Calculus of Variations and Optimal Control

1 Introduction

The calculus of variations is concerned with the determination of extrema (maxima and minima) or stationary values of functionals. A functional can be defined as a function of several other functions. Hence the calculus of variations can be used to solve trajectory optimization problems.

A simple problem in the theory of the calculus of variations with no constraints can be stated as follows:

Find a function $u(x)$ that minimizes the functional (integral)

$$A = \int_{x_1}^{x_2} F(x, u, u', u'', \cdots)dx, \tag{1}$$

where A and F can be called functionals (functions of other functions). Here x is the independent variable,

$$u = u(x), \quad u' = \frac{du(x)}{dx}, \quad u'' = \frac{d^2u(x)}{dx^2}, \quad \cdots . \tag{2}$$

In mechanics, the functional usually possesses a clear physical meaning. For example, in the mechanics of deformable solids, the potential energy (P) plays the role of the functional (P is a function of the displacement components u, v, and w, which, in turn, are functions of the coordinates x, y, and z).

The integral in Eq. (1) is defined in the region or domain $[x_1, x_2]$. Let the values of u be prescribed on the boundaries as $u(x_1) = u_1$ and $u(x_2) = u_2$. These are called the boundary conditions of the problem. One of the procedures that can be used to solve the problem in Eq. (1) will be as follows:

1. Select a series of trial or tentative solutions $u(x)$ for the given problem and express the functional A in terms of each of the tentative solutions.

2. Compare the values of A given by the different tentative solutions.

3. Find the correct solution to the problem as that particular tentative solution which makes the functional A assume an extreme or stationary value.

The mathematical procedure used to select the correct solution from a number of tentative solutions is called the **calculus of variations**.

2 Stationary Values of Functionals

Any tentative solution $\bar{u}(x)$ in the neighborhood of the exact solution $u(x)$ may be represented as

$$\bar{u}(x) = u(x) + \delta u(x). \tag{3}$$

The variation in u, i.e., δu is defined as an infinitesimal, arbitrary change in u for a fixed value of the variable x, i.e., for $\delta x = 0$. Here δ is called the variational operator (similar to the differential operator d). The operation of variation is commutative with both integration and differentiation, that is,

$$\delta\left(\int F dx\right) = \int (\delta F) dx, \quad \delta\left(\frac{du}{dx}\right) = \frac{d}{dx}(\delta u). \tag{4}$$

Also, we define the variation of a function of several variables or a functional in a manner similar to the calculus definition of a total differential:

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u'' + \frac{\partial F}{\partial x}\delta x. \tag{5}$$

If we are finding variation of F for a fixed value of x, then $\delta x = 0$.

Now, let us consider the variation in A (δA) corresponding to variations in the solution (δu). If we want the condition for the stationariness of A, we take the necessary condition as the vanishing of first derivative of A,

$$\delta A = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u''\right) dx = \int_{x_1}^{x_2} \delta F dx = 0. \tag{6}$$

Integrate the second and third terms by parts* to obtain

$$\begin{aligned}\int_{x_1}^{x_2} \frac{\partial F}{\partial u'}\delta u' dx &= \int_{x_1}^{x_2} \frac{\partial F}{\partial u'}\delta\left(\frac{\partial u}{\partial x}\right) dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial u'}\frac{\partial}{\partial x}(\delta u) dx \\ &= \frac{\partial F}{\partial u'}\delta u\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\delta u dx. \end{aligned} \tag{7}$$

$$\begin{aligned}\int_{x_1}^{x_2} \frac{\partial F}{\partial u''}\delta u'' dx &= \int_{x_1}^{x_2} \frac{\partial F}{\partial u''}\frac{\partial}{\partial x}(\delta u') dx \\ &= \frac{\partial F}{\partial u''}\delta u'\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial u''}\right)\delta u' dx \\ &= \frac{\partial F}{\partial u''}\delta u'\bigg|_{x_1}^{x_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial u''}\right)\delta u\bigg|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right)\delta u dx. \end{aligned} \tag{8}$$

