermitian matrices. It is easy to see that
$$\det A(x) = -\sum_{\mu,\nu=0}^{3} \eta_{\mu,\nu} x^\mu x^\nu.$$
Let $SL^*(2, \mathbb{C}) = \{A \in GL(2, \mathbb{C}) | \det A = \pm 1\}$. Then the following $\mathbb{Z}_2$-covering
$$\begin{aligned} \pi : SL^*(2, \mathbb{C}) \quad &\to \quad SO(3,1) \\ N \quad &\to \quad \{A(x) \to \det(N) N A(x) N^\dagger\} \end{aligned}$$
identifies $\mathrm{Spin}(3,1)$ with $SL^*(2, \mathbb{C})$. The two cases $\det A = \pm 1$ correspond to matrix M in $SO(3,1)$ with $M_{00} > 0$ or $M_{00} < 0$. Sometimes $\mathrm{Spin}(3,1)$ just refers to the universal cover $SL(2, \mathbb{C})$ of the connected component $SO^+(3,1)$ of $SO(3,1)$ containing identity.

- Spin(6). Since $Cl_6^0 \cong Cl_5 \cong \mathbb{C}(4)$, we get a map $\mathrm{Spin}(6) \to SU(4)$ which turns out to be an isomorphism. This can be explicitly realized as follows.

  Let $V = \mathbb{C}^4$ with basis $\{e_1, e_2, e_3, e_4\}$. Let h be the standard hermitian metric such that $h(e_i, e_j) = \delta_{ij}$. It induces a $SU(4)$-equivariant $\mathbb{C}$-conjugate linear isomorphism

$$\alpha : V \to V^*, \quad \alpha(v) \to h(v, -).$$

  Let $\omega = -e_1 \wedge e_2 \wedge e_3 \wedge d_4 \in \wedge^4 V$. Let us consider the composition

$$J : \wedge^2 V \overset{-\wedge^2\alpha}{\to} \wedge^2 V^* \overset{\lrcorner\omega}{\to} \wedge^2 V.$$

  Then J is $SU(4)$-equivariant, $\mathbb{C}$-conjugate linear, and $J^2 = 1$. In particular, J defines a real structure on $\wedge^2 V$, whose real points can be identified with $\mathbb{R}^6$. The induced hermitian metric on $\wedge^2 V$ gives a real metric on $\mathbb{R}^6$, which is preserved by $SU(4)$. This gives a double cover

$$SU(4) \to SO(6)$$

  and identifies $\mathrm{Spin}(6) \cong SU(4)$.
- Spin(5, 1). Recall the reprsentation $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$ above. This identifies

$$SL(2, \mathbb{H}) \cong \left(A \in SL(4, \mathbb{C}) | \bar{A} = \Gamma A \Gamma^{-1}\right), \quad \Gamma = \begin{pmatrix} \gamma & \\ & \gamma \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

  Consider the $SL(2, \mathbb{H})$-equivariant $\mathbb{C}$-conjugate linear map

$$J : \wedge^2 \mathbb{C}^4 \to \wedge^2 \mathbb{C}^4, \quad J(a \wedge b) = \Gamma^{-1}\bar{a} \wedge \Gamma^{-1}\bar{b}.$$

  Let $\langle -, - \rangle$ be the $SL(4, \mathbb{C})$-invariant pairing on $\wedge^2 \mathbb{C}^4$

$$\langle -, - \rangle : \wedge^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^4 \to \wedge^4 \mathbb{C}^4 \cong \mathbb{C}.$$

  We identify $\mathbb{R}^6$ with real points of $\wedge^2 \mathbb{C}^4$ with respect to J. In terms of standard basis $\{e_i\}_{i=1}^4$ of $\mathbb{C}^4$,

$$\mathbb{R}^6 = \mathrm{Span}_{\mathbb{R}} \left\{e_1 \wedge e_2, e_1 \wedge e_3 + e_2 \wedge e_4, \mathrm{i}(e_1 \wedge e_3 - e_2 \wedge e_4), e_1 \wedge e_4 - e_2 \wedge e_3, \mathrm{i}(e_1 \wedge e_4 + e_2 \wedge e_3), e_3 \wedge e_4\right\}.$$

  It is easy to check that $\langle -, - \rangle$ gives $\mathbb{R}^6$ an inner product with signature $(-1, 1, 1, 1, 1, 1)$. $SL(2, \mathbb{H})$ acts on $\mathbb{R}^6$ and preserves $\langle -, - \rangle$, leading to

$$SL(2, \mathbb{H}) \to SO(5, 1).$$

3. Spinor

Definition 3.1. Let A be an algebra or group over k. Let $k \subset K$. Then a K-representation of A is a k-linear homomorphism

$$A \to \mathrm{Hom}_K(W, W)$$

for a K-vector space W. Equivalently, W is a representation of $A \otimes_k K$.

In this section $k = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We are interested in k-representations of $\mathrm{Spin}(p, q)$.

3.1. **Real spin representation.** Let $e_1, \cdots, e_{p+q}$ be orthonormal basis of $\mathbb{R}^{n=p+q}$. Define volume forms

$$\omega = e_1 \cdots e_{p+q}, \quad \omega_{\mathbb{C}} = i^{[\frac{n+1}{2}]+q}\omega.$$

ω is a central element of $Cl(V,q)$ if n is odd. We have

$$\omega_{\mathbb{C}}^2 = 1, \quad \omega^2 = \begin{cases} (-1)^q & n \equiv 0,3 \mod 4 \\ (-1)^{q+1} & n \equiv 1,2 \mod 4 \end{cases}$$

Definition 3.2. For $n \equiv 3 \mod 4, q$ even, or $n \equiv 1 \mod 4, q$ odd, we define the **chirality decomposition**

$$Cl_{p,q} = Cl^+_{p,q} \oplus Cl^-_{p+q}, \quad Cl^{\pm}_{p,q} = \frac{1\pm\omega}{2} Cl_{p,q}.$$

Similarly, for n odd in the complex case, we have chirality decomposition

$$\mathbb{C}l_n = \mathbb{C}l^+_n \oplus \mathbb{C}l^-_n, \quad \mathbb{C}l^{\pm}_n = \frac{1\pm\omega_{\mathbb{C}}}{2}\mathbb{C}l_n.$$

Example 3.3. Table 1 illustrates chirality decompositions in Euclidean and Minkowsky spaces.

