Introduction to Functional Analysis

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Spring 2025

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1 Banach space theory

1.1 Quotient spaces, Baire category and uniform boundedness

Theorem. Let $\|\cdot\|$ be a **seminorm** on a vector space V. If we define $E = \{v \in V : \|v\| = 0\}$, then E is a subspace of V, and the function on V/E defined by

$$\|v + E\| = \|v\|$$

for any $v + E \in V/E$ defines a **norm**.

Theorem (Baire Category Theorem)**.** Let M be a complete metric space, and let $\{C_n\}_{n=0}^{\infty}$ be a collection of closed subsets of M such that $M = \bigcup_{n=0}^{\infty} C_n$. Then at least one of the C_n contains an open ball $B(x, r) = \{y \in M : d(x, y) < r\}$.

Theorem (Uniform Boundedness Theorem)**.** Let B be Banach space and V a normed vector space. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{B}(B, V)$. Then if for all $b \in B$ we have $\sup_n \|T_n b\| < \infty$ (that is, this sequence is pointwise boudned), then $\sup_n \|T_n\| < \infty$ (the operator norms are bounded).

Proof. For each $k \in \mathbb{N}$, define

$$C_k = \left\{ b \in B : \|b\| \leq 1, \sup_{n \in \mathbb{N}} \|T_n b\| \leq k \right\}.$$

This set is closed for each $k \in \mathbb{N}$, but by assumption, we have

$$\{b \in B : \|b\| \leq 1\} = \bigcup_{k=0}^{\infty} C_k.$$

The left hand side is a closed subset of B, and is thus a complete metric space. By Baire Category Theorem, there exists $k \in \mathbb{N}$ such that C_k contains an open ball $B(b_0, \delta_0)$. Then, if $b \in B(0, \delta_0)$, we have $b_0 + b \in B(b_0, \delta_0)$ and thus

$$\sup_{n \in \mathbb{N}} \|T_n(b_0 + b)\| \leq k.$$

It follows that

$$\sup_{n \in \mathbb{N}} \|T_n b\| \leq \sup_{n \in \mathbb{N}} \|T_n(b_0 + b)\| + \sup_{n \in \mathbb{N}} \|T_n b_0\| \leq 2k.$$

Suppose $\|b\| = 1$, then $\frac{\delta_0}{2} b \in B(0, \delta_0)$ and thus for all $n \in \mathbb{N}$, we have

$$\left\| T_n \left(\frac{\delta_0}{2} b \right) \right\| \leq 2k.$$

Therefore,

$$\sup_{n \in \mathbb{N}} \|T_n\| \leq \frac{4k}{\delta_0}.$$

□

2 Hilbert space theory

2.1 Basic Hilbert space theory

Definition (Pre-Hilbert space)**.** A **pre-Hilbert** space H is a vector space over $\mathbb{C}$ with a **Hermitian inner product**, which is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ satisfying the following properties.

1. For all $\lambda_1, \lambda_2 \in C$ and $v_1, v_2, w \in H$, we have
$$\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, 2 \rangle + \lambda_2 \langle v_2, w \rangle .$$
2. For all $v, w \in H$, we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
3. For all $v \in H$, we have $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$.

Definition. Let H be a pre-Hilbert space. For all $v \in H$, we define
$$\|v\| = \langle v, v \rangle^{\frac{1}{2}} .$$

Theorem (Cauchy-Schwarz inequality)**.** Let H be a pre-Hilbert space. For all $u, v \in H$, we have
$$|\langle u, v \rangle| \leq \|u\| \|v\| .$$

Proof. Define $f(t) = \|u + tv\|^2$. Notice that
$$\begin{aligned} f(t) &= \langle u + tv, u + tv \rangle \\ &= \langle u, u \rangle + t^2 \langle v, v \rangle + t \langle u, v \rangle + t \langle v, u \rangle \\ &= \|u\|^2 + t^2 \|v\|^2 + 2t \operatorname{Re}(\langle u, v \rangle). \end{aligned}$$
This implies that
$$0 \leq f(t_{\min}) = \|u\|^2 - \frac{\operatorname{Re}(\langle u, v \rangle)^2}{\|v\|^2}.$$
It follows that
$$|\operatorname{Re}(\langle u, v \rangle)| \leq \|u\| \|v\| .$$
This is almost what we want. To finish up, first note that if $\langle u, v \rangle = 0$ then there is nothing to prove, so suppose $\langle u, v \rangle \neq 0$, and define
$$\lambda = \frac{\overline{\langle u, v \rangle}}{|\langle u, v \rangle|}.$$
Note that we have $|\lambda| = 1$ and we have the chain of equalities of real numbers:
$$|\langle u, v \rangle| = \lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \operatorname{Re} \langle \lambda u, v \rangle \leq \|\lambda u\| \|v\| .$$
However, $\|\lambda u\| = \|u\|$, so the proof is complete.

□

Theorem. If H is a pre-Hilbert space, then $\|\cdot\|$ is a norm on H.

Proof. Note that
$$\|v\| = 0 \iff \langle v, v \rangle = 0 \iff v = 0.$$
Now if $\lambda \in \mathbb{C}$ and $v \in H$, then
$$\langle \lambda v, \lambda v \rangle = \lambda \overline{\lambda} \langle v, v \rangle = |\lambda|^2 \|v\|^2 .$$
Therefore, $\|\lambda v\| = |\lambda| \|v\|$.

Finally, let $u, v \in H$, then

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v\rangle \\ &\le \|u\|^2 + \|v\|^2 + 2\left|\langle u, v\rangle\right| \\ &\le \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\| \\ &= (\|u\| + \|v\|)^2.\end{aligned}$$

This completes the proof. □

Theorem. If $u_n \to u$ and $v_n \to v$ in a pre-Hilbert space H, then $\langle u_n, v_n\rangle \to \langle u, v\rangle$.

Proof. If $u_n \to u$ and $v_n \to v$, then $\|u_n - u\| \to 0$ and $\|v_n - v\| \to 0$. It follows that

$$\begin{aligned}\left|\langle u_n, v_n\rangle - \langle u, v\rangle\right| &= \left|\langle u_n - u, v_n\rangle - \langle u, v - v_n\rangle\right| \\ &\le \left|\langle u_n - u, v_n\rangle\right| + \left|\langle u, v - v_n\rangle\right| \\ &\le \|u_n - u\|\,\|v_n\| + \|u\|\,\|v - v_n\| \\ &\le \|u_n - u\| \sup_{k\in\mathbb{N}} \|v_k\| + \|u\|\,\|v - v_n\| \\ &\to 0\end{aligned}$$

as $n \to \infty$. This completes the proof. □

Definition (Hilbert space)**.** A **Hilbert space** is a pre-Hilbert space that is complete with repsect to the norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.