*If $u = u(x)$ and $du = u'(x)dx$, while $v = v(x)$ and $dv = v'(x)dx$, then integration by parts states that:

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx, \quad \int udv = uv - \int vdu.$$

Thus

$$\delta A = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right) \right] \delta u dx$$
$$+ \left[\frac{\partial F}{\partial u'} - \frac{d}{dx}\left(\frac{\partial F}{\partial u''}\right)\right] \delta u \Bigg|_{x_1}^{x_2} + \left[\left(\frac{\partial F}{\partial u''}\right) \delta u' \right]\Bigg|_{x_1}^{x_2} = 0. \tag{9}$$

Since δu is arbitrary thus may not be zero, each term must vanish individually:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right) = 0 \tag{10}$$

$$\left[\frac{\partial F}{\partial u'} - \frac{d}{dx}\left(\frac{\partial F}{\partial u''}\right)\right] \delta u \Bigg|_{x_1}^{x_2} = 0 \tag{11}$$

$$\left(\frac{\partial F}{\partial u''}\right) \delta u' \Bigg|_{x_1}^{x_2} = 0 \tag{12}$$

Equation (10) will be the governing differential equation for the given problem and is called **Euler equation** or **Euler-Lagrange equation**. Equations (11) and (12) give the boundary conditions. The conditions in Eq. (11) and (12) are called natural boundary conditions (if they are satisfied, they are called free boundary conditions). If the natural boundary conditions are not satisfied, we should have

$$\delta u(x_1) = 0, \quad \delta u(x_2) = 0, \quad \delta u'(x_1) = 0, \quad \delta u'(x_2) = 0, \tag{13}$$

in order to satisfy Eqs. (11) and (12). These are called geometric or forced boundary conditions.

Example 1. Brachistochrone Problem: Suppose there is an incline such as that shown in Figure 1. When a ball rolls from A to B, which curve yields the shortest duration?

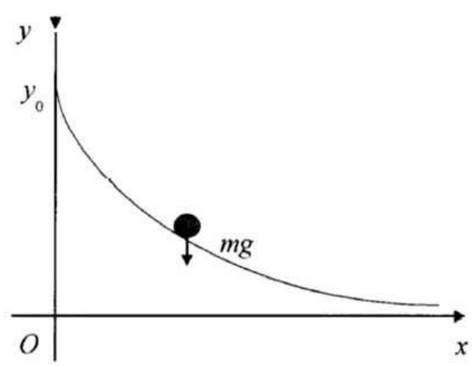

Figure 1: Curve of minimum time of descent.

Solution 1. The integral from the time-step when the ball is at the starting location, t_A, and the time-step of arrival, t_B , is the duration of motion, T,

$$T = \int_{t_A}^{t_B} dt. \tag{14}$$

Let us investigate the expression of this dt using x, y and y' . If the infinitesimal element is taken as ds, then the following relationship may be established by Pythagoras' theorem.

$$ds = \sqrt{d_x^2 + d_y^2} = \sqrt{1 + y'^2}dx, \quad y' = \frac{dy}{dx}. \tag{15}$$

The speed of the ball, v, may be found by taking the time derivative of the distance along the curve. This may be written as follows.

$$v = \frac{ds}{dt} = \sqrt{1 + y'^2}\frac{dx}{dt}. \tag{16}$$

Equation (14) may be rewritten as follows, using Equations (15) and (16).

$$T = \int_{t_A}^{t_B} dt = \int_{x_A}^{x_B} \frac{dt}{ds}\frac{ds}{dx}dx = \int_{x_A}^{x_B} \frac{1}{v}\sqrt{1 + y'^2}dx. \tag{17}$$

If the y coordinate is taken as being in the downwards direction due to gravity, then the distance fallen, y, and the speed, v, must obey the principle of energy conservation so the equation

$$\frac{1}{2}mv^2 = mgy, \tag{18}$$

is satisfied. Rearranging yields $v = \sqrt{2gy}$. Substituting this into Equation (17), yields the following equation.

$$T = \int_{0}^{x_B} \sqrt{\frac{1 + y'^2}{2gy}}dx, \tag{19}$$

where we have set $x_A = 0$ by translating the coordinate system.