Proposition 3.4. *If $n \equiv 3 \mod 4, q$ even, or $n \equiv 1 \mod 4, q$ odd, then $Cl_{p,q}$ has two inequivalent irreducible real representations. Otherwise $Cl_{p,q}$ has an unique irreducible real representation.*

Proof. This follows from the fact that $K(n)$ are simple $\mathbb{R}$-algebras for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$. □

Proposition 3.5. *Let $n \equiv 3 \mod 4, q$ even, or $n \equiv 1 \mod 4, q$ odd. Let W be the unique irreducible representation of $Cl_{p+1,q}$, $\omega_{p+1,q}$ be the corresponding volume form. Then*

(1) *$\omega^2_{p+1,q} = 1$, which induces a decomposition $W = W^+ \oplus W^-$ as $Cl^0_{p+1,q}$-modules.*
(2) *Under $Cl_{p,q} \cong Cl^0_{p+1,q}$, $W^{\pm}$ are the two inequivalent irreducible real representations of $Cl_{p,q}$.*

Proof. (1) is obvious. (2) is proved by comparing the volume forms. □

Lemma 3.6. *Let q even, $n \equiv 3 \mod 4$, or q odd, $n \equiv 1 \mod 4$. Let $W^{\pm}$ be the two irreducible representations of $Cl_{p,q}$. Let*

$$\Delta^{\pm} : \mathrm{Spin}(p,q) \subset Cl^0_{p,q} \subset Cl_{p,q} \to GL(W^{\pm}, \mathbb{R})$$

be the induced real representations. Then $\Delta^{\pm}$ are equivalent real representations of $\mathrm{Spin}(p,q)$.

Proof. The reflection automorphism switches $Cl^+_{p,q} \leftrightarrow Cl^-_{p,q}$ since $\hat{\omega} = -\omega$. It follows that

$$Cl^0_{p,q} = \{x \oplus \hat{x} | x \in Cl^+_{p,q}\}.$$

The lemma follows immediately. □

Definition 3.7. We define the real spinor representation $S = S_{p,q}$ of $\mathrm{Spin}(p,q)$ as the induced representation

$$\Delta_{p,q} : \mathrm{Spin}(p,q) \to GL(S, \mathbb{R})$$

from an irreducible representation S of $Cl_{p,q}$ under $\mathrm{Spin}(p,q) \subset Cl^0_{p,q} \subset Cl_{p,q}$.

Lemma 3.6 implies that this definition is well-defined for any (p,q). By construction, $\Delta_{p,q}$ does not come from a representation of $SO(p,q)$. Proposition 3.4 implies that $S_{p,q}$ is reducible when $n \equiv 0 \mod 4, q$ even or $n \equiv 2 \mod 4, q$ odd.

Example 3.8. Euclidean space.

n	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
$S_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_\pm$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_\pm$	$\mathbb{R}^{16}$
Irreducible $\mathbb{R}$-spinors	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_\pm$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_\pm$

Using $Cl_{n-1,0} = Cl^0_{n,0}$, we find

- $n \equiv 3,5,6,7 \mod 8$. S is irreducible, quaternion for $n = 3,5$, complex for $n = 6$, real for $n = 7$.
- $n \equiv 1 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{R}$-representations.
- $n \equiv 2 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{C}$-representations.
- $n \equiv 4 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{H}$-representations.
- $n \equiv 8 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{R}$-representations.

Example 3.9. Minkowski space.

n	1	2	3	4	5	6	7	8
$Cl_{n-1,1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$S_{n-1,1}$	$\mathbb{R}_\pm$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$\mathbb{H}^2_\pm$	$\mathbb{H}^4$	$\mathbb{C}^8$	$\mathbb{R}^{16}$
Irreducible $\mathbb{R}$-spinors	$\mathbb{R}$	$\mathbb{R}_\pm$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$\mathbb{H}^2_\pm$	$\mathbb{H}^4$	$\mathbb{C}^8$

- $n \equiv 1,5,7,8 \mod 8$. S is irreducible, quaternion for $n = 5,7$, complex for $n = 8$, real for $n = 1$.
- $n = 3 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{R}$-representations.
- $n = 4 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{C}$-representations.
- $n = 2 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{R}$-representations.
- $n = 6 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{H}$-representations.

3.2. **Complex spin representation.** Consider $V_\mathbb{C} = \mathbb{C}^n$, and $\omega_\mathbb{C}$ be the volume form as above, $\omega^2_\mathbb{C} = 1$. We have the chirality decomposition

$$\mathbb{C}l_n = \mathbb{C}l^+_n \oplus \mathbb{C}l^-_n, \quad n \text{ odd}.$$

The following proposition is the complex analogue of the previous discussion.

Proposition 3.10. *For n even, $\mathbb{C}l_n$ has a unique irreducible representation W. Moreover,*

1. *W is decomposed $W = W^+ \oplus W^-$ as $\mathbb{C}l^0_{n-1}$-modules.*
2. *Under $\mathbb{C}l_{n-1} \cong \mathbb{C}l^0_n$, $W^\pm$ are the two inequivalent irreducible representations of $\mathbb{C}l_{n-1}$.*
3. *$W^\pm$ are equivalent $\mathrm{Spin}_\mathbb{C}(n-1)$ representations under $\mathrm{Spin}_\mathbb{C}(n-1) \subset \mathbb{C}l^0_{n-1} \subset \mathbb{C}l_{n+1}$.*

Definition 3.11. Let $p + q = n$. We define the complex spinor representation $\mathbb{S} = \mathbb{S}_n$ of $\mathrm{Spin}_\mathbb{C}(n)$

$$\Delta^\mathbb{C}_n : \mathrm{Spin}_\mathbb{C}(n) \to GL(\mathbb{S}, \mathbb{C})$$

to be the induced one from an irreducible representation $\mathbb{S}$ of $\mathbb{C}l_n$ under $\mathrm{Spin}_\mathbb{C}(n) \subset \mathbb{C}l^0_n \subset \mathbb{C}l_n$.

