

Mathematical Studies Analysis

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Spring 2025

Contents

1	Advanced topics in metric space theory	3
1.1	Baire category	3
1.2	Open mapping theorem	4
1.3	Hahn-Banach theorem and duality	8
2	Differential Calculus	12
2.1	Inverse and implicit function theorem	12
3	Measure and integration	14
3.1	Introduction to abstract measure theory	14
3.1.1	Basic definitions	14
3.1.2	Measures	17
3.1.3	Outer measures and Carathéodory construction	17
3.1.4	Constructing outer measures	18
3.2	Lebesgue and Hausdorff measure	21
3.3	Measurable and μ -measurable functions	21
3.4	Lebesgue-Bochner Integral	26
3.4.1	Integration of \mathbb{R} -valued functions	27
3.4.2	Bochner integration	33
3.5	Constructing product measures	36
3.6	Area formula and change of variable formula	39

1 Advanced topics in metric space theory

1.1 Baire category

Definition. Let X be a metric space.

1. We say that $E \subset X$ is nowhere dense if $(\overline{E})^\circ = \emptyset$.
2. We say that $E \subset X$ is meager in X if

$$E = \bigcup_{\alpha \in A} E_\alpha,$$

where A is a countable set and $E_\alpha \subset X$ is nowhere dense for every $\alpha \in A$.

Theorem. Prove that the following are equivalent for $E \subset X$:

1. E is nowhere dense
2. \overline{E} is nowhere dense
3. $(\overline{E})^c$ is open and dense in X .

Proof. (1) \implies (2). Suppose E is nowhere dense, then $(\overline{E})^\circ = \emptyset$. Note that the closure of \overline{E} is just \overline{E} itself. It follows that \overline{E} is also nowhere dense.

(2) \implies (3). Suppose \overline{E} is nowhere dense. Note that \overline{E} is closed, so $(\overline{E})^c$ is open. Let $x \in X$ be arbitrary. Since \overline{E} is nowhere dense, $x \notin (\overline{E})^\circ$. This implies that for arbitrary $\varepsilon > 0$, we have $B(x, \varepsilon) \not\subset \overline{E}$. This is equivalent to $B(x, \varepsilon) \cap (\overline{E})^c \neq \emptyset$. Hence, $(\overline{E})^c$ is dense in X .

(3) \implies (1). Suppose $(\overline{E})^c$ is dense in X . Let $x \in X$ and $\varepsilon > 0$ be arbitrary. It follows that $B(x, \varepsilon) \cap (\overline{E})^c \neq \emptyset$. This is equivalent to $B(x, \varepsilon) \not\subset \overline{E}$. Therefore, $(\overline{E})^\circ = \emptyset$ and E is nowhere dense. □

Theorem (Baire category theorem). Let X be a complete metric space. Suppose that for each $n \in \mathbb{N}$, $U_n \subset X$ is open and dense in X . Prove that $\bigcap_{n=0}^{\infty} U_n$ is dense in X . Hint: use the shrinking closed set property.

Proof. Consider any $x \in X$ and arbitrary $\varepsilon > 0$, it suffices to show that $U_n \cap B(x, \varepsilon) \neq \emptyset$ for each $n \in \mathbb{N}$. Now inductively choosing a sequence $x_i \in X$ and $\varepsilon_i > 0$ such that for each $i \in \mathbb{N}$, $B[x_i, \varepsilon_i] \subset U_i$, $B[x_{i+1}, \varepsilon_{i+1}] \subset B[x_i, \varepsilon_i] \subset B(x, \varepsilon)$, and $\varepsilon_i < 2^{-i}\varepsilon$.