The selection of the integrand

$$F(x, y, y') = \sqrt{\frac{1 + y'^2}{2gy}}, \tag{20}$$

in order to minimize T is a problem in the calculus of variations. Now $F(x, y, y')$ may be substituted into the Euler-Lagrange equation,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \tag{21}$$

but x is not explicitly contained in F, i.e., it is a function of y and y' alone. The following transformation of Euler's equation may therefore be used.†

$$F - y'\frac{\partial F}{\partial y'} = C_1, \tag{31}$$

where C_1 is a constant. Substituting for F in this equation,

$$\sqrt{\frac{1+y'^2}{2gy}} - \frac{y'^2}{\sqrt{2gy(1+y'^2)}} = \frac{1}{\sqrt{2gy(1+y'^2)}} = C_1. \tag{32}$$

Squaring both sides, the equation may be rearranged as follows. Since the right hand side is constant, so we may write it as C_2. That is

$$y(1+y'^2) = \frac{1}{2gC_1^2} = C_2. \tag{33}$$

Equation (33) is rearranged as follows.

$$y' = \sqrt{\frac{C_2 - y}{y}}. \tag{34}$$

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†**Beltrami Identify**: From

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \tag{22}$$

consider

$$\frac{dF}{dx} = \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' + \frac{\partial F}{\partial x}. \tag{23}$$

then

$$\frac{\partial F}{\partial y}y' = \frac{dF}{dx} - \frac{\partial F}{\partial y'}y'' - \frac{\partial F}{\partial x}. \tag{24}$$

Multiply Eq. (22) by y'

$$y'\frac{\partial F}{\partial y} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \quad y'\frac{dF}{dy} = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right), \tag{25}$$

then

$$\frac{dF}{dx} - \frac{\partial F}{\partial y'}y'' - \frac{\partial F}{\partial x} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{26}$$

Apply the chain rule

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}y'\right) = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)y' + \frac{\partial F}{\partial y'}y'', \quad \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)y' = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}y'\right) - \frac{\partial F}{\partial y'}y'', \tag{27}$$

then

$$-\frac{\partial F}{\partial x} + \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0. \tag{28}$$

For $\partial F/\partial x = 0$, where F is independent of x, then

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0, \tag{29}$$

hence

$$F - y'\frac{\partial F}{\partial y'} = C. \tag{30}$$

The domain of the curve is taken as $C_2 \geq y \geq 0$ such that the ball would roll. The initial condition is taken as $y = 0$ when $\theta = 0$ such that $dx/dy = \tan\theta$, where θ is the angle between the vertical axis and the tangent to the current point (x, y). Recall that $y' = dy/dx$, then

$$y' = \sqrt{\frac{C_2 - y}{y}} = \frac{\cos\theta}{\sin\theta}, \tag{35}$$

$$y\cos^2\theta = C_2\sin^2\theta - y\sin^2\theta, \tag{36}$$

$$y(\cos^2\theta + \sin^2\theta) = y = C_2\sin^2\theta. \tag{37}$$

Consider

$$\frac{dy}{d\theta} = C_2\frac{d\sin^2\theta}{d\sin\theta}\frac{d\sin\theta}{d\theta} = 2C_2\sin\theta\cos\theta, \tag{38}$$

and

$$\frac{dx}{d\theta} = \frac{dx}{dy}\frac{dy}{d\theta} = \sqrt{\frac{y}{C_2 - y}}\frac{dy}{d\theta} = \frac{\sin\theta}{\cos\theta}2C_2\sin\theta\cos\theta = 2C_2\sin^2\theta = C_2(1-\cos(2\theta)).^{\ddagger} \tag{43}$$

Integrating, we have

$$x = C_2\int 1 - \cos(2\theta)d\theta = \frac{C_2}{2}(2\theta - \sin(2\theta)) + C_3.^{\S} \tag{45}$$

Here to set the constant of integration equal to zero, i.e, $C_3 = 0$, we used the fact that $x = 0$ corresponds to $y = 0$, and hence to $\theta = 0$, incidentally showing that the tangent to the curve is vertical at the starting point.