Example 3.12. In the complex case, we have the following table

n	2m	2m+1
$\mathbb{C}l_n$	$\mathbb{C}(2^m)$	$\mathbb{C}(2^m)\oplus\mathbb{C}(2^m)$
$\mathbb{S}_n$	$\mathbb{C}^{2^m}$	$\mathbb{C}^{2^m}$
Irreducible $\mathbb{C}$-spinors	$\mathbb{C}^{2^{m-1}}_\pm$	$\mathbb{C}^{2^m}$

We have the following concrete realization of the above complex represenations.

- $\boxed{n=2m}$. $V=\mathbb{R}^{2m}=\mathbb{C}^m, V_{\mathbb{C}}=V\otimes_{\mathbb{R}}\mathbb{C}=V_{\mathbb{C}}^{1,0}\oplus V_{\mathbb{C}}^{0,1}$. Let us represent
$$V=\mathrm{Span}_{\mathbb{R}}\{dx^i,dy^i\}_{1\le i\le m},\quad V_{\mathbb{C}}^{1,0}=\mathrm{Span}_{\mathbb{C}}\{dz^i\}_{1\le i\le m},\quad V_{\mathbb{C}}^{0,1}=\mathrm{Span}_{\mathbb{C}}\{d\bar{z}^i\}_{1\le i\le m}.$$
Then the Clifford action $Cl(V)\to \mathrm{End}_{\mathbb{C}}(\wedge^* V_{\mathbb{C}}^{1,0})$ has a geometric realization
$$dx^i\to dz^i-\iota_{z^i},\quad dy^i\to \frac{1}{\sqrt{-1}}(dz^i+\iota_{z^i}).$$
The two irreducible $\mathbb{C}$-spinors $\mathbb{S}_{\pm}$ are given by
$$\boxed{\mathbb{S}_+=\wedge^{even}V_{\mathbb{C}}^{1,0},\quad \mathbb{S}_-=\wedge^{odd}V_{\mathbb{C}}^{1,0},\quad \mathbb{S}_n\cong\wedge^*V_{\mathbb{C}}^{1,0}.}$$
Similarly there is a Clifford acton $Cl(V)\to \mathrm{End}_{\mathbb{C}}(\wedge^* V_{\mathbb{C}}^{0,1})$ realized by
$$dx^i\to d\bar{z}^i-\iota_{\bar{z}^i},\quad dy^i\to -\frac{1}{\sqrt{-1}}(d\bar{z}^i+\iota_{\bar{z}^i}).$$
$\wedge^*V_{\mathbb{C}}^{1,0}$ and $\wedge^*V_{\mathbb{C}}^{0,1}$ are isomorphic Cl_{2m}-modules under the complex conjugation.

Proposition 3.13. *Let $\mathbb{S}_n$ be the complex spinor representation. Then we have isomorphic* $\mathrm{Spin}(n)$*-modules*
$$\mathbb{S}_n\otimes_{\mathbb{C}}\mathbb{S}_n\cong\begin{cases}\wedge^*\mathbb{C}^n, & \text{for } n \text{ even.}\\ \wedge^{even}\mathbb{C}^n\cong\wedge^{odd}\mathbb{C}^n, & \text{for } n \text{ odd.}\end{cases}$$

Proof. Assume $n=2m$. By dimension reason, $\mathbb{S}_{2m}\otimes_{\mathbb{C}}\mathbb{S}_{2m}$ is the irreducible $\mathbb{C}l_{2m}\otimes_{\mathbb{C}}\mathbb{C}l_{2m}\cong\mathbb{C}l_{4m}$-module. On the other hand, $\mathbb{C}l_{2m}$ is the irreducible $\mathbb{C}l_{2m}\otimes_{\mathbb{C}}\mathbb{C}l_{2m}$-module by
$$\Phi:\mathbb{C}l_n\otimes_{\mathbb{C}}\mathbb{C}l_n\to\mathrm{End}_{\mathbb{C}}(\mathbb{C}l_n),\quad \Phi_{x,y}(u)=xuy^t.$$
It follows that we have equivalent $\mathbb{C}l_{2m}\otimes_{\mathbb{C}}\mathbb{C}l_{2m}$-modules
$$\mathbb{S}_{2m}\otimes_{\mathbb{C}}\mathbb{S}_{2m}\cong\mathbb{C}l_{2m}.$$
Restricting to $\mathrm{Spin}(2m)$-modules, and observing that for $x\in\mathrm{Spin}(2m)$, $x^t=x^{-1}, x=\hat{x}$, we find
$$\mathbb{S}_{2m}\otimes_{\mathbb{C}}\mathbb{S}_{2m}\cong\wedge^*\mathbb{C}^{2m}\quad\text{as }\mathrm{Spin}(2m)\text{-modules.}$$

Assume $n=2m+1$. $\mathbb{C}l_{2m+1}=\mathbb{C}l_{2m+1}^+\oplus\mathbb{C}l_{2m+1}^-$, with diagonal embedding
$$\mathbb{C}l_{2m+1}^0=\{x\oplus\hat{x}|x\in\mathbb{C}l_{2m+1}^+\}.$$
Let $\mathbb{S}_{2m+1}$ denote the irreducible representation of $\mathbb{C}l_{2m+1}^+\cong\mathbb{C}l_{2m+1}^0$. Similar argument as above shows
$$\mathbb{S}_{2m+1}\otimes_{\mathbb{C}}\mathbb{S}_{2m+1}\cong\mathbb{C}l_{2m+1}^0=\wedge^{even}\mathbb{C}^{2m+1}.$$
□

3.3. **Spinors in physics.**

Definition 3.14. Let V be a $\mathbb{C}$-representation of a real group G.