Since U_0 is dense in X , $B(x, \varepsilon) \cap U_0 \neq \emptyset$. Note that both U_0 and $B(x, \varepsilon)$ are open, so we can choose $x_0 \in B(x, \varepsilon) \cap U_0$ and $\varepsilon_0 > 0$ so small that $B[x_0, \varepsilon_0] \subset B(x, \varepsilon) \cap U_0$ and $\varepsilon_0 < \varepsilon$. Now suppose for $0 \leq i \leq n$, we have chosen $x_i \in X$ and $\varepsilon_i > 0$ such that $B[x_i, \varepsilon_i] \subset U_i$ and $\varepsilon_i < 2^{-i}\varepsilon$ for all $0 \leq i \leq n$, $B[x_{i+1}, \varepsilon_{i+1}] \subset B[x_i, \varepsilon_i]$ for all $0 \leq i < n$. Since U_{n+1} is dense in X , $B(x_n, \varepsilon_n) \cap U_{n+1} \neq \emptyset$. Note also both U_{n+1} and $B(x_n, \varepsilon_n)$ are open. Therefore, choose $x_{n+1} \in B(x_n, \varepsilon_n) \cap U_{n+1}$ and $\varepsilon_{n+1} > 0$ so small that $B[x_{n+1}, \varepsilon_{n+1}] \subset B(x_n, \varepsilon_n) \cap U_{n+1}$ and $\varepsilon_{n+1} < \frac{\varepsilon_n}{2}$. It follows that $B[x_{n+1}, \varepsilon_{n+1}] \subset U_{n+1}$ and $B[x_{n+1}, \varepsilon_{n+1}] \subset B[x_n, \varepsilon_n] \subset B(x, \varepsilon)$. Also, $\varepsilon < \frac{\varepsilon_n}{2} < 2^{-n-1}\varepsilon$. Now we have successfully constructing the desired sequence.

Since X is complete, $\bigcap_{n=0}^{\infty} B[x_n, \varepsilon_n] = \{z\}$ for some $z \in X$. Note that for each n , we have $z \in B[x_n, \varepsilon_n] \subset U_n$. Also, $z \in B[x_n, \varepsilon_n] \subset B(x, \varepsilon)$. Therefore, $z \in U_n \cap B(x, \varepsilon)$ for each $n \in \mathbb{N}$ and $\bigcap_{n=0}^{\infty} U_n$ is dense in X . □

Remark. An equivalent statement of the theorem is the following:

Let X be a complete metric space and $\{C_n\}$ a countable collection of closed subsets of X such that $X = \bigcup_{n \in \mathbb{N}} C_n$. Then at least one of the C_n contains an open ball.

1.2 Open mapping theorem

Linear surjections

Theorem (Open mapping theorem). Let X, Y be Banach spaces over a common field and assume that $T \in \mathcal{L}(X; Y)$. Prove that the following are equivalent.

1. T is surjective.
2. There exists $\delta > 0$ such that $B_Y(0, \delta) \subset \overline{T(B_X(0, 1))}$.
3. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $B_Y(0, \delta) \subset T(B_X(0, \varepsilon))$.
4. T is an open map: if $U \subset X$ is open, then $T(U) \subset Y$ is open.
5. There exists $C \geq 0$ such that for each $y \in Y$ there exists $x \in X$ such that $Tx = y$ and

$$\|x\|_X \leq C \|y\|_Y.$$

HINT: Prove that (1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (1), keeping in mind the following suggestions.

1. For (1) \implies (2): Study the sets $C_n = \overline{T(B_X(0, n))} \subset Y$ for $n \geq 1$.
2. For (2) \implies (3): Prove that $\overline{T(B_X(0, 1))} \subset T(B_X(0, 3))$ by considering $y \in \overline{T(B_X(0, 1))}$ and inductively constructing $\{x_j\}_{j=0}^\infty \subset X$ such that $\|x_j\|_X < 2^{-j}$ and $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$ for all $m \in \mathbb{N}$.

Proof. (1) \implies (2). Following the hint, for $n \geq 1$ let $C_n = \overline{T(B_X(0, n))}$. Then each of the C_n are closed. Since T is surjective, $Y = \bigcup_{n=1}^\infty C_n$. Suppose for contradiction that each C_n are nowhere dense. It then follows that C_n^c are dense in Y . By Baire Category Theorem, $\bigcap_{n=1}^\infty C_n^c$ is dense in Y . However, $\bigcap_{n=1}^\infty C_n^c = (\bigcup_{n=1}^\infty C_n)^c = \emptyset$, a contradiction. Therefore, at least one C_n is not nowhere dense. That is, there exists some $n \geq 1$, $\overline{T(B_X(0, n))}$ contains an open ball. However, this is the same set as $n\overline{T(B_X(0, 1))}$. Therefore, $\overline{T(B_X(0, 1))}$ contains an open ball $B_Y(y_0, 4r)$ for some $y_0 \in Y$ and $r > 0$.