From $dy/d\theta = 2C_2\sin\theta\cos\theta$, we have

$$y = C_2\int 2\sin\theta\cos\theta d\theta = C_2\int\sin(2\theta)d\theta = C_2\left(-\frac{1}{2}\cos(2\theta) + C_4\right).^{\P} \tag{47}$$

‡From trigonometry identities,

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \tag{40}$$

if $\alpha = \beta = \theta$, then

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta. \tag{41}$$

Furthermore, from

$$\cos^2\theta + \sin^2\theta = 1, \Rightarrow \cos^2\theta = 1 - \sin^2\theta. \tag{42}$$

Hence

$$\cos(2\theta) = 1 - 2\sin^2\theta, \Rightarrow 1 - \cos(2\theta) = 2\sin^2\theta. \tag{43}$$

§Consider $d2\theta/d\theta = 2$, then $d2\theta = 2d\theta$, hence

$$x = \frac{C_2}{2}\int(1 - \cos(2\theta))d(2\theta) = \frac{C_2}{2}\left\{\int d2\theta - \int\cos(2\theta)d(2\theta)\right\} = \frac{C_2}{2}\{2\theta - \sin(2\theta)\} + C_3. \tag{45}$$

¶From $d2\theta/d\theta = 2$, then

$$\int\sin(2\theta)d\theta = \int\frac{1}{2}\sin(2\theta)d(2\theta) = -\frac{1}{2}\cos(2\theta) + C_4. \tag{47}$$

At $y = 0$, we have $\theta = 0$, then $C_4 = 1/2$. Hence

$$y = \frac{C_2}{2}(1 - \cos(2\theta)). \tag{48}$$

If we specify, for instance, $y_B = 1$ and $\theta_B = \pi/2$ (horizontal movement), then

$$1 = \frac{C_2}{2}(1 - \cos(\pi)) = C_2. \tag{49}$$

Then for the ball to fall $y_B = 1$ unit down, it will reach the point at the angle $\theta_B = \pi/2$ and locate at $x_B = 1.57$ units.

3 Optimal Control

Consider a general dynamic system of order n,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(0) = \mathbf{x}_{init}, \tag{50}$$

where $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are n-dimensional state and m-dimensional input vectors, respectively. The objective is to determine $\mathbf{u}(t)$, $0 \leq t \leq t_f$, such that the following objective function is minimized or maximized:

$$J(\mathbf{u}(t)) = G(\mathbf{x}(t_f)) + \int_0^{t_f} F(\mathbf{x}, \mathbf{u}) dt. \tag{51}$$

Note that state equations serve as constraints for the optimization of J.

3.1 Necessary Conditions for Optimality

Let $\mathbf{u}_0(t)$ be a candidate for the optimal input vector, and let the corresponding state vector be $\mathbf{x}_0(t)$, i.e.,

$$\dot{\mathbf{x}}_0(t) = \mathbf{f}(\mathbf{x}_0(t), \mathbf{u}_0(t)). \tag{52}$$

In order to see whether $\mathbf{u}_0(t)$ is indeed an optimal solution, this candidate optimal input is perturbed by a small amount $\delta\mathbf{u}(t)$; i.e.,

$$\mathbf{u}(t) = \mathbf{u}_0(t) + \delta\mathbf{u}(t). \tag{53}$$

The change in the value of the objective function can be written as

$$\delta J = J(\mathbf{u}_0(t) + \delta\mathbf{u}(t)) - J(\mathbf{u}_0(t)) \tag{54}$$

which can also be obtained from (51) as

$$\begin{aligned} \delta J = {} & \left(\frac{\partial G}{\partial \mathbf{x}}\right) \delta\mathbf{x}(t_f) + \int_0^{t_f} \left[\left(\frac{\partial F}{\partial \mathbf{x}}\right) \delta\mathbf{x} + \left(\frac{\partial F}{\partial \mathbf{u}}\right) \delta\mathbf{u}\right] dt \\ & + \left[\left(\frac{\partial G}{\partial \mathbf{x}}\right) \mathbf{f}(t_f) + F(t_f)\right] \delta t_f. \end{aligned} \tag{55}$$