Example. Some examples of Hilbert spaces:

- $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) : z_j \in \mathbb{C}\}$ with $\langle z, w\rangle = \sum_j z_j \overline{w_j}$ is a Hilbert space.
- $\ell^2 = \left\{a = \{a_k\}_{k=0}^{\infty} : a_k \in \mathbb{C},\ \sum_{k=0}^{\infty} |a_k|^2 < \infty\right\}$ with $\langle a, b\rangle = \sum_{k=0}^{\infty} a_k \overline{b_k}$ is a Hilbert space.
- If $E \subset \mathbb{R}$ is measurable, then $L^2(E) = \left\{f : E \to \mathbb{C},\ \int_E |f|^2 < \infty\right\}$ with $\langle f, g\rangle = \int_E f\overline{g}$ is a Hilbert space.

We will show that each separable Hilbert spaces is isometrically isomorphic to either $\mathbb{C}^n$ or ℓ^2.

Now we have seen that ℓ^2 and L^2 spaces are Hilbert spaces. This is expected since the definition of the inner product in these spaces uses the fact that they are ℓ^2 or L^2. A natural question then is whether other ℓ^p or L^p spaces are also Hilbert spaces with respect to some inner product? It turns out there is a simple way to decide whether a norm come from a inner-product, and thus whether a Banach space is a Hilbert space.

Theorem (Parallelogram Law)**.** If H is a pre-Hilbert space, then for all $u, v \in H$, we have

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right).$$

In addition, if H is a normed vector space satisfying this equality, then H is a pre-Hilbert space.

Using the previous theorem, we can verify that ℓ^p and L^p with $p \neq 2$ are **not** Hilbert spaces.

Definition (Orthogonal)**.** If H is a pre-Hilbert space, $u, v \in H$ are **orthogonal** if $\langle u, v\rangle = 0$. We denote this as $u \perp v$.

Definition (Orthonormal sets)**.** If H is a pre-Hilbert space, a subset $\{e_\lambda\}_{\lambda\in\Lambda} \subset H$ is **orthonormal** if for all $\lambda \in \Lambda$, we have $\|e_\lambda\| = 1$ and $\lambda_1 \neq \lambda_2$ implies $e_{\lambda_1} \perp e_{\lambda_2}$.

Remark. we will mainly be interested in the case where we have a countable orthonormal set.

Example. The set $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n\in\mathbb{Z}}$ as elements in $L^2([-\pi,\pi])$ is an orthonormal subset of $L^2([-\pi,\pi])$. Indeed, for any $m,n \in \mathbb{Z}$, we have

$$\int_{-\pi}^{\pi} e^{imx}\overline{e^{inx}} = \int_{-\pi}^{\pi} e^{i(m-n)x} = \begin{cases} 2\pi & (m=n), \\ 0 & (m \neq n). \end{cases}$$

Therefore, $\left\langle \frac{e^{inx}}{\sqrt{2\pi}}, \frac{e^{imx}}{\sqrt{2\pi}} \right\rangle = \delta_{mn}$, and $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n\in\mathbb{Z}}$ is an orthonormal subset of $L^2([-\pi,\pi])$.

Theorem (Bessel)**.** If $\{e_n\}_{n=0}^{\infty}$ is countable orthonormal subset of a pre-Hilbert space H, then for all $u \in H$, we have

$$\sum_{n=0}^{\infty} \left|\langle u, e_n\rangle\right|^2 \leq \|u\|^2 .$$

Proof. We first do the finite case. Suppose $\{e_n\}_{n=1}^{N}$ is an orthonormal subset of H. Then,

$$\begin{aligned} \left\| \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\|^2 &= \left\langle \sum_{n=0}^{N} \langle u, e_n\rangle e_n, \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\rangle \\ &= \sum_{n=0}^{N}\sum_{m=1}^{N} \langle u, e_n\rangle \overline{\langle u, e_m\rangle} \langle e_n, e_m\rangle \\ &= \sum_{n=0}^{N} \left|\langle u, e_n\rangle\right|^2 . \end{aligned}$$

Also,

$$\begin{aligned} \left\langle u, \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\rangle &= \sum_{n=0}^{N} \overline{\langle u, e_n\rangle} \langle u, e_n\rangle \\ &= \sum_{n=0}^{N} \left|\langle u, e_n\rangle\right|^2 . \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &\leq \left\| u - \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\|^2 \\ &= \|u\|^2 + \left\| \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\|^2 - 2\operatorname{Re}\left\langle u, \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\rangle \\ &= \|u\|^2 - \sum_{n=0}^{N} \left|\langle u, e_n\rangle\right|^2 , \end{aligned}$$

as desired.

For the infinite case, just take the limit as $N \to \infty$. □

Definition (Maximal orthonormal subset)**.** An orthonormal subset $\{e_\lambda\}_\lambda$ of a pre-Hilbert space is **maximal** if $u \in H$ and $\langle u, e_\lambda\rangle = 0$ for all $\lambda \in \Lambda$ implies that $u = 0$.

Theorem. Every non-trivial pre-Hilbert space has a maximal orthonormal subset.

This can be proved using Zorn's Lemma. We will prove something less strong but often equally useful by hand, without applying Zorn's Lemma.

Theorem. Every non-trivial separable pre-Hilbert space has a countable maximal orthonormal subset.

Proof. Use the Gram-Schimdt process. Let $\{v_j\}_{j=0}^{\infty}$ be a countable dense subset of H where $v_0 \neq 0$. Claim that for any $n \in \mathbb{N}$, there exists $m(n) \leq n$ and an orthonormal subset $\left\{e_1, \ldots, e_{m(n)}\right\}$ such that

1. $\operatorname{span}\left\{e_1, \ldots, e_{m(n)}\right\} = \operatorname{span}\left\{v_1, \ldots, v_n\right\}$.
2. If $v_n \in \operatorname{span}\left\{v_1, \ldots, v_{n-1}\right\}$, we have
$$\left\{e_1, \ldots, e_{m(n)}\right\} = \left\{e_1, \ldots, e_{m(n-1)}\right\} \cup \emptyset.$$
Otherwise, we have
$$\left\{e_1, \ldots, e_{m(n)}\right\} = \left\{e_1, \ldots, e_{m(n-1)}\right\} \cup e_{m(n)}$$
for some $e_{m(n)} \in H$.

Prove this by induction. For the base case, let $e_1 = \frac{v_1}{\|v_1\|}$. For the inductive step, suppose the claim holds for $n = k$. If $v_{k+1} \in \operatorname{span}\left\{v_1, \ldots, v_k\right\}$, then
$$\operatorname{span}\left\{e_1, \ldots, e_{n(k)}\right\} = \operatorname{span}\left\{v_1, \ldots, v_k\right\} = \operatorname{span}\left\{v_1, \ldots, v_{k+1}\right\}.$$
Now suppose $v_{k+1} \notin \operatorname{span}\left\{v_1, \ldots, v_k\right\}$. Define
$$w_{k+1} = v_{k+1} - \sum_{j=1}^{m(k)} \left\langle v_{k+1}, e_j \right\rangle e_j.$$
Note that $w_{k+1} \neq 0$ and define $e_{m(k+1)} = \frac{w_{k+1}}{\|w_{k+1}\|}$. Then, $\left\|e_{m(k+1)}\right\| = 1$ and for all $1 \leq l \leq m(k)$,
$$\begin{aligned}
\left\langle e_{m(k+1)}, e_l \right\rangle &= \frac{1}{\|w_{k+1}\|} \left\langle v_{k+1} - \sum_{j=1}^{m(k)} \left\langle v_{k+1}, e_j \right\rangle, e_l \right\rangle \\
&= \frac{1}{\|w_{k+1}\|} \left(\left\langle v_{k+1}, e_l \right\rangle - \left\langle v_{k+1}, e_l \right\rangle \right) \\
&= 0.
\end{aligned}$$
Therefore, $e_{m(k+1)}$ is the desired vector we want and we have completed the proof for the claim.