Let $y_1 = Tx_1$ for some $x_1 \in B_X(0, 1)$ such that $\|y_0 - y_1\| < 2r$. It follows that $B_Y(y_1, 2r) \subset B_Y(y_0, 4r) \subset \overline{T(B_X(0, 1))}$. For any $y \in Y$ such that $\|y\| < r$, we have

$$y = -\frac{1}{2}y_1 + \frac{1}{2}(2y + y_1) = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1).$$

However, notice that

$$\frac{1}{2}(2y + y_1) \subset \frac{1}{2}B_Y(y_1, 2r) \subset \frac{1}{2}\overline{T(B_X(0, 1))} = \overline{T(B_X(0, \frac{1}{2}))}.$$

It follows that

$$y = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1) \in -T\left(\frac{x_1}{2}\right) + \overline{T(B_X(0, \frac{1}{2}))}.$$

Note that $-T(\frac{x_1}{2}) \in T(B_X(0, \frac{1}{2}))$. Therefore, $y \in \overline{T(B_X(0, 1))}$. Since y is arbitrary with $\|y\| < r$, we have $B_Y(0, r) \subset \overline{T(B_X(0, 1))}$.

(2) \implies (3). Following the hint, we first show $\overline{T(B_X(0, 1))} \subset T(B_X(0, 3))$. By assumption, we have $B_Y(0, R) \subset \overline{T(B_X(0, 1))}$ for some $R > 0$. It follows from homogeneity that for each $m \in \mathbb{N}$, we have

$$2^{-m}B_Y(0, R) = B_Y(0, 2^{-m}R) \subset 2^{-m}\overline{T(B_X(0, 1))} = \overline{T(B_X(0, 2^{-m}))}.$$

Let $y \in \overline{T(B_X(0, 1))}$ and pick $x_0 \in X$ with $\|x_0\| < 1$ such that $\|y - Tx_0\| < 2^{-1}R$. Now suppose we have chosen x_j for $0 \leq j \leq m$ such that $\|x_j\| < 2^{-j}$ and $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$ for all $m \in \mathbb{N}$. By the inclusion above, we can pick $x_{m+1} \in X$ with $\|x_{m+1}\| < 2^{-m-1}$ such that

$$\left\|y - \sum_{j=0}^m Tx_j - Tx_{m+1}\right\| = \left\|y - \sum_{j=0}^{m+1} Tx_j\right\| < 2^{-m-2}R.$$

Therefore, $y - \sum_{j=0}^{m+1} Tx_j \in B_Y(0, 2^{-m-2})R$. This completes the inductive construction, and we have found a sequence $\{x_j\}$ such that $\|x_j\| < 2^{-j}$ and $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$ for each $m \in \mathbb{N}$. Note that

$$\sum_{j=0}^{\infty} \|x_j\| \leq \sum_{j=0}^{\infty} 2^{-j} = 2,$$

so $\sum_{j=0}^{\infty} x_j$ converges absolutely. Since X is Banach, $\sum_{j=0}^{\infty} x_j$ converges to some $x \in X$ with $\|x\| \leq 2$. Also, since $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$, taking the limit where m approaches infinity we obtain

$$y = \sum_{j=0}^{\infty} Tx_j = T \left(\sum_{j=0}^{\infty} x_j \right) = Tx.$$

Therefore, $y \in T(B_X(0, 3))$ and thus $\overline{T(B_X(0, 1))} \subset T(B_X(0, 3))$.