If the solution of (50) with $\mathbf{u}_0(t)$ given by (53) is $\mathbf{x}_0(t) + \delta\mathbf{x}(t)$, then

$$\frac{d(\mathbf{x}_0(t) + \delta\mathbf{x}(t))}{dt} = \mathbf{f}(\mathbf{x}_0(t) + \delta\mathbf{x}(t), \mathbf{u}_0(t) + \delta\mathbf{u}(t)). \tag{56}$$

Linearizing (56),

$$\frac{d(\delta\mathbf{x}(t))}{dt} = \frac{\partial\mathbf{f}}{\partial\mathbf{x}(t)}\delta\mathbf{x}(t) + \frac{\partial\mathbf{f}}{\partial\mathbf{u}}\delta\mathbf{u}(t). \tag{57}$$

Multiplying (57) by $\boldsymbol{\lambda}^\top(t)$ and integrating from 0 to t_f,

$$\int_0^{t_f} \boldsymbol{\lambda}^\top(t)\frac{d(\delta\mathbf{x}(t))}{dt}dt - \int_0^{t_f} \boldsymbol{\lambda}^\top(t)\left[\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\delta\mathbf{x}(t) + \frac{\partial\mathbf{f}}{\partial\mathbf{u}}\delta\mathbf{u}(t)\right]dt = 0, \tag{58}$$

where $\boldsymbol{\lambda}(t)$ is an n-dimensional vector. Subtract (58) from (55) and evaluating the first integral in (58) by parts‖

$$\begin{aligned}\delta J = &\left[F(t_f) + \frac{\partial G}{\partial\mathbf{x}}\mathbf{f}(t_f)\right]\delta t_f \\ &+ \boldsymbol{\lambda}^\top(0)\delta\mathbf{x}(0) + \left[\frac{\partial G}{\partial\mathbf{x}} - \boldsymbol{\lambda}^\top(t_f)\right]\delta\mathbf{x}(t_f) \\ &+ \int_0^{t_f}\left[\frac{\partial H}{\partial\mathbf{x}}\delta\mathbf{x}(t) + \frac{\boldsymbol{\lambda}^\top(t)}{dt}\delta\mathbf{x}(t) + \frac{\partial H}{\partial\mathbf{u}}\delta\mathbf{u}(t)\right]dt,\end{aligned} \tag{60}$$

where the function H is known as Hamiltonian, defined as follows:

$$H = F(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^\top(t)\mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{61}$$

such that

$$\frac{\partial H}{\partial\mathbf{x}} = \frac{\partial F}{\partial\mathbf{x}} + \lambda^\top(t)\frac{\partial\mathbf{f}}{\partial\mathbf{x}}, \quad \frac{\partial H}{\partial\mathbf{u}} = \frac{\partial F}{\partial\mathbf{u}} + \lambda^\top(t)\frac{\partial\mathbf{f}}{\partial\mathbf{u}}. \tag{62}$$

Because $\boldsymbol{\lambda}(t)$ is arbitrary, it is chosen to satisfy

$$\frac{\boldsymbol{\lambda}^\top(t)}{dt} = -\frac{\partial H}{\partial\mathbf{x}}. \tag{63}$$

Terms outside the integral in (60) are known as boundary conditions terms, which are removed for the specified problem. For example, if t_f and $\mathbf{x}(0)$ are specified, $\delta t_f = 0$ and $\delta\mathbf{x}(0) = 0$, and the third outside term in (60) vanishes under the following condition:

$$\boldsymbol{\lambda}^\top(t_f) = \left.\frac{\partial G}{\partial\mathbf{x}}\right|_{t=t_f}. \tag{64}$$

