Part 1. Strategic Game Foundations

In the **traditional simultaneous-demand bargaining game**, two players negotiate over how to split a fixed resource, such as a sum of money. Each player simultaneously submits a demand specifying the amount of the resource they wish to receive. The outcomes are determined as follows:

- If the sum of the two demands is less than or equal to the total resource, each player receives the amount they demanded.
- If the sum of the demands exceeds the total resource, neither player receives anything.

Since both players make their demands simultaneously without knowing the other's choice, the game introduces strategic uncertainty and emphasizes the importance of anticipating the opponent's behavior . (Osborne and Rubinstein, 1994).

In this project, I extend the classic simultaneous-demand game in two main ways to explore more complex strategic environments. First, I introduce an **incomplete information structure**, where each player has a **private minimum acceptable demand** and only knows the probability distribution over the other player's minimum. This modification requires players to form beliefs about the other's likely demands and make decisions under uncertainty, effectively turning the game into a **Bayesian simultaneous-move scenario**. Second, while the baseline analysis focuses on a total resource of 100, I also plan to investigate the effects of **varying the resource cap**, considering values such as 10, 100, and 1000. This allows for an examination of how the scale of available resources influences equilibrium outcomes, efficiency, and fairness.

Formally, the baseline game (with S = 100) can be described as follows:

- **Players:** Two players, $i \in \{A, B\}$
- **Resource:** Total resource $S = 100$ (baseline; later experiments will vary S)
- **Actions:** Each player simultaneously chooses a demand $d_i \in [0,S]$
- **Information Structure:** Each player has a private minimum demand t_i, and the opponent only knows the other's probability distribution F_j.
- **Payoff Function:**

$$u_i(d_i, d_j) = \begin{cases} d_i, & \text{if } d_i + d_j \leq S \\ 0, & \text{if } d_i + d_j > S \end{cases}$$

- **Timing:** Players make their demands simultaneously; payoffs are realized according to the rule above.

Through this framework, the project investigates how strategic uncertainty and the scale of resources jointly shape player behavior, equilibrium structure, and overall efficiency.

1.1 Theoretical solutions

In this game, each player has a private type representing their minimum acceptable demand, and players do not know the other's exact type. Instead, they only know the probability distribution over the opponent's type. Because of this uncertainty, a player's best choice depends not on the opponent's actual action, but on the expected outcomes across all possible types.

Consequently, the **standard Nash Equilibrium**, which assumes complete information, is **no longer sufficient to describe optimal behavior.** This setting requires an equilibrium concept that accounts for strategic uncertainty due to private information, leading naturally to the **Bayesian Nash Equilibrium (BNE)**. In a BNE, each player chooses a strategy function mapping their type to a demand, and no player can improve their expected payoff by unilaterally changing this function.

1.1.1 Equilibrium Concept: Bayes–Nash Equilibrium (BNE)

- **Definition:** A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ constitutes a Bayes–Nash Equilibrium if, for every player $i \in N$, every type $\theta_i \in \Theta_i$, and given beliefs about other players' types induced by the common prior $p(\theta)$, the strategy $s_i^*(\theta_i)$ maximizes the expected utility of player i:

$$\mathbb{E}_{\theta_{-i}\sim p(\cdot|\theta_i)}\Big[u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i});\theta_i)\Big] \geq \mathbb{E}_{\theta_{-i}\sim p(\cdot|\theta_i)}\Big[u_i(s_i(\theta_i), s_{-i}^*(\theta_{-i});\theta_i)\Big]$$

for all alternative strategies s_i.

Here:
$N = \{1,2\}$ (two bargainers).
Θ_i = type space of player i, e.g., risk aversion or fairness preference.
$S_i : \Theta_i \to A_i$, mapping from types to actions (proposals/acceptances).
u_i = payoff function (depends on final allocation and type).

