Matrix Operations using List2D

Massive Information & Knowledge Engineering

204113 Computer & Programming

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Version 2024

Matrix Representation

Matrix

- A matrix is a two-dimensional data structure where numbers are arranged into rows and columns.

$$\begin{array}{cc} \text{4 columns} & \\ \begin{bmatrix} 2 & -5 & -11 & 0 \\ -9 & 4 & 6 & 13 \\ 4 & 7 & 12 & -2 \end{bmatrix} & \text{3 rows} \end{array}$$

- This matrix is a 3x4 (pronounced "three by four") matrix because it has 3 rows and 4 columns.

Python Matrix

- Python doesn't have a built-in type for matrices. However, we can treat a list of a list as a matrix.
- We can treat this list of a list as a matrix having 2 rows and 3 columns.

$$\begin{bmatrix} 1 & 4 & 5 \\ -5 & 8 & 9 \end{bmatrix}$$

```python
A = [[1, 4, 5, 12],
    [-5, 8, 9, 0],
    [-6, 7, 11, 19]]
print("A =", A)
print("A[1] =", A[1])         # 2nd row
print("A[1][2] =", A[1][2])   # 3rd element of 2nd row
print("A[0][-1] =", A[0][-1]) # Last element of 1st Row

column = []; # empty list
for row in A:
  column.append(row[2])
print("3rd column =", column)
```

```
A = [[1, 4, 5, 12], [-5, 8, 9, 0], [-6, 7, 11, 19]]
A[1] = [-5, 8, 9, 0]
A[1][2] = 9
A[0][-1] = 12
3rd column = [5, 9, 11]
```

Matrix Operations

Matrix Addition

- We can perform matrix addition using a Python nested loop.

```python
# Program to add two matrices using nested loop
X = [[12,7,3],
    [4 ,5,6],
    [7 ,8,9]]

Y = [[5,8,1],
    [6,7,3],
    [4,5,9]]

result = [[0,0,0],
          [0,0,0],
          [0,0,0]]

# iterate through rows
for i in range(len(X)):
   # iterate through columns
   for j in range(len(X[0])):
       result[i][j] = X[i][j] + Y[i][j]

for r in result:
   print(r)
```

```
[17, 15, 4]
[10, 12, 9]
[11, 13, 18]
```


Matrix Addition (2)

- We can also perform matrix addition using a nested list comprehension.

```python
# Program to add two matrices using list comprehension

X = [[12,7,3],
    [4 ,5,6],
    [7 ,8,9]]

Y = [[5,8,1],
    [6,7,3],
    [4,5,9]]

result = [[X[i][j] + Y[i][j]  for j in range(len(X[0]))]\
             for i in range(len(X))]

for r in result:
   print(r)
```


Matrix Transpose

- We can also use nested for loops to iterate through each row and each column. At each point we place the `X[i][j]` element into `result[j][i]`.

```python
# Program to transpose a matrix using a nested loop

X = [[12,7],
    [4 ,5],
    [3 ,8]]

result = [[0,0,0],
          [0,0,0]]

# iterate through rows
for i in range(len(X)):
   # iterate through columns
   for j in range(len(X[0])):
       result[j][i] = X[i][j]

for r in result:
   print(r)
```

```
[12, 4, 3]
[7, 5, 8]
```

Matrix Transpose (2)

- We can use nested list comprehension to transpose a matrix, too.

```python
# Program to transpose a matrix using list comprehension

X = [[12,7],
    [4 ,5],
    [3 ,8]]

res = [[X[i][j] for i in range(len(X))]\
                for j in range(len(X[0]))]
for r in result:
   print(r)
```

```
[12, 4, 3]
[7, 5, 8]
```