Hence, (60) can be written as

$$\delta J = \int_0^{t_f}\frac{\partial H}{\partial\mathbf{u}}\delta\mathbf{u}(t)dt. \tag{65}$$

‖

$$\int_0^{t_f}\lambda^\top(t)\frac{d(\delta\mathbf{x}(t))}{dt}dt = \int uv' = uv - \int vu' = \lambda^\top(t_f)\delta\mathbf{x}(t_f) - \lambda^\top(0)\delta\mathbf{x}(0) - \int_0^{t_f}\frac{d\lambda^\top(t)}{dt}\delta\mathbf{x}(t). \tag{59}$$

3.2 Properties of Hamiltonian

When the function $\mathbf{f}$ is not just an explicit function of time, then from (61),

$$\frac{dH}{dt} = \frac{\partial H}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial H}{\partial \mathbf{u}}\frac{d\mathbf{u}}{dt} + \frac{\partial H}{\partial \boldsymbol{\lambda}}\frac{d\boldsymbol{\lambda}}{dt}, \tag{66}$$

With $\partial H/\partial\boldsymbol{\lambda} = \mathbf{f}^\top$, then substitute (50) and (63) into (66),

$$\frac{dH}{dt} = \frac{\partial H}{\partial \mathbf{u}}\frac{d\mathbf{u}}{dt}. \tag{67}$$

Without any constraints on inputs, $\partial H/\partial\mathbf{u} = 0$. Therefore, for optimal inputs,

$$\frac{dH}{dt} = 0. \tag{68}$$

In other words, Hamiltonian H is a constant along an optimal trajectory. When there are constraints, inputs can be either maximum or minimum constant values. In this case, $d\mathbf{u}/dt = 0$.

3.3 Special Case

Final time t_f not specified; i.e., $\delta t_f \neq 0$. In this case, to remove the corresponding boundary condition term in (60),

$$F(t_f) + \frac{\partial G}{\partial \mathbf{x}}\mathbf{f}(t_f) = 0. \tag{69}$$

When final conditions on states are not specified, the condition (64) holds, and (69) reduces to

$$F(t_f) + \boldsymbol{\lambda}^\top(t_f)\mathbf{f}(t_f) = 0. \tag{70}$$

The condition (70) is valid even when some or all states are specified at the final time.

For example, consider that all states are specified at the final time. In this case, with the first-order term in the Taylor series expansion

$$\mathbf{x}_0(t_f) = \mathbf{x}(t_f + \delta t_f) = \mathbf{x}(t_f) + \dot{\mathbf{x}}(t_f)\delta t_f, \tag{71}$$

$$\delta\mathbf{x}(t_f) = \mathbf{x}(t_f) - \mathbf{x}_0(t_f) = -\mathbf{f}(t_f)\delta t_f. \tag{72}$$

It is interesting to note that $\delta\mathbf{x}(t_f) \neq 0$. Substituting (72) into (60) and using (63),

$$\delta J = \left[F(t_f) + \boldsymbol{\lambda}^\top(t_f)\mathbf{f}(t_f)\right]\delta t_f + \boldsymbol{\lambda}^\top(0)\delta\mathbf{x}(0) + \int_0^{t_f} \frac{\partial H}{\partial \mathbf{u}}\delta\mathbf{u}(t)dt. \tag{73}$$

Therefore, the condition (70) is again needed to remove boundary condition terms.

In summary, $H(t_f) = 0$ when t_f is not specified. Because H is a constant, it is concluded that $dH(t)/dt = 0$ along an optimal trajectory for an autonomous system when t_f is not specified.