Now for every $\varepsilon > 0$, we have $\frac{\varepsilon}{3} \overline{T(B_X(0, 1))} \subset \frac{\varepsilon}{3} T(B_X(0, 3)) = T(B_X(0, \varepsilon))$. By assumption, there exists $\delta > 0$ such that $B_Y(0, \delta) \subset \overline{T(B_X(0, 1))}$. Therefore,

$$B_Y \left(0, \frac{\delta \varepsilon}{3} \right) = \frac{\varepsilon}{3} B_Y(0, \delta) \subset \frac{\varepsilon}{3} \overline{T(B_X(0, 1))} \subset T(B_X(0, \varepsilon)).$$

(3) \implies (4). Let $U \subset X$ be open and $y \in T(U)$. There exists $x \in U$ such that $Tx = y$. Since U is open, there exists $\varepsilon > 0$ such that $B_X(x, \varepsilon) \subset U$. By assumption, there exists $\delta > 0$ such that $B_Y(0, \delta) \subset T(B_X(0, \varepsilon))$. It follows that

$$B_Y(y, \delta) = y + B_Y(0, \delta) \subset Tx + T(B_X(0, \varepsilon)) = T(x + B_X(0, \varepsilon)) \subset T(U).$$

Therefore, $T(U)$ is open and T is an open map.

(4) \implies (5). Since T is an open map, $T(B_X(0, 1))$ is open. Also, $T(0) = 0$ so there exists $r > 0$ such that $B_Y(0, r) \subset T(B_X(0, 1))$. Now let $y \in Y$. Then, $\frac{r}{2\|y\|}y \in B_Y(0, r)$ and there exists $x \in B_X(0, 1)$ such that $Tx = \frac{r}{2\|y\|}y$. It follows that

$$T \left(\frac{2\|y\|}{r}x \right) = y,$$

and since $x \in B_X(0, 1)$,

$$\left\| \frac{2\|y\|}{r}x \right\| = \frac{2\|y\|\|x\|}{r} < \frac{2}{r}\|y\|.$$

Letting $C = \frac{2}{r}$ completes the proof.

(5) \implies (1). Since for each $y \in Y$ there exists $x \in X$ such that $Tx = y$, T is surjective. □

Linear homeomorphisms, norm equivalence, and closed graphs

Theorem. Let X and Y be Banach spaces and suppose that $T \in \mathcal{L}(X, Y)$ is a bijection. Prove that $T^{-1} \in \mathcal{L}(Y, X)$, and in particular T is a linear (and thus bi-Lipschitz) homeomorphism.

Proof. Since $T \in \mathcal{L}(X, Y)$ is a bijection, T is a surjection. It follows that T is an open map. In particular, for any $U \subset X$ open, $T(U) = (T^{-1})^{-1}(U)$ is open. Therefore, T^{-1} is continuous and thus T is a linear homeomorphism. □

Theorem. Let X be a vector space that is complete when equipped with both of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that if there exists a constant $C_1 > 0$ such that $\|x\|_2 \leq C_1 \|x\|_1$ for all $x \in X$, then there exists a constant $C_0 > 0$ such that $C_0 \|x\|_1 \leq \|x\|_2 \leq C_1 \|x\|_1$ for all $x \in X$.

Proof. Let $T : X_1 \rightarrow X_2$, where X_1 and X_2 are X equipped with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, be the identity map. Then for any $x \in X$ with $\|x\|_1 = 1$, we have

$$\|Tx\|_2 = \|x\|_2 \leq C_1 \|x\|_1 = C_1.$$

Therefore, $T \in \mathcal{L}(X_1, X_2)$. T is also surjective. Therefore, there exists a constant $C \geq 0$ such that each $\|x\|_1 \leq C \|x\|_2$. Hence, for each $x \in X$

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C_1 \|x\|_1.$$

Letting $C_0 = \frac{1}{C}$ completes the proof. □

Theorem. Let X and Y be Banach spaces and let $T : X \rightarrow Y$ be linear (just the algebraic condition). Prove that the following are equivalent

1. T is continuous, i.e. $T \in \mathcal{L}(X; Y)$.
2. The graph of T , $\Gamma(T) = \{(x, Tx) : x \in X\} \subset X \times Y$, is closed in $X \times Y$, where $X \times Y$ is endowed with any of the usual p -norms.