Matrix Multiplication

- Multiplication of two matrices X and Y is defined only if the number of columns in X is equal to the number of rows in Y.
- If X is a n x m matrix and Y is a m x p matrix then, XY is defined and has the dimension n x p (but YX is not defined).

```python
# 3x3 matrix
X = [[12,7,3], [4,5,6], [7,8,9]]
# 3x4 matrix
Y = [[5,8,1,2],[6,7,3,0],[4,5,9,1]]
# result is 3x4
result = [[0,0,0,0],[0,0,0,0],[0,0,0,0]]

for i in range(len(X)):
  for j in range(len(Y[0])):
    for k in range(len(Y)):
      result[i][j] += X[i][k] * Y[k][j]

for r in result:
   print(r)
```

```
[114, 160, 60, 27]
[74, 97, 73, 14]
[119, 157, 112, 23]
```


Linear Equations

REF: https://openstax.org/

Systems of Linear Equations

- A system of linear equations consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously.
- To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time.
- Some linear systems may not have a solution and others may have an infinite number of solutions.
- For a linear system to have a unique solution, there must be at least as many equations as numbers of variables.

$$2x + y = 15$$

$$3x - y = 5$$

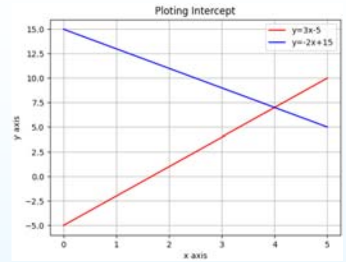
Systems of Linear Equations (2)

$$2x + y = 15$$
$$3x - y = 5$$

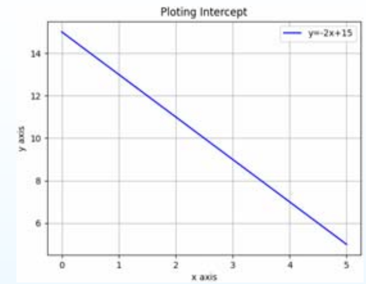
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently.
- In this example, the ordered pair (4, 7) is the only one solution to this system of linear equations.
- We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations.

$$2(4) + (7) = 15 \text{ True}$$
$$3(4) - (7) = 5 \quad \text{True}$$

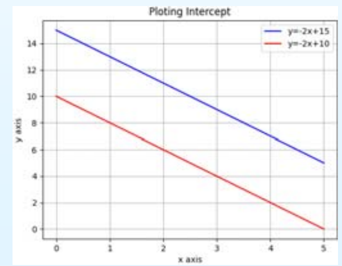
Type of Linear Systems



Consistent System (independent)



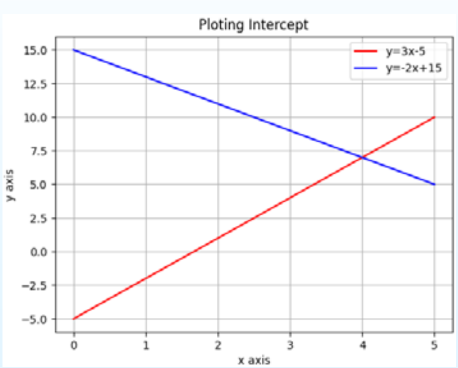
Consistent System (dependent, infinite number of solutions)



Inconsistent System (no solution)

Consistent System (independent)

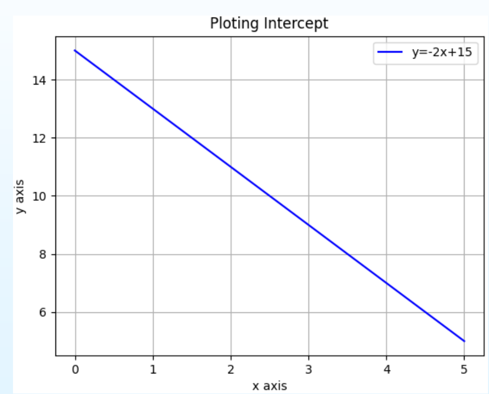
$$\begin{aligned} 2x + y &= 15 \\ 3x - y &= 5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= -2x + 15 \\ y &= 3x - 5 \end{aligned}$$



- In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions.
- A consistent system of equations has at least one solution.
- A consistent system is considered to be an independent system if it has a single solution, such as the example we just explored.
  - The two lines have different slopes and intersect at one point in the plane.