- V is said to be of **real type** if there exists a G-invariant complex structure $J:V\to V$ (i.e. J is complex conjugate linear and $J^2=1$). The real points $\mathrm{Re}(V)=\{v\in V|J(v)=v\}$ is a $\mathbb{R}$-representation of G.
- V is said to be of **quaternionic type** if there exists a G-invariant quaternionic structure $J:V\to V$ (i.e. J is complex conjugate linear and $J^2=-1$).

Various spinors in physics terminology have the following interpretation.

(1) **Dirac spinor**. $\mathbb{S}_{n=p+q}$ gives a $\mathbb{C}$-representation of $\mathrm{Spin}(p,q)$ under an isomorphism

$$\mathrm{Spin}(p,q)\otimes_{\mathbb{R}}\mathbb{C}\cong \mathrm{Spin}_{\mathbb{C}}(n).$$

This representation is called a Dirac spinor.

(2) **Weyl spinor**. When $n=2m$ is even, $\mathbb{S}_n$ is decomposed into two irreducible $\mathbb{C}$-representations

$$\mathbb{S}_n=\mathbb{S}_n^+\oplus\mathbb{S}_n^-.$$

Each $\mathbb{S}_n^\pm$ is called a Weyl spinor.

(3) **Majorana spinor**. If the $\mathbb{C}$-representation $\mathbb{S}_{n=p+q}$ of $\mathrm{Spin}(p,q)$ is of real type, then its real points M_n is called a Majorana spinor.

(4) **Symplectic-Majorana spinor**. If the $\mathbb{C}$-representation $\mathbb{S}_{n=p+q}$ of $\mathrm{Spin}(p,q)$ is of quaternionic type, then $\mathbb{S}_n$ is called a symplectic-Majorana spinor.

(5) **Majorana-Weyl spinor**. When $n=2m=p+q$ is even and the weyl spinors $\mathbb{S}_n^\pm$ are of real type, then the real points are called Majorana-Weyl spinors.

(6) **Symplectic-Majorana-Weyl spinor**. When $n=2m=p+q$ is even and the weyl spinors $\mathbb{S}_n^\pm$ are of quaternionic type, then each $\mathbb{S}_n^\pm$ is called a Symplectic-Majorana-Weyl spinor.

Example 3.15 (Euclidean space)**.** The real and complex representations for Euclidean spaces are summarized as follows.

n	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
$S_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_\pm$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_\pm$	$\mathbb{R}^{16}$
Irred $\mathbb{R}$-spinor	$\mathbb{R}$ (M)	$\mathbb{C}$ (W)	$\mathbb{H}$ (SM)	$\mathbb{H}_\pm$ (SMW)	$\mathbb{H}^2$ (SM)	$\mathbb{C}^4$ (W)	$\mathbb{R}^8$ (M)	$\mathbb{R}^8_\pm$ (MW)
$\mathbb{C}l_n$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$\mathbb{C}(16)$
$\mathbb{S}_n$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^{16}$
Irred $\mathbb{C}$-spinors	$\mathbb{C}$	$\mathbb{C}_\pm$	$\mathbb{C}^2$	$\mathbb{C}^2_\pm$	$\mathbb{C}^4$	$\mathbb{C}^4_\pm$	$\mathbb{C}^8$	$\mathbb{C}^8_\pm$

(1) $n\equiv 8 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Majorana-Weyl** (MW) spinors.

(2) $n\equiv 4 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana-Weyl** (SMW) spinor.

(3) $n\equiv 2,6 \mod 8$. The irreducible chiral $\mathbb{C}$-spinors are **Weyl** spinors. They give rise to equivalent $\mathbb{R}$-spinors which are complex conjugate of each other.

(4) $n\equiv 1,7 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Majorana** spinors.

(5) $n\equiv 3,5 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana** spinors.

Example 3.16 (Minkowski space)**.** The real and complex representations for Minkowski spaces are summarized as follows.

n	1	2	3	4	5	6	7	8
$Cl_{n-1,1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$S_{n-1,1}$	$\mathbb{R}_\pm$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$\mathbb{H}^2_\pm$	$\mathbb{H}^4$	$\mathbb{C}^8$	$\mathbb{R}^{16}$
Irred $\mathbb{R}$-spinor	$\mathbb{R}$ (M)	$\mathbb{R}_\pm$ (MW)	$\mathbb{R}^2$ (M)	$\mathbb{C}^2$ (W)	$\mathbb{H}^2$ (SM)	$\mathbb{H}^2_\pm$ (SMW)	$\mathbb{H}^4$ (SM)	$\mathbb{C}^8$ (W)
$\mathbb{C}l_n$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$\mathbb{C}(16)$
$\mathbb{S}_n$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^{16}$
Irred $\mathbb{C}$-spinors	$\mathbb{C}$	$\mathbb{C}_\pm$	$\mathbb{C}^2$	$\mathbb{C}^2_\pm$	$\mathbb{C}^4$	$\mathbb{C}^4_\pm$	$\mathbb{C}^8$	$\mathbb{C}^8_\pm$

(1) $n\equiv 2 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Majorana-Weyl** (MW) spinors.

(2) $n\equiv 6 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana-Weyl** (SMW) spinor.

(3) $n \equiv 4,8 \mod 8$. The irreducible chiral $\mathbb{C}$-spinors are **Weyl** spinors. They give rise to equivalent $\mathbb{R}$-spinors which are complex conjugate of each other.
(4) $n \equiv 1,3 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Majorana** spinors.
(5) $n \equiv 5,7 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana** spinors.

3.4. **Unitarity.**

Definition 3.17. Let V be a vector space over k [1]. A k-hermitian form h is a $\mathbb{R}$-bilinear pairing

$$h(-,-) : V \otimes_{\mathbb{R}} V \to k$$

such that for any $v_1, v_2 \in V, \lambda \in k$

- $h(v_1, v_2\lambda) = h(v_1, v_2)\lambda$.
- $h(v_1, v_2) = \overline{h(v_2, v_1)}$.

h is called positive-definite if $h(v,v) > 0$ for any nonzero $v \in V$.