Proof. (a) \implies (b). Let $\{(x_n, Tx_n)\}$ be a convergent sequence in $\Gamma(T)$. Since X is Banach, $x_n \rightarrow x$ for some $x \in X$. Since $T \in \mathcal{L}(X; Y)$, we have

$$\lim_{n \rightarrow \infty} Tx_n = T \left(\lim_{n \rightarrow \infty} x_n \right) = Tx.$$

Therefore, $(x_n, Tx_n) \rightarrow (x, Tx) \in \Gamma(T)$, and thus $\Gamma(T)$ is closed.

(b) \implies (a). Let $\pi_1 : \Gamma(T) \rightarrow X$ and $\pi_2 : \Gamma(T) \rightarrow Y$ by $\pi_1(x, Tx) = x$ and $\pi_2(x, Tx) = Tx$. Since $\Gamma(T)$ is a closed in Banach space Y , $\Gamma(T)$ is Banach space. It is clear that both π_1 and π_2 are bounded linear maps. Moreover, π_1 is a bijection. It follows that $S = \pi_1^{-1}$ is a bounded linear map. Therefore, $T = \pi_2 \circ S$ is a bounded linear map. □

Linear injections with closed range

Theorem. Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Prove the following are equivalent.

1. T is injective and $\text{range}(T)$ is closed.
2. $T : X \rightarrow \text{range}(T)$ is a linear homeomorphism.
3. There exists $C \geq 0$ such that $\|x\|_X \leq C \|Tx\|_Y$ for all $x \in X$.

HINT: Prove that (1) \implies (2) \implies (3) \implies (1).

Proof. (1) \implies (2). If T is injective and $\text{range}(T)$ is closed, then $\Gamma(T) = \{(x, Tx) : x \in X\}$ is closed in $X \times Y$. Therefore, $T : X \rightarrow \text{range}(T)$ is a bounded linear map. Since T is injective, this map is actually bijective from X to $\text{range}(T)$. Therefore, T is a linear homeomorphism.

(2) \implies (3). Since T is a bijective bounded linear map, from X to $\text{range}(T)$. There exists a constant $C \geq 0$ such that for each $y \in \text{range}(T)$ there exists a unique $x \in X$ such that $Tx = y$ and $\|x\| \leq C \|y\| = C \|Tx\|$. Since T is a bijection, $\|x\| \leq C \|Tx\|$ for all $x \in X$.

(3) \implies (1). Let $x \in X$ be such that $Tx = 0$. It follows that $\|x\| \leq C \|Tx\| = 0$. Therefore, $x = 0$ and T is injective. To show that $\text{range}(T)$ is closed, consider a convergent sequence $\{y_n\} \subset \text{range}(T)$ with $y_n = Tx_n$. Since for any $n, m \in \mathbb{N}$ we have

$$\|x_n - x_m\| \leq C \|T(x_n - x_m)\| = C \|y_n - y_m\|,$$

$\{x_n\}$ is Cauchy. Since X is Banach, $x_n \rightarrow x$ for some $x \in X$. Therefore, for all $n \in \mathbb{N}$ we have

$$\|y_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \|x_n - x\|,$$

and $y_n \rightarrow Tx$. Hence, $\text{range}(T)$ is closed and the proof is complete. \square

Theorem. Let X and Y be Banach spaces over a common field. Then, the following subsets of $\mathcal{L}(X; Y)$ are open:

1. $\{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\},$
2. $\{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\},$
3. $\mathcal{H}(X; Y) = \{T \in \mathcal{L}(X; Y) : T \text{ is a homeomorphism}\}.$

Proof. 1. Let $T \in \mathcal{L}(X; Y)$ be surjective. By open mapping theorem, there is $\delta > 0$ such that $B_Y(0, \delta) \subset TB_X(0, 1)$. By homogeneity we have $B_Y(0, r) \subset TB_X(0, \alpha r)$ for all $r > 0$ where $\alpha = \delta^{-1}$. Now let $S \in \mathcal{L}(X; Y)$ be such that $\|T - S\| < \beta < (2\alpha)^{-1}$. Claim S is surjective.