Consistent System (dependent)

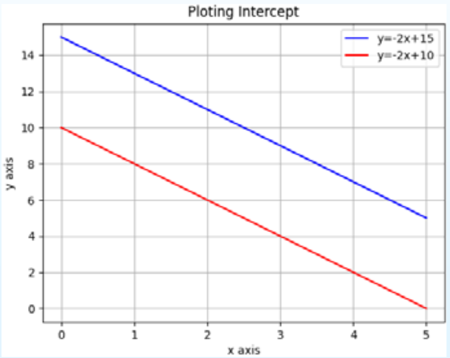
$$\begin{aligned} 2x + y &= 15 \\ 6x + 3y &= 45 \end{aligned} \quad \Rightarrow \quad y = -2x + 15$$



- A consistent system is considered to be a dependent system if the equations have the same slope and the same y-intercepts.
  - In other words, the lines coincide so the equations represent the same line.
  - Every point on the line represents a coordinate pair that satisfies the system.
  - Thus, there are an infinite number of solutions.

Inconsistent System

$$2x + y = 15 \qquad y = -2x + 15$$
$$2x + y = 10 \qquad y = -2x + 10$$



- Another type of system of linear equations is an inconsistent system, which is one in which the equations represent two parallel lines.
- The lines have the same slope but different y-intercepts.
- There are no points common to both lines; hence, there is no solution to the system.

Gaussian Elimination

Gauss (*Gauß*)

- Carl Friedrich Gauss lived during the late 18th century and early 19th century, but he is still considered one of the most prolific German mathematicians in history.
- His contributions to the science of mathematics and physics span fields such as algebra, number theory, analysis, differential geometry, astronomy, and optics, among others.
- His discoveries regarding matrix theory changed the way mathematicians have worked for the last two centuries.

Augmented Matrix

- A matrix can serve as a device for representing and solving a system of equations.
- To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix.
- We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs.
- When a system is written in this form, we call it an augmented matrix.

$$\begin{aligned} 3x + 4y &= 7 \\ 4x - 2y &= 5 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array}\right]$$

Row-echelon Form

- To solve the system of equations, we want to convert the matrix to row-echelon form, in which there are ones down the main diagonal from the upper left corner to the lower right corner, and zeros in every position below the main diagonal.

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$$

- We use row operations corresponding to equation operations to obtain a new matrix that is row-equivalent in a simpler form.
- Here are the guidelines to obtaining row-echelon form.
  - In any nonzero row, the first nonzero number is a 1. It is called a leading 1.
  - Any all-zero rows are placed at the bottom on the matrix.
  - Any leading 1 is below and to the right of a previous leading 1.
  - Any column containing a leading 1 has zeros in all other positions in the column.

Row Operations

- To solve a system of equations we perform the following row operations to convert the coefficient matrix to row-echelon form and do back-substitution to find the solution.
  - Interchange rows. (Notation: $R_i \leftrightarrow R_j$)
  - Multiply a row by a constant. (Notation: cR_i)
  - Add the product of a row multiplied by a constant to another row. (Notation: $R_i + cR_j$)

Gaussian Elimination

- The Gaussian elimination method refers to a strategy used to obtain the row-echelon form of a matrix.
- The goal is to write matrix A with the number 1 as the entry down the main diagonal and have all zeros below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow A = \begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

- The first step of the Gaussian strategy includes obtaining a 1 as the first entry, so that row 1 may be used to alter the rows below.

Gaussian Elimination (how to)

$$A = \begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

- Given an augmented matrix, perform row operations to achieve row-echelon form.
  - The first equation should have a leading coefficient of 1. Interchange rows or multiply by a constant, if necessary.
  - Use row operations to obtain zeros down the first column below the first entry of 1.
  - Use row operations to obtain a 1 in row 2, column 2.
  - Use row operations to obtain zeros down column 2, below the entry of 1.
  - Use row operations to obtain a 1 in row 3, column 3.
  - Continue this process for all rows until there is a 1 in every entry down the main diagonal and there are only zeros below.
  - If any rows contain all zeros, place them at the bottom.

