Masters Bridge Program **Summer 2021**

Session 5: Continuous random variables: part II

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We continue our review of continuous random variables with further exercises on manipulating their probability density functions or cumulative distribution functions.

5.1 Max and min of random variables

Exercise 1 (distribution and density functions of min and max)**.** Let $X_1, \ldots, X_n$ be independent random variables. Define $X_{\text{max}} = \max(X_1, ..., X_n)$ and $X_{\text{min}} = \min(X_1, ..., X_n)$.

(a) Suppose X_i has distribution function F_i. Express the distribution functions of X_{max} and X_{min} in terms of the individual distribution functions $F_1, ..., F_n$.

(b) Now suppose further that each X_i have the same distribution F. Differentiate the distribution function from (a) to find the density functions of X_{max} and X_{min}.

Exercise 2. (min of independent exponential variables). Let $X_1, ..., X_n$ be exponentially distributed random variables. Let the rate parameter of X_i be λ_i.

Show that $X_{\text{min}} = \min(X_1, ..., X_n)$ is also exponentially distruted with rate parameter $\lambda = \lambda_1 + ... + \lambda_n$.

Exercise 3. (min of independent uniform variables). Let $X_1, ..., X_n$ be uniformly distributed on (0, 1). Compute the density function of X_{min}.

5.2 Joint distributions

Given a pair of random variables, their *joint distribution* is the probability distribution over the plane,

$$\mathbb{P}(B) = \mathbb{P}((X, Y) \in B)$$

where $B \subset \mathbb{R}^2$.

The *joint density* $f(x, y)$ gives the probability that (X, Y) are in an infinitesimal neighborhood of (x, y). Probability on sets B are given by the usual integration formula,

$$\mathbb{P}((X, Y) \in B) = \int\int_B f(x, y)\, dx\, dy.$$

We record some other definitions:

- **Marginal distributions**:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

  In other words, given a joint density, we can recover the density on X or Y alone by integrating out the other variable.

- **Independence**: Random variables X and Y are independent if and only if the joint density is a product of the two marginals,

$$f(x,y) = f_X(x)f_Y(y).$$

- **Expectations**: Let g be a function that maps $\mathbb{R}^2 \mapsto \mathbb{R}$. Then

$$\mathbb{E}(g(X,Y)) = \int\int g(x,y)f(x,y)\,dx\,dy.$$

Exercise 4 (two independent uniform random variables)**.** Let X, Y be independent Uniform$(0,1)$ random variables.

(a) Find $\mathbb{P}(X^2 + Y^2 \leq 1)$.

(b) Find $\mathbb{P}(X^2 + Y^2 \leq 1 | X + Y \geq 1)$.

(c) Find $\mathbb{P}(Y \leq X^2)$.

Exercise 5 (uniform on a triangle)**.** Let X, Y uniformly distribution on the region $\{(x,y) : 0 < x < y < 1\}$.

(a) Find $\mathbb{P}(X^2 + Y^2 \leq 1)$.

(b) Find $\mathbb{P}(X^2 + Y^2 \leq 1 | X + Y \geq 1)$.

(c) Find $\mathbb{P}(Y \leq X^2)$.