Remark 3.18. We have the alternate description of hermitian forms

(1) $\mathbb{R}$-hermitian form is the same as an inner product.
(2) $\mathbb{C}$-hermitian form is the same as a I-invariant sympletic pairing $\omega : \wedge^2_{\mathbb{R}} V \to \mathbb{R}$. Here I ($I^2 = -1$) defines the complex structure on V. Then

$$h(v_1, v_2) = \omega(v_1, Iv_2) + \mathrm{i}\,\omega(v_1, v_2).$$

(3) $\mathbb{H}$-hermitian form is the same as a sympletic pairing $\omega_{\mathbb{C}} : \wedge^2_{\mathbb{C}} V \to \mathbb{C}$ such that

$$\omega_{\mathbb{C}}(v_1 J, v_2 J) = \overline{\omega_{\mathbb{C}}(v_1, v_2)}.$$

Here $J : V \to V$ ($J^2 = -1$) is the complex conjugate linear operator defining the quaternionic structure on V. Then

$$h(v_1, v_2) = \overline{\omega_{\mathbb{C}}(v_1, v_2 J)} + \mathrm{j}\,\omega_{\mathbb{C}}(v_1, v_2).$$

Lemma 3.19. *Let G be a finite group or compact Lie group. Let W be a k-representation of G. Then W carries a G-invariant positive definite k-hermitian form.*

Proof. Let h be any positive definite k-hermitian form. Averaging h over G gives a desired hermitian form. □

3.4.1. *Euclidean space.* We consider the Euclidean space $\mathbb{R}^n$ and unitarity of spinors.

Proposition 3.20. *Let W be a k-representation of Cl_n. Then there exists a positive definite k-hermitian form h on W that is invariant under Clifford multiplication by unit vectors $e \in \mathbb{R}^n$, i.e.,*

$$h(e \cdot s_1, e \cdot s_2) = h(s_1, s_2), \quad \forall e \in \mathbb{R}^n, |e|^2 = 1, \quad s_i \in W.$$

In particular, h leads to group homomorphism

$$\mathrm{Spin}(n) \to \begin{cases} SO(W) & k = \mathbb{R} \\ SU(W) & k = \mathbb{C} \\ Sp(W) & k = \mathbb{H}. \end{cases}$$

[1]When $k = \mathbb{H}$, V is a right $\mathbb{H}$-module

Proof. Consider the finite group with presentation

$$\Gamma_n = \left\langle e_1, \cdots, e_n, -1 | e_i^2 = -1, (-1)^2 = 1, e_i e_j = (-1) e_j e_i, (-1) e_i = e_i(-1) \right\rangle.$$

Then W carries a representation of Γ_n such that (-1) acts as $-\mathrm{Id}$. Then h is given by a Γ_n-invariant positive definite k-hermitian form. □

Remark 3.21. For later applications, we collect formulae for real dimensions

$$\begin{cases} \dim_{\mathbb{R}} SO(n) = \frac{1}{2}n(n-1) \\ \dim_{\mathbb{R}} SU(n) = n^2 - 1 \\ \dim_{\mathbb{R}} Sp(n) = n(2n+1). \end{cases}$$

Example 3.22. The irreducible real spinor of Euclidean $\mathbb{R}^6$ is $\mathbb{C}^4$. This leads to an isomorphism

$$\mathrm{Spin}(6) \to SU(4).$$

This is explicitly realized in Example 2.11.

3.4.2. *Minkowski space.* Now we consider the Minkowski space. Let $\{e_i\}_{i=1,\cdots,n}$ be the orthonormal generators of $Cl_{n,0}$. Then the generators $\tilde{e}_0, \tilde{e}_1, \cdots, \tilde{e}_{n-1}$ of $Cl_{n-1,1}$ can be realized inside $\mathbb{C}l_n$ via

$$\tilde{e}_1 = e_1, \cdots, \tilde{e}_{n-1} = e_{n-1}, \tilde{e}_0 = \sqrt{-1} e_n.$$

Let $\mathbb{S}$ be a $\mathbb{C}$-representation of $\mathbb{C}l_n$, with a $\mathbb{C}$-hermitian form h by Proposition 3.20. Then

$$h(\tilde{e}_0 s_1, s_2) = h(s_1, \tilde{e}_0 s_2), \quad h(\tilde{e}_i s_1, s_2) = -h(s_1, \tilde{e}_i s_2), \quad 1 \le i \le n-1.$$

Let us denote

$$\langle s_1, s_2 \rangle_0 = h(\tilde{e}_0 s_1, s_2).$$

Then $\langle s_1, s_2 \rangle$ is a $\mathbb{C}$-hermitian form satisfying

$$\langle s_1, \tilde{e}_i s_2 \rangle_0 = \langle \tilde{e}_i s_1, s_2 \rangle_0, \quad 0 \le i \le n-1.$$

In particular, $\langle -, - \rangle_0$ is $\mathrm{Spin}(n-1,1)$-invariant but not positive definite.

Proposition 3.23. *Let $x \in \wedge^k \mathbb{R}^k \subset Cl_{n-1,1}$. Then*

$$\langle s_1, x \cdot s_2 \rangle_0 = (-)^{k(k-1)/2} \overline{\langle s_2, x \cdot s_1 \rangle_0}.$$

In particular, $\mathrm{i}^{k(k-1)/2} \langle s, xs \rangle_0$ is real.

Remark 3.24. In physics application, this proposition shows the reality of the following expression

$$\int \psi^\dagger \gamma^0 (i\gamma^\mu \partial_\mu - m)\psi.$$

3.5. **Charge conjugation.**

Definition 3.25. Let $\mathbb{S}$ be a k-representation of $\mathrm{Spin}(p,q)$. We define a **charge conjugation** on $\mathbb{S}$ to be a $\mathrm{Spin}(p,q)$-invariant non-degenerate bilinear form $C : \mathbb{S} \otimes_k \mathbb{S} \to k$.

We consider charge conjugation for complex spinors.