Solving a 2x2 System by Gaussian Elimination

$$2x + 3y = 6$$
$$x - y = 1/2$$

Augmented Matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & 1/2 \end{array}\right]$$

$R_1 \Leftrightarrow R_2$

$$\left[\begin{array}{cc|c} 1 & -1 & 1/2 \\ 2 & 3 & 6 \end{array}\right]$$

$R_2 \Rightarrow R_2 - 2R_1$

$$\left[\begin{array}{cc|c} 1 & -1 & 1/2 \\ 0 & 5 & 5 \end{array}\right]$$

$R_2 \Rightarrow \frac{1}{5}R_2$

$$\left[\begin{array}{cc|c} 1 & -1 & 1/2 \\ 0 & 1 & 1 \end{array}\right]$$

Back-substitution

$$y = 1$$
$$x - (1) = \frac{1}{2}$$
$$x = \frac{3}{2}$$

Solving a 3x3 System by Gaussian Elimination

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 2 & -5 & 6 & 6 \\ -3 & 3 & 4 & 6 \end{array}\right]$$

$R_2 \Rightarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ -3 & 3 & 4 & 6 \end{array}\right]$$

$R_3 \Rightarrow R_3 + 3R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -6 & 16 & 15 \end{array}\right]$$

$R_3 \Rightarrow R_3 + 6R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 15 \end{array}\right]$$

$R_3 \Rightarrow \frac{1}{4}R_3$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 15/4 \end{array}\right]$$

Back-substitution

Your work!

Applying 3x3 Matrix to Finance

- Ava invests a total of $10,000 in three accounts, one paying 5% interest, another paying 8% interest, and the third paying 9% interest. The annual interest earned on the three investments last year was $770. The amount invested at 9% was twice the amount invested at 5%. How much was invested at each rate?
- We have a system of three equations in three variables.
  - Let x be the amount invested at 5% interest, let y be the amount invested at 8% interest, and let z be the amount invested at 9% interest.

$$x + y + z = 10000$$
$$0.05x + 0.08y + 0.09z = 770$$
$$2x - z = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 0.05 & 0.08 & 0.09 & 770 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

Applying 3x3 Matrix to Finance

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 0.05 & 0.08 & 0.09 & 770 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

$R_2 \Rightarrow R_2 - .05R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 0 & 0.03 & 0.04 & 270 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

$R_3 \Rightarrow R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 0 & 0.03 & 0.04 & 270 \\ 0 & -2 & -3 & -20000 \end{array}\right]$$

$R_2 \Rightarrow \frac{1}{.03}R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 0 & 1 & \frac{4}{3} & 9000 \\ 0 & -2 & -3 & -20000 \end{array}\right]$$

$R_3 \Rightarrow R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 0 & 1 & 4/3 & 9000 \\ 0 & 0 & -1/3 & -2000 \end{array}\right]$$

Back-substitution

$$z = 6000$$
$$y = 1000$$
$$x = 3000$$

Inverse Matrix

Identity Matrix & Multiplicative Inverse

- The identity matrix, I_n, is a square matrix containing ones down the main diagonal and zeros everywhere else.
- If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that $AB = BA = I_n$, then $B = A^{-1}$, the multiplicative inverse of a matrix A.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix & Multiplicative Inverse (2)

- Show that the given matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \qquad B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} . \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1(-9) + 5(2) & 1(-5) + 5(1) \\ -2(-9) - 9(2) & -2(-5) - 9(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} = \begin{bmatrix} -9(1) - 5(-2) & -9(5) - 5(-9) \\ 2(1) + 1(-2) & 2(5) + 1(-9) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding the Multiplicative Inverse Matrix

- We write the augmented matrix with the identity on the right and A on the left.
- Performing elementary row operations so that the identity matrix appears on the left, we will obtain the inverse matrix on the right.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

Finding the Multiplicative Inverse Matrix (2)

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \Rightarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \Rightarrow R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \Leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right] \Rightarrow \cdots$$

Finding the Multiplicative Inverse Matrix (3)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \Rightarrow R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \Rightarrow R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

$$A^{-1} = B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Solving a Linear Equation System

- Given a system of equations, write the coefficient matrix A, the variable matrix X, and the constant matrix B. Then,

$$AX = B$$

- Multiply both sides by the inverse of A to obtain the solution.