Now let
$$S = \bigcup_{n=0}^{\infty} \left\{e_1, \ldots, e_{m(n)}\right\}.$$
Then S is a countable orthonormal subset of H. Now we show S is maximal. Suppose $u \in H$ and $\langle u, e_l \rangle = 0$. Since $\{v_j\}_{j=0}^{\infty}$ is dense in H, there exists $\left\{v_{j(k)}\right\}_{k=0}^{\infty}$ such that $v_{j(k)} \to u$ as $k \to \infty$. By our claim, we know $v_{j(k)} \in \operatorname{span}\left\{e_1, \ldots, e_{m(j(k))}\right\}$. By Bessel's inequality,
$$\left\|v_{j(k)}\right\|^2 = \sum_{l=1}^{m(j(k))} \left|\left\langle v_{j(k)}, e_l \right\rangle\right|^2 = \sum_{l=1}^{m(j(k))} \left|\left\langle v_{j(k)} - u, e_l \right\rangle\right|^2 \leq \left\|v_{j(k)} - u\right\|^2,$$
where for the first equality we used the fact that $v_{j(k)} \in \operatorname{span}\left\{e_1, \ldots, e_{m(j(k))}\right\}$. Since $v_{j(k)} \to u$ as $k \to \infty$, the inequality implies that $\left\|v_{j(k)}\right\| \to 0$ as $k \to \infty$ and thus $\|u\| = 0$, showing that S is indeed a maximal orthonormal subset of H. □

Corollary. ℓ^2 and L^2 have countable maximal orthonormal subset since they are both separable.

2.2 Orthonormal bases and Fourier Series

Definition (Orthonormal basis)**.** Let H be a Hilbert space. An **orthonormal basis** of H is a countable maximal orthonormal subset $\{e_n\}_n$ of H.

Theorem. If $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis in Hilbert space H, then for all $u \in H$, we have
$$\sum_{n=0}^{\infty} \langle u, e_n \rangle e_n = u.$$
This is the Fourier-Bessel series.

This tells us we can write each element in H as a infinite linear combination of the orthonormal basis.

Proof. We first prove the sequence of partial sums $\{\sum_{n=0}^{m} \langle u, e_n \rangle e_n\}_m$ is Cauchy. Let $\varepsilon > 0$. By Bessel's inequality, we have

$$\sum_{n=0}^{\infty} |\langle u, e_n \rangle|^2 \le \|u\|^2 < \infty.$$

Therefore, there exsits $M \in \mathbb{N}$ such that $N \ge M$ implies $\sum_{n=N+1}^{\infty} |\langle u, e_n \rangle|^2 < \varepsilon^2$. Then for all $m > l \ge M$, we have

$$\left\| \sum_{n=0}^{m} \langle u, e_n \rangle e_n - \sum_{n=0}^{l} \langle u, e_n \rangle e_n \right\|^2 \le \sum_{n=l+1}^{m} |\langle u, e_n \rangle|^2 \le \sum_{n=l+1}^{\infty} |\langle u, e_n \rangle|^2 < \varepsilon^2.$$

Therefore, the sequence of partial sum is Cauchy. Since H is complete, there exists $\overline{u} \in H$ such that $\overline{u} = \sum_{n=0}^{\infty} \langle u, e_n \rangle e_n$. It remains to show that $\overline{u} = u$. By continuity of inner-product, for all $l \in \mathbb{N}$, we have

$$\begin{aligned} \langle u - \overline{u}, e_l \rangle &= \lim_{m\to\infty} \left\langle u - \sum_{n=0}^{m} \langle u, e_n \rangle e_n, e_l \right\rangle \\ &= \lim_{m\to\infty} \left[\langle u, e_l \rangle - \sum_{n=0}^{m} \langle u, e_n \rangle \langle e_n, e_l \rangle \right] \\ &= 0. \end{aligned}$$

Since $\{e_n\}_{n=0}^{\infty}$ is maximal, this implies that $u - \overline{u} = 0$ and the proof is complete. □

Theorem. Let H be a Hilbert space. If H has an orthonormal basis, then H is separable.

Proof. Suppose $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for H. Then

$$S = \bigcup_{m\in\mathbb{N}} \left\{ \sum_{n=0}^{m} q_n e_n : q_n \in \mathbb{Q} + i\mathbb{Q} \right\}$$

is a countable set. Also, by the previous theorem, S is dense in H. □

Remark. Let H be a Hilbert space. H is separable if and only if H has an orthonormal basis.

Theorem (Parseval's identity)**.** If H is a Hilbert space and $\{e_n\}_{n=0}^{\infty}$ is a countable orthonormal basis, then for all $u \in H$, we have

$$\sum_n |\langle u, e_n \rangle|^2 = \|u\|^2$$

Proof. We have $u = \sum_n \langle u, e_n \rangle e_n$. This implies that

$$\begin{aligned} \|u\|^2 &= \lim_{m\to\infty} \left\langle \sum_{n=0}^{m} \langle u, e_n \rangle e_n, \sum_{l=0}^{m} \langle u, e_l \rangle e_l \right\rangle \\ &= \lim_{m\to\infty} \sum_{n=0}^{m} \sum_{l=0}^{m} \langle u, e_n \rangle \overline{\langle u, e_l \rangle} \langle e_n, e_l \rangle \\ &= \lim_{m\to\infty} \sum_{n=0}^{m} |\langle u, e_n \rangle|^2 \\ &= \sum_{n=0}^{\infty} |\langle u, e_n \rangle|^2 . \end{aligned}$$

□

Theorem. If H is an infinte dimensional separable Hilbert space, then H is isometrically isomorphic to ℓ^2. That is, there exists bijective boudned linear map $T : H \to \ell^2$ such that for all $u, v \in H$, we have

$$\|Tu\|_{\ell^2} = \|u\|_H \text{ and } \langle Tu, Tv \rangle_{\ell^2} = \langle u, v \rangle_H .$$

Proof. Since H is separable, there exists an orthonormal basis $\{e_n\}_{n=0}^{\infty}$. For all $u \in H$, the previous theorem gives

$$\|u\| = \left(\sum_{n=0}^{\infty} |\langle u, e_n\rangle|^2\right)^{\frac{1}{2}}.$$

Define $T : H \to \ell^2$ by

$$Tu = \{\langle u, e_n\rangle\}_{n=0}^{\infty} \in \ell^2.$$

It is easy to check that T is the desired isometric isomorphism. □

Next we use the theories we learned in a more concrete setting — the Fourier series.