Definition 3.26. Let $Cl(V)$ be a Clifford algebra, W be a Clifford k-module. We define two Clifford structures ons its k-linear dual, denoted by $W^\vee, W^\triangledown$ respectively, via

$$W^\vee: \quad (x\cdot\varphi)(w) = \varphi(x^t\cdot w), \quad \forall x\in W, x\in Cl(V).$$

and

$$W^\triangledown: \quad (x\cdot\varphi)(w) = \varphi(\hat{x}^t\cdot w), \quad \forall x\in W, x\in Cl(V).$$

They induce identical spin representations. A **charge conjugation** C on W is a Clifford module isomorphism between W and $W^\vee$ or $W^\triangledown$.

Equivalently, C can be viewed as a non-degenerate bilinear form $C: W\otimes_k W\to k$ such that

$$C(e\cdot s_1, s_2) = \eta C(s_1, e\cdot s_2) \quad \forall s_i\in W, e\in V.$$

Here η is either $+1$ or -1. We divide our discussion into cases when n is even or odd.

3.5.1. *n=2m.*

Lemma/Definition 3.27. *Let $\mathbb{S}_{2m}$ be the complex spin representation. Then there exists unique (up to rescaling) $\mathbb{C}l_{2m}$-module isomorphisms*

$$C_+: \mathbb{S}_{2m}\to\mathbb{S}_{2m}^\vee, \quad C_-: \mathbb{S}_{2m}\to\mathbb{S}_{2m}^\triangledown.$$

The corresponding charge conjugation is denoted by

$$C_\pm: \mathbb{S}_{2m}\otimes_{\mathbb{C}}\mathbb{S}_{2m}\to\mathbb{C}.$$

It satisfies the following symmetry properties

$$\boxed{C_\pm(\alpha,\beta) = (-)^{m(m\mp1)/2}C_\pm(\beta,\alpha)}, \quad \alpha,\beta\in\mathbb{S}_{2m}.$$

Proof. The definition of $\mathbb{C}_\pm$ follows from the uniqueness of $\mathbb{C}l_{2m}$-representation.

To see the symmetry property, we use the presentation in Example 3.12. Let $V=\mathbb{R}^{2m}=\mathbb{C}^m, V_{\mathbb{C}} = V\otimes_{\mathbb{R}}\mathbb{C} = V_{\mathbb{C}}^{1,0}\oplus V_{\mathbb{C}}^{0,1}$. Let us represent $\mathbb{S}_{2m} = \wedge^* V_{\mathbb{C}}^{1,0}$. Then the pairing

$$C_\pm: \wedge^*V_{\mathbb{C}}^{1,0}\otimes\wedge^*V_{\mathbb{C}}^{1,0}\to\wedge^m V_{\mathbb{C}}^{1,0}\cong\mathbb{C}$$

is given by

$$C_+(\alpha,\beta) = (\alpha^t\wedge\beta)^{top}, \quad C_-(\alpha,\beta) = (\hat{\alpha}^t\wedge\beta)^{top}, \quad \alpha,\beta\in\wedge^*V_{\mathbb{C}}^{1,0}.$$

Here $(dz_{i_1}\cdots dz_{i_k})^t = dz_{i_k}\cdots dz_{i_1}$, $\quad \left(\widehat{dz_{i_1}\cdots dz_{i_k}}\right)^t = (-1)^k dz_{i_k}\cdots dz_{i_1}$. The symmetry property follows. $\square$

Definition 3.28. Let $V_{\mathbb{C}} = \mathbb{C}^{2m}$. We define the pairing

$$\Gamma_\pm^k: \mathbb{S}_{2m}\otimes_{\mathbb{C}}\mathbb{S}_{2m}\to\wedge^k V_{\mathbb{C}}^\vee$$

by the formula

$$\Gamma_\pm^k(s_1,s_2)(\alpha) = C_\pm(s_1,\alpha\cdot s_2)$$

where $\alpha\cdot s_2$ is the Clifford action.

Proposition 3.29. $\Gamma_\pm^k$ *has the following symmetry property:*

$$\boxed{\Gamma_\pm^k(s_1,s_2) = (-1)^{k(k\mp1)/2+m(m\mp1)/2}\Gamma_\pm^k(s_2,s_1)}, \quad s_i\in\mathbb{S}_{2m}.$$

Proof.

$$\begin{aligned}\Gamma^k_\pm(s_1,s_2)(\alpha) &= C_\pm(s_1,\alpha s_2) = (\pm)^k C_\pm(\alpha^t s_1, s_2) = (\pm)^k(-)^{k(k-1)/2}C_\pm(\alpha s_1, s_2)\\ =&(\pm)^k(-)^{k(k-1)/2}(-)^{m(m\mp1)/2}C_\pm(s_2,\alpha s_1) = (-1)^{k(k\mp1)/2+m(m\mp1)/2}\Gamma^k_\pm(s_2,s_1)(\alpha).\end{aligned}$$

□

Remark 3.30. Note that for $m+k$ is odd, the pairing $\Gamma^k_\pm$ is in fact on different chiral spinors

$$\Gamma^k_\pm : \mathbb{S}_+\otimes\mathbb{S}_- \to \wedge^k V_{\mathbb{C}}^\vee.$$

On the other hand, when $m+k$ is even, then $\Gamma^k_\pm$ have the same symmetry property.

3.5.2. $n=2m+1$. Let us first describe the charge conjugation. Let $\mathbb{S}=\mathbb{S}_{2m+1}$ be the irreducible representation of $\mathbb{C}l^+_{2m+1}$. Observe that the volume form has the property

$$\omega^t = (-1)^m\omega, \quad \hat{\omega}^t = -(-1)^m\omega.$$

This implies that

$$\mathbb{S}_{2m+1} \cong \begin{cases} \mathbb{S}^\vee_{2m+1} & m \text{ even} \\ \mathbb{S}^{\hat\vee}_{2m+1} & m \text{ odd}\end{cases}.$$

Definition 3.31. When $n=2m+1$, there is only one charge conjugation (up to rescaling) by

$$\begin{aligned} C_+ &: \mathbb{S}_{2m+1}\otimes\mathbb{S}_{2m+1}\to\mathbb{C}, \quad m \text{ even}\\ C_- &: \mathbb{S}_{2m+1}\otimes\mathbb{S}_{2m+1}\to\mathbb{C}, \quad m \text{ odd}\end{aligned}$$

We will just denote it by $C:\mathbb{S}_{2m+1}\otimes\mathbb{S}_{2m+1}\to\mathbb{C}$.