Let $y \in Y$, inductively construct sequences $\{x_n\}$ and $\{y_n\}$. First let $y_0 = y$. Then, $\|y_0\| \in B(0, 2\|y_0\|)$. Select $x_0 \in X$ be such that $Tx_0 = y_0$ and $\|x_0\| \leq 2\alpha\|y_0\|$. Suppose we have selected y_i, x_i for $0 \leq i \leq n$. Set $y_{n+1} = y_n - Sx_n$ and select x_{n+1} be such that $Tx_{n+1} = y_{n+1}$ and $\|x_{n+1}\| \leq 2\alpha\|y_{n+1}\|$. Then, we have

$$\|y_{n+1}\| = \|Tx_n - Sx_n\| \leq \|T - S\| \|x_n\| < 2\alpha\beta\|y_n\|$$

and

$$\|x_{n+1}\| = 2\alpha\|y_{n+1}\| \leq 2\alpha\|T - S\| \|x_n\| < 2\alpha\beta\|x_n\|.$$

Note that $2\alpha\beta < 1$ and X is Banach, define

$$x = \sum_{n=0}^{\infty} x_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N x_n.$$

Also note that $\lim_{n \rightarrow \infty} y_n = 0$. It follows that

$$Sx = \sum_{n=0}^{\infty} Sx_n = \sum_{n=0}^{\infty} (y_n - y_{n+1}) = y_0 - \lim_{n \rightarrow \infty} y_{n+1} = y.$$

Therefore S is surjective and the set of surjective bounded linear maps are open.

2. Suppose $T \in \mathcal{L}(X; Y)$ is injective with closed range. Then, closed range theorem gives $C > 0$ such that $\|x\| \leq C\|Tx\|$ for all $x \in X$. Now suppose $S \in \mathcal{L}(X; Y)$ is such that $\|T - S\| < (2C)^{-1}$. Claim that S is also injective with closed range. Indeed,

$$\begin{aligned} \|x\| &\leq C\|Tx\| \leq C\|Sx\| + C\|(T - S)x\| \\ &\leq C\|Sx\| + \frac{1}{2}\|x\|. \end{aligned}$$

This shows that $\|x\| \leq 2C\|Sx\|$ for all $x \in X$. By closed range theorem, S is injective with closed range. This implies that the set of injective bounded linear operator with closed range is open.

3. This directly follows from

$$\mathcal{H}(X; Y) = \{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\} \cap \{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\}.$$

\square

Theorem. Let X and Y be Banach spaces over a common field. Then the following holds.

1. The sets

$$\mathcal{L}_R(X; Y) = \{T \in \mathcal{L}(X; Y) : \text{there exists } S \in \mathcal{L}(Y; X) \text{ such that } ST = I_X\}$$

and

$$\mathcal{L}_L(X; Y) = \{T \in \mathcal{L}(X; Y) : \text{there exists } S \in \mathcal{L}(Y; X) \text{ such that } TS = I_Y\}$$

are open.

2. The following inclusion holds:

$$\mathcal{L}_L(X; Y) \subset \{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\}$$

and

$$\mathcal{L}_R(X; Y) \subset \{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\}.$$

3. The sets $\mathcal{L}_L(X; Y) \setminus \mathcal{L}_R(X; Y)$ and $\mathcal{L}_R(X; Y) \setminus \mathcal{L}_L(X; Y)$ are open.

Proof. 1. Let $T_0 \in \mathcal{L}_R$ and $S_0 \in \mathcal{L}(Y; X)$ be such that $T_0 S_0 = I_Y$. Note that $I_X \in \mathcal{H}(X)$ and when $\|P\| < 1$ for $P \in \mathcal{L}(X)$, we have $I_X + P \in \mathcal{H}(X)$. Suppose now $T \in \mathcal{L}(X; Y)$ and $\|T\| < \|S_0\|^{-1}$. It follows that $I_X + S_0 T \in \mathcal{H}(X)$. For such T , we then have

$$T_0 + T = T_0(I_X + S_0 T).$$

Also,

$$(T_0 + T)(I_X + S_0 T)^{-1} S_0 = T_0(I_X + S_0 T)(I_X + S_0 T)^{-1} S_0 = T_0 S_0 = I_Y.$$

Therefore, $T_0 + T \in \mathcal{L}_R$ for $T \in B(T_0, \|S_0\|^{-1})$ and \mathcal{L}_R is open.