Theorem. The subset $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ is an orthonormal subset of $L^2([-\pi, \pi])$.

Definition. Let $f \in L^2([-\pi, \pi])$. Then the **n-th Fourier coefficient** of f is

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\, dt.$$

The **N-th Fourier sum** of f is

$$S_n f(x) = \sum_{|n|\le N} \widehat{f}(n)e^{inx} = \sum_{|n|\le N}\left\langle f, \frac{e^{int}}{\sqrt{2\pi}}\right\rangle \frac{e^{inx}}{\sqrt{2\pi}}.$$

The **Fourier series** of f is the formal series $\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{-inx}$.

The natural question now is whether we have for all $f \in L^2([-\pi, \pi])$,

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{inx}.$$

That is, whether we have the following convergence in L^2.

$$\lim_{N\to\infty} \|f - S_N f\|_2 = 0.$$

This question is then equivalent to whether $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ is maximal in $L^2([-\pi, \pi])$. That is, whether $\widehat{f}(n) = 0$ for all $n \in \mathbb{N}$ implies $f = 0$.

The answer to the question is yes, but it is going to take some work. We first do some simple calculation.

Theorem. For all $f \in L^2([-\pi, \pi])$ and for all $N \in \mathbb{N}$, we have

$$S_N f(x) = \int_{-\pi}^{\pi} D_N(x - t)f(t)\, dt,$$

where

$$D_N(x) = \begin{cases} \frac{2N+1}{2\pi} & (x = 0) \\ \frac{\sin\left(N+\frac{1}{2}\right)x}{2\pi \sin\frac{x}{2}} & (x \neq 0) \end{cases}$$

it the **Dirichlet kernel**. Figure 1 shows a plot of $D_N(x)$ on $[-\pi, \pi]$ for $N = 1, 2, 3$. Note that D_N is a smooth function.

Proof. If $f \in L^2([-\pi, \pi])$, we have

$$\begin{aligned} S_N f(x) &= \sum_{|n|\le N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\, dt\right)e^{inx} \\ &= \int_{-\pi}^{\pi} f(t)\left(\frac{1}{2\pi}\sum_{|n|\le N} e^{in(x-t)}\right) dt. \end{aligned}$$

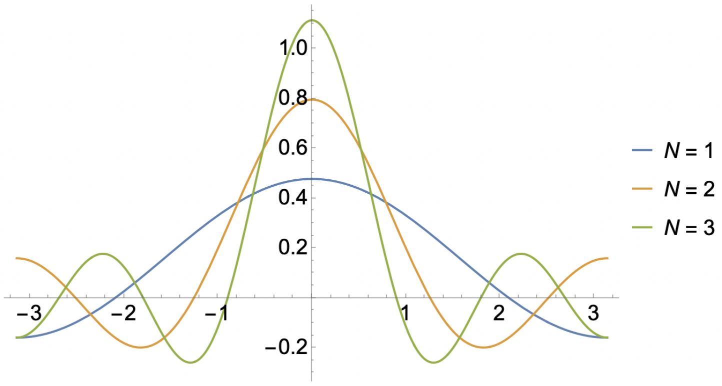

Figure 1: Plot of Dirichlet kernel $D_N(x)$ on $[-\pi, \pi]$ for $N = 1, 2, 3$.

Let $D_N(x) = \frac{1}{2\pi} \sum_{|n| \le N} e^{in(x-t)}$. Then for $x \neq 0$, we have

$$\begin{aligned} D_N(x) &= \frac{1}{2\pi} \sum_{n=-N}^{N} e^{-inx} \\ &= \frac{1}{2\pi} e^{-iNx} \sum_{n=0}^{2N} \left(e^{ix}\right)^n \\ &= \frac{1}{2\pi} e^{-iNx} \frac{1 - e^{i(2N+1)x}}{1 - e^{ix}} \\ &= \frac{1}{2\pi} \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}} \\ &= \frac{1}{2\pi} \frac{2i \sin(N + \frac{1}{2})x}{2i \sin \frac{x}{2}} \\ &= \frac{1}{2\pi} \frac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}}, \end{aligned}$$

as desired. For $x = 0$, we also clearly have $D_N(0) = \frac{(2N+1)}{2\pi}$. The proof is thus complete. □

Definition. If $f \in L^2([-\pi, \pi])$, we define the N**-th Cesaro-Fourier mean** of f by

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{k=0}^{N} S_k f(x).$$

The idea behind defining the Cesaro mean is that if the original sequence converges, the Cesaro mean also converge to the same limit. However, Cesaro have even better property — the Cesaro mean can converge even if the original sequence does not converge. Therefore, it has better convergence properties and hopefully we can show it converge to f in L^2 more easily. The goal now is then to show

$$\|\sigma_N f - f\|_2 \to 0 \text{ as } N \to \infty.$$

This would tell us if all Fourier coefficients are zero, then the Cesaro means are zero, and the limit above would tell us f is zero.

2.3 Fejer's theorem and convergence of Fourier series

In this section, we will show that if $f \in L^2([-\pi, \pi])$, then $\|\sigma_N f - f\|_2 \to 0$ as $N \to \infty$.

First we will rewrite the Cesaro Fourier mean, just like what we did for the partial Fourier sum using the Dirichlet kernel.

Theorem. For any $f \in L^2([-\pi, \pi])$, we have

$$\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(x-t) f(t)\, dt,$$

where

$$K_N(x) = \begin{cases} \frac{N+1}{2\pi} & (x = 0) \\ \frac{1}{2\pi(N+1)} \frac{\sin^2 \frac{N+1}{2} x}{\sin^2 \frac{x}{2}} & (x \neq 0) \end{cases}$$

is the Fejer kernel.

Moreover, we have

1. $K_N(x) \geq 0$, $K_N(x) = K_N(-x)$, and $K_N(x)$ is 2π periodic.
2. $\int_{-\pi}^{\pi} K_N(t)\, dt = 1$.
3. If $\delta \in (0, \pi]$, then for all $\delta \leq |x| \leq \pi$, we have

$$|K_N(x)| \leq \frac{1}{2\pi(N+1)\sin^2 \frac{\delta}{2}}.$$

A plot for $K_N(x)$ is shown in Figure 2.

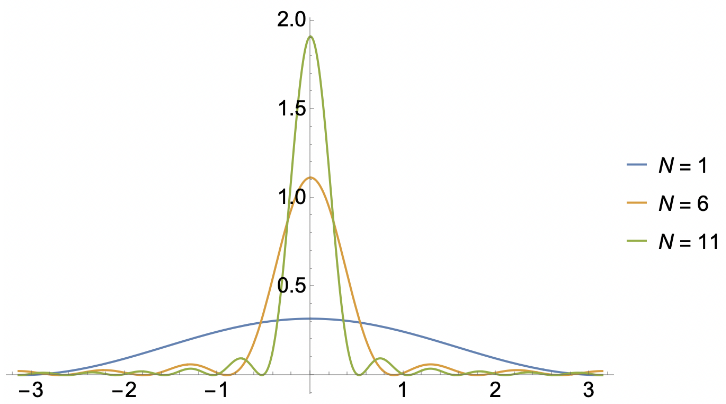

Figure 2: Plot of Dirichlet kernel $D_N(x)$ on $[-\pi, \pi]$ for $N = 1, 6, 11$.