$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$\boldsymbol{X = A^{-1}B}$$

- If the coefficient matrix does not have an inverse, does that mean the system has no solution?
  - No, if the coefficient matrix is not invertible, the system could be inconsistent and have no solution, or be dependent and have infinitely many solutions.
- Can we solve for X by finding the product BA^{-1}?
  - No, recall that matrix multiplication is not commutative, so $A^{-1}B \neq BA^{-1}$.

Solving a Linear Equation System (2)

- Solve the following system using the inverse of a matrix.

$$5x + 15y + 56z = 35$$
$$-4x - 11y - 41z = -26$$
$$-x - 3y - 11z = -7$$

- We first write the equation in form of $AX = B$.

$$\begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

- Then, we find the inverse of A by augmenting with the identity.

$$\left[\begin{array}{ccc|ccc} 5 & 15 & 56 & 1 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Solving a Linear Equation System (3)

$$\left[\begin{array}{ccc|ccc} 5 & 15 & 56 & 1 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \Rightarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 56/5 & 1/5 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \Rightarrow R_2+4R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 56/5 & 1/5 & 0 & 0 \\ 0 & 1 & 19/5 & 4/5 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \Rightarrow R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 56/5 & 1/5 & 0 & 0 \\ 0 & 1 & 19/5 & 4/5 & 1 & 0 \\ 0 & 0 & 1/5 & 1/5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \Rightarrow 5R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 56/5 & 1/5 & 0 & 0 \\ 0 & 1 & 19/5 & 4/5 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right] \xrightarrow{R_1 \Rightarrow R_1-3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1/5 & -11/5 & -3 & 0 \\ 0 & 1 & 19/5 & 4/5 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

Solving a Linear Equation System (4)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1/5 & -11/5 & -3 & 0 \\ 0 & 1 & 19/5 & 4/5 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right] \xrightarrow{R_1 \Rightarrow R_1+\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 19/5 & 4/5 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

$$\xrightarrow{R_2 \Rightarrow R_2-\frac{19}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & 1 & -19 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Determinant of a Matrix

Determinant of a 2×2 Matrix

- The determinant of a 2×2 matrix, given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

- Notice the change in notation. There are several ways to indicate the determinant, including $\det(A)$ and replacing the brackets in a matrix with straight lines, $|A|$.

Cramer's Rule for 2×2 Systems

- Cramer's Rule is a method that uses determinants to solve systems of equations that have the same number of equations as variables.
- Consider a system of two linear equations in two variables.

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

- The solution using Cramer's rule is given as

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, D \neq 0; \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, D \neq 0$$

- If we are solving for x, the x column is replaced with the constant column. If we are solving for y, the y column is replaced with the constant column.

Cramer's Rule for 2×2 Systems (Example)

Using Cramer's Rule to Solve a 2 × 2 System

Solve the following 2×2 system using Cramer's Rule.

$$12x + 3y = 15$$
$$2x - 3y = 13$$

Solve for x.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{-45 - 39}{-36 - 6} = \frac{-84}{-42} = 2$$

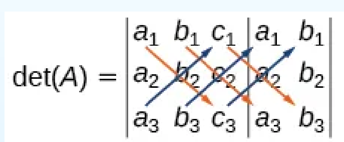
Solve for y.