1.1.2 Analytical solution

- **Equilibrium Characterization:**

In the incomplete-information Bayesian version, each player has a private minimum acceptable demand t_i, unknown to the opponent, who only knows the probability distribution over t_i. Players choose a strategy function $d_i(t_i)$ mapping their type to a demand.

A Bayesian Nash Equilibrium (BNE) is characterized by the property that, for every player i and type t_i:

$$d_i(t_i) = \arg\max_{d_i} \mathbb{E}_{t_{-i}}\big[u_i(d_i, d_{-i}(t_{-i}))\big]$$

  - That is, each player's chosen demand maximizes expected payoff given beliefs about the other player's type and strategy.
  - Strategies $d_i(t_i)$ are monotonically increasing in own type: higher minimum acceptable demands lead to higher chosen demands.
  - Equilibrium balances the risk of zero payoff (over-demanding) with the potential gain of high demand. Unlike complete-information NE,

equilibrium is not a fixed pair of demands, but a function mapping private types to optimal demands.

- **Efficiency:**
  - Pareto Efficiency:
    - Any outcome where total demand ≤ S is Pareto efficient: increasing one player's payoff requires decreasing the other's.
    - Outcomes where total demand > S (zero payoff) are Pareto inefficient because both could be strictly better by lowering demands.
  - Utilitarian Efficiency:
    - Expected total payoff is maximized when strategies avoid over-demanding while allocating as much of S as possible.
    - The BNE strategies implicitly attempt to maximize expected sum of payoffs, given uncertainty about opponent type.
- **Fairness:**
  - Equity
    - Because each player's demand depends on their private type, realized payoffs may differ. Higher-type players receive more.
    - In expectation, fairness improves if type distributions are symmetric.
  - Envy-Freeness:
    - An allocation is envy-free if a player prefers her own payoff to the other player's payoff.
    - In this game, envy-freeness is not guaranteed, because different types may lead to asymmetric payoffs and the zero-payoff rule can generate situations where both players are worse off.

1.1.3 Interpretation

- **Realism:**
  - This Bayesian bargaining formulation reflects realistic negotiation environments where players cannot fully observe each other's constraints.
  - For example, in salary negotiations or business contracts, each side has a private minimum acceptable level (type t_i) but only probabilistic beliefs about the other's needs.
  - By incorporating incomplete information, the game captures how strategic demands emerge under uncertainty, rather than assuming perfect foresight as in the complete-information version.
- **Multiplicity of Equilibria:**
  - The game admits multiple Bayesian Nash Equilibria (BNE). In some equilibria, players demand conservatively, keeping the sum $d_A + d_B$ safely below the cap S, which guarantees positive payoffs but leaves resources underutilized.

- In other equilibria, players demand aggressively near the cap, achieving high expected payoffs but running a greater risk of exceeding S and ending with zero. This multiplicity illustrates the tension between safety and ambition in bargaining with incomplete information.

- **Refinements:**
  - Refinements such as Perfect Bayesian Equilibrium or Sequential Equilibrium are relevant if we consider dynamic alternating-offer protocols.
  - These refinements incorporate belief updating and credible threats, narrowing down plausible equilibria and providing more predictive power. In our static one-shot proposals, BNE suffices, but dynamic versions or introducing time discounting could benefit from these refinements.

- **Bounded Rationality and Computational Tractability:**
  - In real behavior, players may not solve the Bayesian best-response problem exactly. Instead, they adopt heuristics such as "demand close to half of S" or "demand slightly above expected type." These heuristics may explain systematic deviations from equilibrium predictions:
    - Over-demanding → frequent zero-payoff outcomes.
    - Fair-split heuristics → coordination on 50-50 even when types differ, driven by norms rather than expected payoff maximization.
  - Solving for BNE requires integrating best responses over the distribution of types, which is computationally demanding even in the two-player case. Strategy adjustments must be recomputed for each cap, highlighting the need for computational tools such as NashPy for matrix-based equilibria and GTE for extensive-form comparisons. The tractability issue becomes especially salient when extending the type space or introducing richer belief structures.