Note that $K_N(x)$ is concentrated at 0 when N is very large. In this case, we have

$$\begin{aligned} \sigma_N f(x) &= \int_{-\pi}^{\pi} K_N(x-t) f(t)\, dt \\ &\approx f(x) \int_{-\pi}^{\pi} K_N(t)\, dt \\ &= f(x). \end{aligned}$$

This provides a rough intuition behind the Fejer kernel. The fact that K_N is non-negative makes a huge difference compared to the Dirichlet kernel, since it gives much better properties.

Proof. Recall that

$$S_k f(x) = \int_{-\pi}^{\pi} D_k(x-t) f(t)\, dt,$$

where

$$D_k(t) = \begin{cases} \frac{2N+1}{2\pi} & (t = 0), \\ \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})t}{\sin \frac{t}{2}} & (t \neq 0). \end{cases}$$

It follows that

$$\begin{aligned}\sigma_N f(x) &= \frac{1}{N+1}\sum_{k=0}^{N} S_k f(x)\\ &= \int_{-\pi}^{\pi} \frac{1}{N+1}\sum_{k=0}^{N} D_k(x-t)f(t)\,dt.\end{aligned}$$

Then for $x \neq 0$, we have

$$\begin{aligned}K_N(x) &= \frac{1}{N+1}\sum_{k=0}^{N} D_k(x)\\ &= \frac{1}{2\pi(N+1)}\frac{1}{2\sin^2\frac{x}{2}}\sum_{k=0}^{N} 2\sin\frac{x}{2}\sin\left(k+\frac{1}{2}\right)x\\ &= \frac{1}{2\pi(N+1)}\frac{1}{2\sin^2\frac{x}{2}}\sum_{k=0}^{N}[\cos kx-\cos(k+1)x]\\ &= \frac{1}{2\pi(N+1)}\frac{1}{\sin^2\frac{x}{2}}\frac{1-\cos(N+1)}{2}\\ &= \frac{1}{2\pi(N+1)}\frac{\sin^2\frac{N+1}{2}x}{\sin^2\frac{x}{2}}.\end{aligned}$$

It follows immediately that $K_N(x) \geq 0$, $K_N(x)$ is even and 2π periodic.

For property 2, note that for all k,

$$\int_{-\pi}^{\pi} D_k(t)\,dt = \int_{-\pi}^{\pi}\sum_{n=-k}^{k} e^{int}\,dt = 1.$$

Then,

$$\int_{-\pi}^{\pi} K_N(t)\,dt = \frac{1}{N+1}\sum_{k=0}^{N}\int_{-\pi}^{\pi} D_k(t)\,dt = 1,$$

as desired.

For property 3, let $\delta \in (0,\pi]$. Note that $\sin^2\frac{x}{2}$ is even and increasing on $[0,\pi]$. It follows that $\delta \leq |x| \leq \pi$ implies $\sin^2\frac{x}{2} \geq \sin^2\frac{\delta}{2}$. Therefore,

$$K_N(x) \leq \frac{1}{2\pi(N+1)}\frac{\sin^2\frac{N+1}{2}x}{\sin^2\frac{\delta}{2}} \leq \frac{1}{2\pi(N+1)\sin^2\frac{\delta}{2}}.$$

□

Since the continuous functions that vanishes at both end points is dense in $L^2([-\pi,\pi])$, it make sense to first prove the theorem for continuous functions. We have the following theorem by Fejer.

Theorem (Fejer)**.** If $f \in C([-\pi,\pi])$ is 2π-periodic, $f(\pi) = f(-\pi)$, then $\sigma_N f \to f$ uniformly on $[-\pi,\pi]$.

Proof. First we extennd f by periodicity to all of $\mathbb{R}$. Then $f \in C(\mathbb{R})$, 2π-periodic. This implies that f is uniformly continuous and bounded.

Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|y-z| < \delta$ implies that $|f(y)-f(z)| < \frac{\varepsilon}{2}$. Choose $M \in \mathbb{N}$ such that

$$\frac{2\,\|f\|_\infty}{(N+1)\sin^2\frac{\delta}{2}} < \frac{\varepsilon}{2}.$$

for all $N \geq M$. Also, since f and K_N are both 2π-periodic, we have

$$\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(x-t)f(t)\ dt = \int_{x-\pi}^{x+\pi} K_N(\tau)f(x-\tau)\ d\tau = \int_{-\pi}^{\pi} K_N(\tau)f(x-\tau)\ d\tau.$$

Then for all $N \geq M$ and for all $x \in [-\pi, \pi]$, we have

$$\begin{aligned}
|\sigma_N f(x) - f(x)| &= \left| \int_{-\pi}^{\pi} K_N(t) f(x-t)\ dt - \int_{\pi}^{\pi} K_N(t) f(x)\ dt \right| \\
&\leq \int_{-\pi}^{\pi} K_N(t) \left| f(x-t) - f(x) \right|\ dt \\
&\leq \int_{|t|<\delta} K_N(t) \left| f(x-t) - f(x) \right|\ dt + \int_{\delta \leq |t| \leq \pi} K_N(t) \left| f(x-t) - f(x) \right|\ dt \\
&\leq \frac{\varepsilon}{2} \int_{|t|<\delta} K_N(t)\ dt + 2 \left\| f \right\|_\infty \int_{\delta \leq |t| \leq \pi} \frac{1}{2\pi(N+1)\sin^2 \frac{\delta}{2}}\ dt \\
&\leq \frac{\varepsilon}{2} + \frac{2 \left\| f \right\|_\infty}{(N+1)\sin^2 \frac{\delta}{2}} \\
&\leq \varepsilon.
\end{aligned}$$

□

Remark. The same proof can be modified if instead of $K_N(x) \geq 0$, we have

$$\sup_{N \in \mathbb{N}} \int_{-\pi}^{\pi} |K_N(x)|\ dx < \infty.$$

Note that

$$\int_{-\pi}^{\pi} |D_N(x)|\ dx \sim \log N,$$

so we cannot reproduct the proof using Dirichlet kernel.

We only need some last bit of information to conclude the answer of our main question.

Theorem. For all $f \in L^2([-\pi, \pi])$, we have $\left\| \sigma_N f \right\|_2 \leq \left\| f \right\|_2$.