Lemma 3.32. *The charge conjugation C has the following symmetry property*

$$\boxed{C(\alpha,\beta) = (-)^{m(m+1)/2}C(\beta,\alpha)}.$$

Proof. Let us consider the embedding

$$j : \mathbb{C}l_{2m} \cong \mathbb{C}l^0_{2m+1} \subset \mathbb{C}l_{2m+1}.$$

It is easy to see that

$$j(\hat{x}^t) = j(x)^t = \overline{j(x)}^t.$$

Therefore the symmetry property of C on $\mathbb{S}_{2m+1}$ is the same as C_- on $\mathbb{S}_{2m}$ for any m. □

Definition 3.33. Let $V_{\mathbb{C}} = \mathbb{C}^{2m+1}$. We can define the pairing

$$\Gamma^k : \mathbb{S}_{2m+1}\otimes\mathbb{S}_{2m+1}\to\wedge^k V^\vee_{\mathbb{C}}.$$

Similarly,

Proposition 3.34. Γ^k *has the following symmetry property:*

$$\boxed{\Gamma^k(s_1,s_2) = (-1)^{k(k-1)/2+mk+m(m+1)/2}\Gamma^k(s_2,s_1)}, \quad s_i\in\mathbb{S}_{2m+1}.$$

Note that $(-1)^{k(k-1)/2+mk+m(m+1)/2} = (-1)^{(m-k)(m-k+1)/2}$.

3.5.3. *Majorana spinor revisited.* Now we revisit the meaning of Majorana spinor for $\mathrm{Spin}(p,q)$. Let e_i be the Clifford generator of $Cl(p,q)$ such that

$$e_i^2 = \begin{cases} -1 & \text{if } 1 \leq i \leq p \\ +1 & \text{if } p+1 \leq i \leq n = p+q. \end{cases}$$

Let $\mathbb{S}_n$ be the complex spin representation. Let $(-,-)$ be a hermitian form on $\mathbb{S}_n$ such that

$$(e_i \cdot s_1, s_2) = \eta\,(s_1, e_i \cdot s_2), \quad 1 \leq i \leq n, s_1, s_2 \in \mathbb{S}_n.$$

Here $\eta = \pm 1$ is a fixed sign (we can choose $\eta = (-1)^{q+1}$). In the Euclidean case, $(-,-)$ is the hermitian inner product with $\eta = -1$. In the Minkowski case, $(-,-) = \langle -,- \rangle_0$ defined in Section 3.4.2 with $\eta = 1$.

Let $C(-,-)$ be a charge conjugation. We define a complex conjugate linear map $* : \mathbb{S}_n \to \mathbb{S}_n, s \to s^*$

$$h(s_1, s_2) = C(s_1^*, s_2), \quad \forall s_i \in \mathbb{S}_n.$$

For any unit generator e_i and $s \in \mathbb{S}_n$,

$$(e_i \cdot s)^* = \pm e_i \cdot s^*.$$

where the sign $\pm$ depends on the signature and charge conjugation. In particular, $*$ is $\mathrm{Spin}(n)$-equivariant.

Majorana-type conditions for $\mathbb{S}_n$ appear precisely when $*^2 = \pm 1$. Precisely,

$$(*)^2 = \begin{cases} 1 & \text{Majorana} \\ -1 & \text{Symplectic-Majorana} \end{cases}.$$

- When $*^2 = 1$, $*$ defines a real structure. The Majorana spinors can be expressed by
$$s^* = s, \quad s \in \mathbb{S}_n.$$
- When $*^2 = -1$, $*$ defines a quaternionic structure. We need several spinors $\mathbb{S}_n^{\oplus N}$ to impose the symplectic-Majorana condition
$$s^* = \Omega s, \quad s \in \mathbb{S}_n^{\oplus N},$$
where Ω is a anti-symmetric $N \times N$-matrix with $\Omega\overline{\Omega} = -1$.

4. Poincaré group

The **Poincaré group** is the isometry group of $\mathbb{R}^{p,q}$. We work with its universal cover and denote by

$$\mathrm{Poin}(p,q) = \mathbb{R}^{p,q} \rtimes \mathrm{Spin}(p,q).$$

In physics, particles are organized into unitary representations of Poincaré group. There is an essential difference between Euclidean and Minkowski cases: $\mathrm{Spin}(d)$ is a compact simple Lie group while $\mathrm{Spin}(d-1,1)$ is a non-compact simple Lie group. It is known that every non-trivial irreducible unitary representation of a non-compact simple Lie group is infinite dimensional, while for compact Lie groups they are all finite dimensional. We will focus on the Minkowski space in this section.

4.1. **Poincaré algebra.** Let $\mathrm{poin}(d-1,1)$ be the Lie algebra of $\mathrm{Poin}(d-1,1)$, called the **Poincaré algebra**.

Let us choose linear coordinates x^μ of $\mathbb{R}^{d-1,1}$ with metric

$$\eta = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^{d-1})^2.$$

A basis of $\mathrm{poin}(d-1,1)$ can be represented by

$$\mathbf{P}_\mu = -i\frac{\partial}{\partial x^\mu}, \quad \mathbf{M}_{\mu\nu} = -i\left(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}\right)$$

satisfying the Poincaré algebra relations

$$\begin{aligned} \left[\mathbf{P}_\mu, \mathbf{P}_\nu\right] &= 0 \\ \left[\mathbf{M}_{\mu\nu}, \mathbf{P}_\rho\right] &= i\eta_{\mu\rho}\mathbf{P}_\nu - i\eta_{\nu\rho}\mathbf{P}_\mu \\ \left[\mathbf{M}_{\mu\nu}, \mathbf{M}_{\rho\sigma}\right] &= i\eta_{\mu\rho}\mathbf{M}_{\nu\sigma} - i\eta_{\nu\rho}\mathbf{M}_{\mu\sigma} - (\rho \leftrightarrow \sigma) \end{aligned}$$