4 Linear Quadratic Regulator (LQR)

Consider the linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_{init}. \tag{74}$$

The objective is to drive the state vector $\mathbf{x}(t)$ to the origin of the state space (zero state vector) from any nonzero initial values of states. The following objective function is chosen:

$$J = \frac{1}{2}\mathbf{x}^\top(t_f)\mathbf{S}(t_f)\mathbf{x}(t_f) + \frac{1}{2}\int_0^{t_f} \left[\mathbf{x}^\top(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\top(t)\mathbf{R}\mathbf{u}(t)\right] dt, \tag{75}$$

where the final time t_f is fixed. Without any loss of generality, matrices $\mathbf{S}$, $\mathbf{Q}$, and $\mathbf{R}$ are chosen to be symmetric. In addition, $\mathbf{S}$ and $\mathbf{Q}$ are chosen to be positive semidefinite, and $\mathbf{R}$ to be positive definite. Symbolically, these are expressed as

$$\mathbf{S} = \mathbf{S}^\top \geq 0, \quad \mathbf{Q} = \mathbf{Q}^\top \geq 0, \quad \mathbf{R} = \mathbf{R}^\top > 0. \tag{76}$$

The problem is to find $\mathbf{u}(t)$, $0 \leq t \leq t_f$; such that the objective function (75) is minimized.

4.1 Open-Loop Optimal Control

For this optimal control problem, the Hamiltonian (61) is

$$H = \frac{1}{2}\left[\mathbf{x}^\top(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\top\mathbf{R}\mathbf{u}(t)\right] + \boldsymbol{\lambda}^\top(t)\left[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right]. \tag{77}$$

The necessary condition for optimality yields

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{R}\mathbf{u}(t) + \mathbf{B}^\top\boldsymbol{\lambda}(t) = 0, \tag{78}$$

or

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^\top\boldsymbol{\lambda}(t). \tag{79}$$

The dynamics of is given by (63) with final conditions (64). Hence,

$$\frac{d\boldsymbol{\lambda}}{dt} = -\mathbf{Q}\mathbf{x}(t) - \mathbf{A}^\top\boldsymbol{\lambda}(t), \quad \boldsymbol{\lambda}(t_f) = \mathbf{S}\mathbf{x}(t_f). \tag{80}$$

Equation (74) and (80) represent a two-point boundary value problem (TPBVP) which can be solved to find $\boldsymbol{\lambda}(t)$ and $\mathbf{x}(t)$. Putting in the matrix form,

$$\begin{bmatrix}\dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}}\end{bmatrix} = \mathbf{M}\begin{bmatrix}\mathbf{x} \\ \boldsymbol{\lambda}\end{bmatrix}, \tag{81}$$

where

$$\mathbf{M} = \begin{bmatrix}\mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top \\ -\mathbf{Q} & -\mathbf{A}^\top\end{bmatrix}. \tag{82}$$

Solving (81),

$$\begin{bmatrix}\mathbf{x}(t)\\ \boldsymbol{\lambda}(t)\end{bmatrix} = e^{\mathbf{M}t}\begin{bmatrix}\mathbf{x}(0)\\ \boldsymbol{\lambda}(0)\end{bmatrix}. \tag{83}$$

To determine $\boldsymbol{\lambda}(0)$, the matrix is partitioned as follows:

$$e^{\mathbf{M}t} = \begin{bmatrix}\mathbf{E}_{11}(t) & \mathbf{E}_{12}(t)\\ \mathbf{E}_{21}(t) & \mathbf{E}_{22}(t)\end{bmatrix}. \tag{84}$$

From (83) and (84),

$$\mathbf{x}(t) = \mathbf{E}_{11}(t)\mathbf{x}(0) + \mathbf{E}_{12}(t)\boldsymbol{\lambda}(0), \tag{85}$$

$$\boldsymbol{\lambda}(t) = \mathbf{E}_{21}(t)\mathbf{x}(0) + \mathbf{E}_{22}(t)\boldsymbol{\lambda}(0). \tag{86}$$