Proof. Suppose first the $f \in C([-\pi, \pi])$ and 2π-periodic. Then $\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(t) f(x-t)\ dt$. It follows that

$$\begin{aligned}
\int_{-\pi}^{\pi} |\sigma_N f(x)|^2\ dx &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x-s)\overline{f(x-t)} K_N(s) K_N(t)\ ds\ dt\ dx \\
&= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_N(s) K_N(t) \left[\int_{-\pi}^{\pi} f(x-s)\overline{f(x-t)}\ dx \right]\ ds\ dt \\
&\leq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_N(s) K_N(t) \left\| f(\cdot - s) \right\|_2 \left\| f(\cdot - t) \right\|_2\ ds\ dt \\
&\leq \left\| f \right\|_2^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_N(s) K_N(t)\ ds\ dt \\
&= \left\| f \right\|_2,
\end{aligned}$$

where we used Cauchy-Schwarz inequality. This implies that $\left\| \sigma_N f \right\|_2 \leq \left\| f \right\|_2$.

Now for the general case, by density there exists sequence $\{f_n\}_{n=0}^{\infty}$ of 2π-periodic continuous function that $\left\| f_n - f \right\|_2 \to 0$. Then, $\left\| \sigma_N f_n - \sigma_N f \right\| \to 0$ as $n \to \infty$. Therefore,

$$\left\| \sigma_N f \right\|_2 = \lim_{n \to \infty} \left\| \sigma_N f_n \right\|_2 \leq \lim_{n \to \infty} \left\| f_n \right\|_2 = \left\| f \right\|_2.$$

□

Theorem. For all $f \in L^2([-\pi,\pi])$, we have $\|\sigma_N f - f\|_2 \to 0$ as $N \to \infty$. In particular, as a immediate corollary, if $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.

Proof. Let $f \in L^2([-\pi,\pi])$ and $\varepsilon > 0$. Again by density there exists $g \in C([-\pi,\pi])$ 2π-periodic such that $\|f-g\|_2 \le \frac{\varepsilon}{3}$. Since $\sigma_N g \to g$ uniformly on $[-\pi,\pi]$, there exists $M \in \mathbb{N}$ such that for all $N \ge M$ and all $x \in [-\pi,\pi]$, we have

$$|\sigma_N g(x) - g(x)| < \frac{\varepsilon}{3\sqrt{2\pi}}.$$

Then for all $N \ge M$,

$$\begin{aligned}\|\sigma_N f - f\|_2 &\le \|\sigma_N(f-g)\|_2 + \|\sigma_N g - g\| + \|g-f\|_2 \\ &\le 2\|f-g\|_2 + \left(\int_{-\pi}^{\pi} |\sigma_N g(x) - g(x)|^2 \ dx\right)^{\frac{1}{2}} \\ &\le \varepsilon.\end{aligned}$$

□

Remark. We have shown that for all $f \in L^2([-\pi,\pi])$, $\|S_N f - f\|_2 \to 0$. This does not say $S_N f$ converge to f almost everywhere. However, by a theorem by Carleson, for all $f \in L^2([-\pi,\pi])$, we actually do have $S_N f \to f$ almost everywhere. Also, for all $1 < p < \infty$, $\|S_N f - f\|_p \to 0$. This is not true for $p = 1$ or $p = \infty$.

2.4 Minimizers, orthogonal complements, and Riesz representation theorem

Length minimizers

Theorem. Suppose H a Hilbert space and $C \subset H$ is a subset such that

1. $C \ne \emptyset$.
2. C is closed.
3. C is convex. That is, if $v_1, v_2 \in C$ and $t \in [0,1]$, then $tv_1 + (1-t)v_2 \in C$.

Then, there exists a unique $v \in C$ such that $\|v\| = \inf_{u \in C} \|u\|$.

Proof. Let $d = \inf_{u\in C} \|u\|$, which we know exists. Then there exists sequence $\{u_n\}_{n=0}^{\infty} \subset C$ such that $\|u_n\| \to d$.

Claim that $\{u_n\}_{n=0}^{\infty}$ is Cauchy. Let $\varepsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that $n \ge N$ implies

$$2\|u_n\|^2 < 2d^2 + \frac{\varepsilon^2}{2}.$$

It follows that for all $n, m \ge N$, we have

$$\|u_n - u_m\|^2 = 2\|u_n\|^2 + 2\|u_m\|^2 - 4\left\|\frac{u_n + u_m}{2}\right\|^2,$$

by the Parallelogram law. Note that $\frac{u_n+u_m}{2} \in C$. Therefore,

$$\|u_n - u_m\|^2 \le 2d^2 + \frac{\varepsilon^2}{2} + 2d^2 + \frac{\varepsilon^2}{2} - 4d^2 = \varepsilon^2.$$

This shows that $\{u_n\}_{n=0}^{\infty}$ is Cauchy.

Since H is Hilbert space, there exists $v \in H$ such that $u_n \to v$. Since C is closed, $v \in C$. It is also clear that $\|v\| = d$. To show this element is unique, suppose $v, \overline{v} \in C$ and $\|v\| = \|\overline{v}\| = d$. Then,

$$\|v - \overline{v}\|^2 = 2\|v\|^2 + 2\|\overline{v}\|^2 - 4\left\|\frac{v+\overline{v}}{2}\right\| \le 4d^2 - 4d^2 = 0.$$

This implies that $v = \overline{v}$ and the proof is complete. □

Orthocomplements

Theorem. If H is a Hilbert space, $W \subset H$ is a subspace, then

$$W^{\perp} = \{u \in H : \langle u, w \rangle = 0 \text{ for all } w \in V\}$$

is a closed linear subspace of H.

Moreover, if W is closed, then

$$H = W \oplus W^{\perp}.$$

That is, for all $u \in H$, there exists unique $w \in W$ and $w^{\perp} \in W^{\perp}$ such that $u = w + w^{\perp}$.

Proof. It is easy to show that $W^{\perp}$ is a subspace of H, and $W \cap W^{\perp} = \{0\}$. To show $W^{\perp}$ is closed, let $\{u_n\}_{n=0}^{\infty}$ be a sequence in $W^{\perp}$ and $u \in H$ such that $u_n \to u$. We need to show that $u \in W^{\perp}$. Let $w \in W$, then

$$\langle u, w \rangle = \lim_{n \to \infty} \langle u_n, w \rangle = 0.$$

Therefore, $u \in W^{\perp}$ and $W^{\perp}$ is a closed linear subspace of H.

Now suppose W is closed. If $W = H$, then $W^{\perp} = \{0\}$ and $H = W \oplus W^{\perp}$. Now assume that $W \neq H$. Let $u \in H \setminus W$ and define

$$C = u + W = \{u + w : w \in W\}.$$

Note that $u \in C$ so $C \neq \emptyset$. Also, C is convex, since if $u + w_1 \in C$, $u + w_2 \in C$, and $t \in [0,1]$, then

$$t(u + w_1) + (1-t)(u + w_2) = u + (tw_1 + (1-t)w_2) \in u + W.$$

Claim that C is also closed. Suppose $\{u + w_n\}_{n=0}^{\infty} \subset C$ is such that $u + w_n \to v$ for some $v \in H$. We want to show that $v \in C$. This implies that $w_n \to v - u$ and since W is closed, $v - u \in W$. It follows that $v = u + (v - u)$ so $v \in C$.