There are two Casimir operators

$$C_1 = -\mathbf{P}^2 = -\mathbf{P}^\mu \mathbf{P}_\mu, \quad C_2 = -\frac{1}{2}\mathbf{P}^2 \mathbf{M}_\mu \mathbf{M}^{\mu\nu} + \mathbf{M}_{\mu\rho}\mathbf{P}^\rho \mathbf{M}^{\mu\sigma}\mathbf{P}_\sigma.$$

Remark 4.1. When $d = 4$, C_2 is the square of the Pauli-Lubanski vector

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{M}^{\nu\rho}\mathbf{P}^\sigma.$$

W^μ commutes with $\mathbf{P}_\mu$, transfers as a vector under $\mathbf{M}_{\mu\nu}$, and with its own commutator relation

$$\left[W_\mu, W_\nu\right] = i\epsilon_{\mu\nu\rho\sigma}W^\rho \mathbf{P}^\sigma.$$

C_1, C_2 essentially classify unitary irreducible representations in four dimension. More Casimir operators are present in higher dimensions.

4.2. **Unitary representation.** We discuss Wigner's classification of nonnegative-energy irreducible unitary representations of $\mathrm{Poin}(d-1,1)$ in terms of induced representations of the little group.

The first Casimir operator

$$C_1 = -\mathbf{P}^\mu \mathbf{P}_\mu = m^2$$

has the physics interpretation of **mass** and $\mathbf{P}^0$ is the **energy**. We only consider non-negative energy representations, i.e., $\mathbf{P}^0 \geq 0$. The second Casimir operator C_2 is the spin operator. The representations will be characterized by the mass and spin.

Let H be a irreducible unitary representation of $\mathrm{Poin}(d-1,1)$. Since the translation subgroup is Abelian, we can decompose into common eigenvalues

$$\mathrm{H} = \bigoplus_{p\in\mathcal{O}} \mathrm{H}_p, \quad \mathbf{P}^\mu = p^\mu \text{ on } \mathrm{H}_p\,.$$

Here $\mathcal{O}$ is a $SO(d-1,1)$-orbit in $\mathbb{R}^{d-1,1}$. The eigenvalue $p = \{p^\mu\}$ is called the **momentum**. Let $\mathrm{Stab}_{\mathcal{O}} \subset \mathrm{Spin}(d-1,1)$ be the stablizer subgroup of the orbit $\mathcal{O}$. This is Wigner's little group. Then H is induced by a representation V of $\mathrm{Stab}_{\mathcal{O}}$

$$\mathrm{H} = \mathrm{Spin}(d-1,1) \otimes_{\mathrm{Stab}_{\mathcal{O}}} V$$

which carries a natural Poincaré group action. $\dim V$ is often called the **physics degree of freedom**.

- $m^2 > 0$. This case is called $\boxed{\textit{massive}}$. $\mathcal{O}$ is the orbit of $p = (m, 0, \cdots, 0)$. $\mathrm{Stab}_{\mathcal{O}} = \mathrm{Spin}(d-1)$.
- $m^2 = 0$. This case is called $\boxed{\textit{massless}}$.

- If $p \neq 0$, then $\mathcal{O}$ is the orbit of $(E, 0, \cdots, 0, E)$. $\mathrm{Stab}_{\mathcal{O}} = \mathrm{Poin}(d-2, 0)$, which can be seen by using the light cone coordinate

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^{d-1} \pm x^0).$$

In this light cone frame, $\mathcal{O}$ is the orbit of $(p^-, 0, \cdots, 0)$. Then $\mathrm{Stab}_{\mathcal{O}}$ is generated by $\{\mathbf{M}_{mn}, \mathbf{M}_{m+}\}_{1 \leq m,n \leq d-2}$. The representation of $\mathrm{Poin}(d-2, 0)$ is further induced: let ξ^m be the eigenvalue of $\mathbf{M}_{m+}$.
  - $\xi \neq 0$. The little group is $\mathrm{Spin}(d-3)$. This case is called $\boxed{infinite\ spin}$.
  - $\xi = 0$. The little group is $\mathrm{Spin}(d-2)$. This case is called $\boxed{helicity}$.
- If $p = 0$, then $\mathcal{O}$ is the origin. $\mathrm{Stab}_{\mathcal{O}} = \mathrm{Spin}(d-1, 1)$. This case is called $\boxed{zero\ momentum}$.

- $m^2 < 0$. This case is called $\boxed{tachyonic}$. $\mathcal{O}$ is the orbit of $(0, \cdots, 0, \sqrt{-m^2})$. $\mathrm{Stab}_{\mathcal{O}} = \mathrm{Spin}(d-2, 1)$.

4.3. **Coleman-Mandula Theorem.** Let H be a unitary representation of $\mathrm{Poin}(d-1,1)$. The physics system is described by the **S-matrix**, which is a $\mathrm{Poin}(d-1,1)$-equivariant unitary operator

$$S : \mathrm{Sym}(\mathrm{H}) \to \mathrm{Sym}(\mathrm{H}).$$

H is $\mathbb{Z}_2$-graded and Sym is the graded symmetric product. By a symmetry of the S-matrix, we mean an operator $B : \mathrm{Sym}(H) \to \mathrm{Sym}(H)$ which is a derivation and commutes with S.

Under a suitable assumption in the massive case, Coleman-Mandula Theorem says that the Lie algebra of all even symmetries of the S-matrix ($d > 2$) of is a direct sum

$$\mathrm{poin}(d-1,1) \oplus I$$

I is called **internal symmetry**, which does not mix with Poincaré group. In the case when only massless representations exist, Poincaré algebra may be enlarged to conformal algebra.

However, if we allow odd symmetries, then there exists nontrivial extensions of Poincaré algebra. They are called super Poincaré algebras and classified by the Haag-Lopuszanski-Sohnius Theorem.

5. To be continued