Imposing the condition $\boldsymbol{\lambda}(t_f) = \mathbf{S}\mathbf{x}(t_f)$, then

$$\mathbf{E}_{21}(t_f)\mathbf{x}(0) + \mathbf{E}_{22}(t_f)\boldsymbol{\lambda}(0) = \mathbf{S}[\mathbf{E}_{11}(t_f)\mathbf{x}(0) + \mathbf{E}_{12}(t_f)\boldsymbol{\lambda}(0). \tag{87}$$

From (87),

$$\boldsymbol{\lambda}(0) = [\mathbf{E}_{22}(t_f) - \mathbf{S}\mathbf{E}_{12}(t_f)]^{-1}[\mathbf{S}\mathbf{E}_{11}(t_f) - \mathbf{E}_{21}(t_f)]\mathbf{x}(0). \tag{88}$$

Substituting this $\boldsymbol{\lambda}(0)$ into (83), $\boldsymbol{\lambda}(t)$ is obtained. Then, the optimal is found from (79). However, this TPBVP must be solved again if initial conditions change. Furthermore, the control inputs are implementable in an open-loop fashion only, as they are not in the forms of functions of states.

4.2 Closed-Loop Optimal Control

Use the following transformation

$$\boldsymbol{\lambda}(t) = \mathbf{S}\mathbf{x}(t), \tag{89}$$

where $\mathbf{S}(t)$ is a symmetric $n \times n$ matrix. Substituting (89) into (80),

$$\frac{d\mathbf{S}(t)}{dt}\mathbf{x}(t) + \mathbf{S}(t)\frac{d\mathbf{x}}{dt} = -\mathbf{Q}\mathbf{x}(t) - \mathbf{A}^\top\mathbf{S}(t)\mathbf{x}(t). \tag{90}$$

Using (74) and (79),

$$\left[\frac{d\mathbf{S}}{dt} + \mathbf{S}\mathbf{A} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{S} + \mathbf{Q} + \mathbf{A}^\top\mathbf{S}\right]\mathbf{x}(t) = 0. \tag{91}$$

Because (91) is true for all $\mathbf{x}(t)$, then

$$\frac{d\mathbf{S}}{dt} + \mathbf{S}\mathbf{A} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{S} + \mathbf{Q} + \mathbf{A}^\top\mathbf{S} = 0, \tag{92}$$

and $\mathbf{S}(t_f) = \mathbf{S}$ from (64). Equation (92) is known as the Riccati equation. Also from (79),

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t), \quad \mathbf{K}(t) = \mathbf{R}^{-1}\mathbf{B}^\top\mathbf{S}(t). \tag{93}$$

The structure of (93) indicates that $\mathbf{K}(t)$ is the optimal state feedback gain matrix. Because the solution of $\mathbf{S}(t)$ does not depend on system states, this gain is optimal for all initial conditions on states.

Example 2. Consider a first-order system

$$\dot{x} = x + u, \tag{94}$$

and the objective function

$$J = \frac{5}{2}x^2(t_f) + \frac{1}{2}\int_0^{t_f} 2x^2 + 3u^2 dt, \tag{95}$$

find the closed-loop optimal control gain.

Solution 2. Here $A = 1$, $B = 1$, $Q = 2$, $R = 3$, $S = 5$. Therefore, the Riccati equation becomes

$$\dot{S} = -2S + \frac{S^2}{3} - 2, \quad S(t_f) = 5. \tag{96}$$

Then, the optimal control law will be

$$u(t) = -\frac{S(t)}{3}x(t), \quad 0 \le t \le t_f. \tag{97}$$

If $t_f \to \infty$, the Ricatti equation becomes

$$0 = -2S + \frac{S^2}{3} - 2, \tag{98}$$

giving $S = 3 \pm \sqrt{15}$. In order to have a stable closed-loop system, the positive value must be chosen. The optimal state feedback law is

$$u(t) = -Kx(t), \quad K = 1 + \frac{\sqrt{15}}{3}. \tag{99}$$

the closed-loop system dynamics is

$$\dot{x}(t) = -\frac{\sqrt{15}}{3}x(t), \tag{100}$$

which is stable.