Since C is nonempty, closed, and convex, there exists unique element $v \in C$ such that

$$\|v\| = \inf_{w \in W} \|u + w\|.$$

Note that $v \in C$ so $u - v \in W$. Also, $u = (u - v) + v$. Claim that $v \in W^{\perp}$. Let $w \in W$ and

$$f(t) = \|v + tw\|^2 = \|v\|^2 + t^2 \|w\|^2 + 2t \operatorname{Re} \langle v, w \rangle.$$

Then $f(t)$ has a minimum at $t = 0$, which implies $f'(0) = \langle v, w \rangle = 0$. Repeat the previous argument with iw in place of w to obtain $\operatorname{Re} \langle v, iw \rangle = \operatorname{Im} \langle v, w \rangle = 0$. This shows that $w \in W^{\perp}$ and thus $H = W + W^{\perp}$.

To show the decomposition is unique, suppose $u = w_1 + w_1^{\perp} = w_2 + w_2^{\perp}$. This implies that

$$w_2 - w_1 = w_1^{\perp} - w_2^{\perp} \in W \cap W^{\perp}.$$

However, $W \cap W^{\perp} = \{0\}$, so $w_1 = w_2$ and $w_1^{\perp} = w_2^{\perp}$. □

Theorem. If $W \subset H$ is a subspace, then

$$\overline{W} = \left(W^{\perp}\right)^{\perp},$$

where $\overline{W}$ is the closure of W.

Proof. Homework. □

Definition (Projection)**.** A bounded linear operator $P : H \to H$ is a **projection** if $P^2 = P$.

Theorem. Let H be a Hilbert space, $W \subset H$ be a closed subspace. Then by the previous theorem we have $H = W \oplus W^{\perp}$. Define $\Pi_W : H \to H$ in the following way: for $v = w + w^{\perp}$, define

$$\Pi_W(v) = w.$$

Then Π_W is a projection.

Proof. It is easy to veirfy that Π_W is linear and $\Pi_W^2 = \Pi_W$. Claim Π_W is bounded. Suppose $v = w + w^{\perp}$. It follows that

$$\|v\|^2 = \left\|w + w^{\perp}\right\|^2 = \|w\|^2 + \left\|w^{\perp}\right\|^2 \geq \|w\|^2 .$$

This shows that $\|\Pi_W(v)\| \leq \|v\|$ so Π_W is a bounded linear operator. □

Riesz representation theorem

Theorem (Riesz representation theorem)**.** If H is a Hilbert space, then for all $f \in H'$, there exists a unique $v \in H$ such that

$$f(u) = \langle u, v\rangle \text{ for all } u \in H.$$

Proof. For uniqueness, suppose $f(u) = \langle u, v\rangle = \langle u, \widetilde{v}\rangle$. This implies that $\langle u, v - \widetilde{v}\rangle = 0$ for all $u \in H$. Setting $u = v - \widetilde{v}$ gives $v = \widetilde{v}$.

Now we show existence. If $f = 0$, let $v = 0$. Suppose now $f \neq 0$. Then there exists $u_1 \in H$ such that $f(u_1) \neq 0$. Now let $u_0 = \frac{u_1}{f(u_1)}$, then $f(u_0) = 1$. Let

$$C = \{u \in H : f(u) = 1\} = f^{-1}(\{1\}).$$

Then C is a nonempty and closed subset of H. Claim that C is also convex. If $u_1, u_2 \in C$ and $t \in [0, 1]$, then

$$f(tu_1 + (1 - t)u_2) = tf(u_1) + (1 - t)f(u_2) = 1.$$

Therefore, C is also convex. This implies that there exists $v_0 \in C$ such that

$$v_0 = \inf_{u \in C} \|u\| .$$

Note that $v_0 \neq 0$ and let $v = \frac{v_0}{\|v_0\|^2}$. Claim this is the desired vector. Let $N = f^{-1}(\{0\})$. Then, $C = v_0 + N$ and $\|v_0\| = \inf_{w \in N} \|v_0 + w\|$. By a similar argument as a previous theorem, we have $v_0 \in N^{\perp}$. Let $u \in H$, then

$$f(u - f(u)v_0) = f(u) - f(u)f(v_0) = 0.$$

Therefore, $u - f(u)v_0 \in N$. Since $v_0 \in N^{\perp}$, we have $\langle u - f(u)v_0, v_0\rangle = 0$. This implies that

$$\begin{aligned}
\langle u, v\rangle &= \frac{1}{\|v_0\|^2} \langle u, v_0\rangle \\
&= \frac{1}{\|v_0\|^2} \left(\langle u - f(u)v_0, v_0\rangle + f(u) \langle v_0, v_0\rangle\right) \\
&= f(u),
\end{aligned}$$

completing the proof. □

2.5 Adjoint of a bounded linear operator on a Hilbert space

Theorem. Let H be a Hilbert space and $A : H \to H$ be a bounded linear operator. Then there exists a unique bounded linear operator $A^* : H \to H$ (the **adjoint** of A) such that for all $u, v \in H$,

$$\langle Au, v\rangle = \langle u, A^*v\rangle .$$

Moreover, $\|A^*\| = \|A\|$.

Proof. The uniqueness of A^* follows from a similar argument of uniqueness for Riesz representation theorem.

Let $v \in H$. Define $f_v : H \to \mathbb{C}$ by
$$f_v(u) = \langle Au, v \rangle .$$
Then for any $u_1, u_2 \in H$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, we have
$$\begin{aligned} f_v(\lambda_1 u_1 + \lambda_2 u_2) &= \langle A(\lambda_1 u_1 + \lambda_2 u_2), v \rangle \\ &= \lambda_1 \langle Au_1, v \rangle + \lambda_2 \langle Au_2, v \rangle \\ &= \lambda_1 f_v(u_1) + \lambda_2 f_v(u_2). \end{aligned}$$
Therefore, f_v is a linear map. Next, suppose $\|u\| = 1$, then
$$|f_v(u)| = |\langle Au, v \rangle| \le \|Au\| \, \|v\| \le \|A\| \, \|v\| .$$
This shows that $f_v \in H'$. Then, by Riesz representation theorem, there is a unique $A^*v \in H$ such that for all $u \in H$,
$$f_v(u) = \langle Au, v \rangle = \langle u, A^*v \rangle .$$
Claim that $v \mapsto A^*v$ is linear. Let $v_1, v_2 \in H$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, then for all $u \in H$, we have
$$\begin{aligned} \langle u, A^*(\lambda_1 v_2 + \lambda_2 v_2) \rangle &= \langle Au, \lambda_1 v_2 + \lambda_2 v_2 \rangle \\ &= \overline{\lambda_1} \langle Au, v_1 \rangle + \overline{\lambda_2} \langle Au, v_2 \rangle \\ &= \overline{\lambda_1} \langle u, A^*v_1 \rangle + \overline{\lambda_2} \langle u, A^*v_2 \rangle \\ &= \langle u, \lambda_1 A^*v_1 + \lambda_2 A^*v_2 \rangle . \end{aligned}$$
This implies that
$$A^*(\lambda_1 v_2 + \lambda_2 v_2) = \lambda_1 A^*v_1 + \lambda_2 A^*v_2.$$
Therefore, A^* is a linear map. Next we show A^* is bounded and $\|A^*\| = \|A\|$. Suppose $\|v\| = 1$. If $A^*v = 0$, then clearly $\|A^*v\| \le \|A\|$. Suppose now $A^*v \neq 0$, then
$$\|A^*v\|^2 = \langle A^*v, A^*v \rangle = \langle AA^*v, v \rangle \le \|AA^*v\| \, \|v\| \le \|A\| \, \|A^*v\| .$$
Therefore, $\|A^*\| \le \|A\|$ and A^* is a bounded linear operator.

