

Session 6: Dependence between random variables

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6.1 Conditional densities

Recall that for a discrete random variable X , then the conditional distribution of Y is given by

$$\mathbb{P}(Y = y|X = x) = \frac{\mathbb{P}(Y = y, X = x)}{\mathbb{P}(X = x)}.$$

However, if X is a continuous random variable, then the denominator $X = x$ has probability 0, which results in an undefined expression.

In order to define conditional distributions for continuous random variables X and Y , we can express the conditional distribution in terms of densities.

Conditional density of Y given $X = x$. For random variables X and Y with joint density $f(x, y)$, for each x with marginal density $f_X(x) > 0$, the conditional density of Y given $X = x$ is the probability density function

$$f_Y(y|X = x) = \frac{f(x, y)}{f_X(x)}. \quad (6.1)$$

Then for any set B ,

$$\mathbb{P}(Y \in B|X = x) = \int_B f_Y(y|X = x) dy$$

Exercise 1 (Uniform on a triangle). Let X, Y uniformly distribution on the region $\{(x, y) : 0 < x < y < 2\}$. Find $\mathbb{P}(Y > 1|X = x)$.

We can re-arrange Equation (6.1) to arrive at the multiplication rule for densities,

$$f(x, y) = f_X(x)f_Y(y|X = x).$$

Exercise 2 (Gamma and uniform). Suppose X has a Exponential(λ) distribution, and that given $X = x$, Y is uniform on $(0, x)$.

- (a) Find the joint density of (X, Y) .
- (b) Find marginal density of Y .

6.2 Conditional expectations

Let g be a function that maps $\mathbb{R} \mapsto \mathbb{R}$. The **conditional expectation of $g(Y)$ given $X = x$** is

$$\mathbb{E}(g(Y)|X = x) = \int g(y)f_Y(y|X = x) dx.$$

Exercise 3. Show,

(a) **Average conditional probability:**

$$\mathbb{P}(Y \in B) = \int \mathbb{P}(Y \in B|X = x)f_X(x) dx$$

(b) **The tower property:**

$$E(Y) = \int \mathbb{E}(Y|X = x)f_X(x) dx.$$

We can succinctly write the tower property as $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X))$.

Exercise 4 (Expectation of a product). Let X and Y be random variables, and h a function of X . Show that

$$\mathbb{E}(h(X)Y) = \mathbb{E}(h(X)\mathbb{E}(Y|X)).$$

Exercise 5 (Conditional variance). can't decide if i want to include this (6.2.18).

Exercise 6 (Predictions). In a previous session, we saw that if we want to predict Y by a constant b , in order to minimize the MSE

$$\text{MSE}(b) = \mathbb{E}((Y - b)^2),$$

we should set $b = \mathbb{E}(Y)$.

Now suppose the data come in pairs (X, Y) , and we get to observe X in order to predict Y . We want to construct a predictor, which is a function $g(X)$, to predict Y .

If we again want to minimize the MSE,

$$\text{MSE}(g(X)) = \mathbb{E}((Y - g(X))^2),$$

argue that the optimal g is $g(x) = \mathbb{E}(Y|X = x)$.