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{156 - 30}{-36 - 6} = -\frac{126}{42} = -3$$

Determinant of a 3×3 Matrix

- Determinant of a 3×3 matrix, given $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is defined as

$$\det(A) = \left|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right|\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}\right|$$



$$\det(A) = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

- Can we use this method to find the determinant of a larger matrix?
  – No!

Cramer's Rule for 3×3 Systems

- Consider a 3 × 3 system of equations.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}, D \neq 0.$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_1 \\ a_3 & b_3 & d_1 \end{vmatrix}$$

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Cramer's Rule for 3×3 Systems (Example)

Find the solution to the given 3 × 3 system using Cramer's Rule.

$$x+y-z=6$$
$$3x-2y+z=-5$$
$$x+3y-2z=14$$

$$D=\begin{vmatrix}1&1&-1\\3&-2&1\\1&3&-2\end{vmatrix}, D_x=\begin{vmatrix}6&1&-1\\-5&-2&1\\14&3&-2\end{vmatrix}, D_y=\begin{vmatrix}1&6&-1\\3&-5&1\\1&14&-2\end{vmatrix}, D_z=\begin{vmatrix}1&1&6\\3&-2&-5\\1&3&14\end{vmatrix}$$

Then,

$$x=\frac{D_x}{D}=\frac{-3}{-3}=1$$
$$y=\frac{D_y}{D}=\frac{-9}{-3}=3$$
$$z=\frac{D_z}{D}=\frac{6}{-3}=-2$$

Cramer's Rule with an Inconsistent System

Solve the system of equations using Cramer's Rule.

$$3x-2y=4 \quad (1)$$
$$6x-4y=0 \quad (2)$$

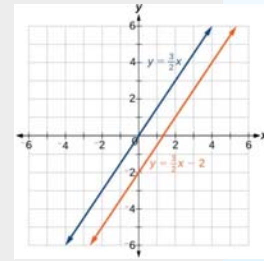
We begin by finding the determinants $D, D_x,$ and D_y.

$$D=\begin{vmatrix}3&-2\\6&-4\end{vmatrix}=3(-4)-6(-2)=0$$

We know that a determinant of zero means that either the system has no solution or it has an infinite number of solutions. To see which one, we use the process of elimination. Our goal is to eliminate one of the variables.

1. Multiply equation (1) by -2.
2. Add the result to equation (2).

$$\begin{array}{r}-6x+4y=-8\\ \underline{6x-4y=0}\\ 0=-8\end{array}$$



We obtain the equation $0=-8$, which is false. Therefore, the system has no solution. Graphing the system reveals two parallel lines. See Figure 1.

Cramer's Rule with a Dependent System

Solve the system with an infinite number of solutions.

$$x-2y+3z=0 \quad (1)$$
$$3x+y-2z=0 \quad (2)$$
$$2x-4y+6z=0 \quad (3)$$

Let's find the determinant first. Set up a matrix augmented by the first two columns.

$$\left|\begin{array}{ccc|cc}1&-2&3&1&-2\\3&1&-2&3&1\\2&-4&6&2&-4\end{array}\right|$$

Then,

$$1(1)(6)+(-2)(-2)(2)+3(3)(-4)-2(1)(3)-(-4)(-2)(1)-6(3)(-2)=0$$

As the determinant equals zero, there is either no solution or an infinite number of solutions. We have to perform elimination to find out.

1. Multiply equation (1) by -2 and add the result to equation (3):

$$\begin{array}{r}-2x+4y-6z=0\\ \underline{2x-4y+6z=0}\\ 0=0\end{array}$$

2. Obtaining an answer of $0=0$, a statement that is always true, means that the system has an infinite number of solutions. Graphing the system, we can see that two of the planes are the same and they both intersect the third plane on a line. See Figure 2.

Properties of Determinants

1. If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
2. When two rows are interchanged, the determinant changes sign.
3. If either two rows or two columns are identical, the determinant equals zero.
4. If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.
5. The determinant of an inverse matrix A^{-1} is the reciprocal of the determinant of the matrix A.
6. If any row or column is multiplied by a constant, the determinant is multiplied by the same factor.

To be continue..

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