Masters Bridge Program **Summer 2021**

Session 5: Continuous random variables: part II

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We continue our review of continuous random variables with further exercises on manipulating their probability density functions or cumulative distribution functions.

5.1 Max and min of random variables

Exercise 1 (distribution and density functions of min and max)**.** Let $X_1, \ldots, X_n$ be independent random variables. Define $X_{\text{max}} = \max(X_1, ..., X_n)$ and $X_{\text{min}} = \min(X_1, ..., X_n)$.

(a) Suppose X_i has distribution function F_i. Express the distribution functions of X_{max} and X_{min} in terms of the individual distribution functions $F_1, ..., F_n$.

(b) Now suppose further that each X_i have the same distribution F. Differentiate the distribution function from (a) to find the density functions of X_{max} and X_{min}.

Exercise 2. (min of independent exponential variables). Let $X_1, ..., X_n$ be exponentially distributed random variables. Let the rate parameter of X_i be λ_i.

Show that $X_{\text{min}} = \min(X_1, ..., X_n)$ is also exponentially distruted with rate parameter $\lambda = \lambda_1 + ... + \lambda_n$.

Exercise 3. (min of independent uniform variables). Let $X_1, ..., X_n$ be uniformly distributed on (0, 1). Compute the density function of X_{min}.

5.2 Joint distributions

Given a pair of random variables, their *joint distribution* is the probability distribution over the plane,

$$\mathbb{P}(B) = \mathbb{P}((X, Y) \in B)$$

where $B \subset \mathbb{R}^2$.

The *joint density* $f(x, y)$ gives the probability that (X, Y) are in an infinitesimal neighborhood of (x, y). Probability on sets B are given by the usual integration formula,

$$\mathbb{P}((X, Y) \in B) = \int\int_B f(x, y)\, dx\, dy.$$

We record some other definitions:

- **Marginal distributions**:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\ dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\ dx.$$

  In other words, given a joint density, we can recover the density on X or Y alone by integrating out the other variable.

- **Independence**: Random variables X and Y are independent if and only if the joint density is a product of the two marginals,

$$f(x,y) = f_X(x) f_Y(y).$$

- **Expectations**: Let g be a function that maps $\mathbb{R}^2 \mapsto \mathbb{R}$. Then

$$\mathbb{E}(g(X,Y)) = \int\int g(x,y) f(x,y)\ dx\, dy.$$

Exercise 4 (two independent uniform random variables)**.** Let X, Y be independent Uniform$(0,1)$ random variables.

(a) Find $\mathbb{P}(X^2 + Y^2 \leq 1)$.

(b) Find $\mathbb{P}(X^2 + Y^2 \leq 1 | X + Y \geq 1)$.

(c) Find $\mathbb{P}(Y \leq X^2)$.

Exercise 5 (uniform on a triangle)**.** Let X, Y uniformly distribution on the region $\{(x,y) : 0 < x < y < 1\}$.

(a) Find the joint density of (X,Y).

(b) Find marginal densities of X and Y.

(c) Are X and Y independent?

(d) Find $E(XY)$.

Exercise 6 (Independent exponential variables)**.** Let X and Y be independent and exponentially distributed random variables with parameters λ and μ, respectively. Calculate $\mathbb{P}(X < Y)$.

5.3 Sums of random variables

Recall that we have already seen that expectations are additive: for any two random variables X, Y (regardless of independence), we have

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

We now give a formula for densities.

If X, Y has density $f(x, y)$, then $X + Y$ has density

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z - x)\, dx,$$

If X and Y are independent, then the formula simplifies to

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx,$$

Exercise 7 (Sum of exponential random variables)**.** Let X and Y be independent exponentially distributed random variables with rate λ. Compute the density $X + Y$.

We record a fact about Normal random variables, to save us the tedium of doing integrals:

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and independently, let $Y \sim \mathcal{N}(\lambda, \tau^2)$. Then $X + Y$ is Normally distributed with mean $\mu + \lambda$ and variance $\sigma^2 + \tau^2$.

Exercise 8 (Distribution of heights)**.** Suppose heights in a large populatoin are approximately normally distributed with a mean of 1.78m with a standard deviaion of 5cm. Suppose a group of 100 people are picked at random from this population.

(a) What is the probability that the tallest person in this group is over 1.93m?

(b) What is the probability that the average height of people in the group is over 1.8cm?

(c) Suppose the distribution of heights was not normal, but some other distribution with the given mean and SD. Would the probability to (a) or (b) change, or whould they remain approximately the same?

Exercise 9 (Catching BART)**.** Suppose BART is scheduled to arrive at Downtown Berkeley station at 8:10am, but its actual arrival time is normally distributed with mean 8:10am and standard deviation 40 seconds. Suppose I try to arrive at the BART station at 8:09am, but my arrival time is actually a normal distribution with mean 8:09am with standard deviation 30 seconds.

(a) What percentage of the time do I arrive at the corner before BART is scheduled to arrive?

(b) What percentage of time do I arrive at the station before BART?

(c) If arrive at the stop at 8:09am, but BART still hasn't come by 8:12AM, what is the probabiliy that I have already missed BART?