Note that for all $u, v \in H$,
$$\langle A^*u, v \rangle = \overline{\langle v, A^*u \rangle} = \overline{\langle Av, u \rangle} = \langle u, Av \rangle .$$
This implies that $A = (A^*)^*$. Therefore, $\|A\| = \|(A^*)^*\| \le \|A^*\|$ and thus $\|A^*\| = \|A\|$. ☐

Example. Suppose $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$ and $(Au)_i = \sum_j A_{ij} u_j$. Then,
$$\langle Au, v \rangle = \sum_i (Au)_i \overline{v_i} = \sum_{i,j} A_{ij} u_j \overline{v_i} = \sum_j u_j \overline{\sum_i \overline{A_{ij}} v_i} = \sum_j u_j \overline{A^* v_j}.$$
This implies that $(A^*)_{ij} = \overline{A_{ji}}$.

Example. Suppose $\{A_{ij}\}_{i,j=0}^{\infty}$ is a double sequence in $\mathbb{C}^n$ such that
$$\sum_{i,j} |A_{ij}|^2 = \lim_{N \to \infty} \sum_{i=0}^{N} \sum_{j=0}^{N} |A_{ij}|^2 < \infty.$$
Define $A : \ell^2 \to \ell^2$ by
$$Aa = \sum_{j=0}^{\infty} A_{ij} a_j,$$
where $a = \{a_j\}_{j=0}^{\infty} \in \ell^2$. Then, $A \in \mathcal{B}(\ell^2, \ell^2)$ and $(A^*)_{ij} = \overline{A_{ji}}$.

Example. Suppose $K \in C([0,1] \times [0,1])$ and define $A : L^2([0,1]) \to L^2([0,1])$ by

$$Af(x) = \int_0^1 K(x,y)f(y)\, dy.$$

Then we can verify that

$$A^*g(x) = \int_0^1 \overline{K(y,x)}g(y)\, dy.$$

Theorem. Suppose H is a Hilbert space and $A : H \to H$ is a bounded linear operator. Then,

$$(\operatorname{range} A)^\perp = \ker(A^\perp).$$

Proof. $v \in \ker(A^*)$ if and only if $\langle u, A^*v\rangle = 0$ for all $u \in H$. This is equivalent to $\langle Au, v\rangle = 0$ for all $u \in H$, and this is equivalent to $v \in (\operatorname{range} A)^\perp$. □

Corollary. Suppose range A is closed, then A is surjective if and only if A^* is injective.

With these useful tools we developed, we are soon going to discuss the solvability of linear equations, that is, equations in the form of $Au = v$. We take for granted that a bounded linear operator takes bounded sets to bounded sets in finite-dimensional spaces, and so we can find a convergent subsequence using Heine-Borel. So the point is that there is some compactness hidden in here in $\mathbb{R}^n$ and $\mathbb{C}^n$, so we need to study some facts about how compactness and Hilbert spaces before we can talk about solvability of equations.

Definition (Compact)**.** If X is a metric space, we say $K \subset X$ is **compact** if every sequence in K has a subsequence converging to an element in K.

Example. Suppose H is an infinite dimensional separable Hilbert space, then $B[0,1]$ is not compact.

Proof. Let $\{e_n\}_{n=0}^\infty$ be an orthonormal subset of H. Then for all $n \neq k$, we have

$$\|e_n - e_k\|^2 = \|e_n\|^2 + \|e_k\|^2 + 2\operatorname{Re}\langle e_n, e_k\rangle = 2.$$

Therefore, $\{e_n\}_{n=0}^\infty$ cannot have a convergent subsequence. □

Recall from Arzela-Ascoli, the extra condition for a set to be compact, other than closed and bounded, is **equicontinuity**. This motivates the folloiwng definition.

Definition (Equi-small tails)**.** Let H be a Hilbert space. A subset $K \subset H$ has **equi-small tails** with respect to a countable orthonormal subset $\{e_k\}_{k=0}^\infty$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $v \in K$,

$$\sum_{k>N} |\langle v, e_k\rangle|^2 < \varepsilon^2.$$

Example. A finite set has equi-small tails with respect to any countable orthonormal subset $\{e_n\}_{n=0}^\infty$.

Theorem. Let H be a Hilbert space and $\{v_n\}_{n=0}^\infty$ a sequence with $v_n \to v$. Let $\{e_k\}_{k=0}^\infty$ be a countable orthonormal subset. Then,

1. $K = \{v_n : n \in \mathbb{N}\} \cup \{v\}$ is compact.
2. K has equi-small tails with respect to $\{e_k\}_{k=0}^\infty$.

Proof. 1. ***** TO-DO *****

2. Since $v_n \to v$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $\|v_n - v\| < \frac{\varepsilon}{2}$. Also, choose $N \in \mathbb{N}$ so large that
$$\sum_{k>N} |\langle v, e_n\rangle|^2 + \max_{1\leq n\leq M-1} \sum_{k>N} |\langle v_n, e_k\rangle|^2 < \varepsilon^2.$$
Then,
$$\sum_{k>N} |\langle v, e_n\rangle| < \frac{\varepsilon^2}{4} < \varepsilon^2.$$
and for all $1 \leq n \leq M-1$,
$$\sum_{k>N} |\langle v_n, e_k\rangle|^2 < \frac{\varepsilon^2}{4} < \varepsilon^2.$$
If $n \geq M$, by Bessel's inequality we have
$$\begin{aligned}\left(\sum_{k>N} |\langle v_n, e_k\rangle|^2\right)^{\frac{1}{2}} &= \left(\sum_{k>N} |\langle v_n - v, e_k\rangle + \langle v, e_k\rangle|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k>N} |\langle v_n - v, e_k\rangle|^2\right)^{\frac{1}{2}} + \left(\sum_{k>N} |\langle v, e_k\rangle|^2\right)^{\frac{1}{2}} \\ &\leq \|v_n - v\| + \frac{\varepsilon}{2} \\ &< \varepsilon.\end{aligned}$$
This shows that K has equi-small tails with respect to $\{e_k\}_{k=0}^{\infty}$.

□