Now let $T_0 \in \mathcal{L}_L$ and $S_0 \in \mathcal{L}(Y; X)$ be such that $S_0 T_0 = I_X$. Again, for $T \in \mathcal{L}(X; Y)$ with $\|T\| < \|S_0\|^{-1}$, we have

$$T_0 + T = (I_X + T S_0) T_0.$$

and

$$S_0(I_X + T S_0)^{-1}(T_0 + T) = I_X.$$

Therefore, \mathcal{L}_R is also open.

2. Let $T \in \mathcal{L}_R$ and $S \in \mathcal{L}(Y; X)$ be such that $TS = I_Y$. Then for any $y \in Y$ let $x = Sy$. It follows that $Tx = TSy = y$. Also, $\|x\| \leq \|S\| \|y\|$ so the 4th item in open mapping theorem guarantees that T is surjective. Hence, $\mathcal{L}_L \subset \{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\}$.

Now let $T \in \mathcal{L}_L$ and $S \in \mathcal{L}(Y; X)$ such that $ST = I_X$. Now for any $x \in X$, we have $\|x\| = \|STx\| \leq \|S\| \|Tx\|$. Then the closed range theorem guarantees that T is injective with closed range. Hence, $\mathcal{L}_R \subset \{T \in \mathcal{L}_R(X; Y) : T \text{ is injective with closed range}\}$.

3. *** TO-DO ***

□

1.3 Hahn-Banach theorem and duality

Theorem (Hahn-Banach theorem in \mathbb{R}). Let X be a real vector space and suppose $p : X \rightarrow \mathbb{R}$ is such that

$$p(tx + (1 - t)y) \leq tp(x) + (1 - t)p(y)$$

for all $t \in [0, 1]$ and $x, y \in X$.

Suppose Y subspace of X and $l : Y \rightarrow \mathbb{R}$ is a linear map such that $l \leq p$ on Y . Then there exists linear map $L : X \rightarrow \mathbb{R}$ such that $L \leq p$ on X and $L = l$ on Y .

Proof. Let

$$P = \{(Z, \lambda) : Y \subset Z \subset X, \lambda \text{ linear functional on } Z, \lambda \leq p \text{ on } Z \text{ and } l = \lambda \text{ on } Y\}$$

Define partial order $(Z_1, \lambda_1) \preceq (Z_2, \lambda_2)$ if and only if $Z_1 \subset Z_2$ and $\lambda_1 = \lambda_2$ on Z_1 . It is easy to verify that this is a partial order. Towards using Zorn's Lemma, let $C \subset P$ be a chain and define

$$U = \bigcup_{(Z, \lambda) \in C} Z, \quad \Lambda = \bigcup_{(Z, \lambda) \in C} \lambda.$$

It is easy to verify that (U, Λ) is an upper bound for the chain. By Zorn's Lemma, P has a maximal element (M, L) . It remains to show that $M = X$.

Suppose for contradiction that $M \neq X$. Pick $x_0 \in X \setminus M$. For any $x, y \in M$, we have

$$\begin{aligned} \beta L(x) + \alpha L(y) &= L(\beta x + \alpha y) \\ &= \frac{1}{\alpha + \beta} L\left(\frac{\beta}{\alpha + \beta}x + \frac{\alpha}{\alpha + \beta}y\right) \\ &\leq (\alpha + \beta)p\left(\frac{\beta}{\alpha + \beta}x + \frac{\alpha}{\alpha + \beta}y\right) \\ &= (\alpha + \beta)p\left(\frac{\beta}{\alpha + \beta}(x - \alpha x_0) + \frac{\alpha}{\alpha + \beta}(y + \beta x_0)\right) \\ &\leq \beta p(x - \alpha x_0) + \alpha p(y + \beta x_0). \end{aligned}$$

This implies that

$$\sup_{\substack{x \in M \\ \alpha > 0}} \frac{1}{\alpha} [L(x) - p(x - \alpha x_0)] \leq \inf_{\substack{y \in M \\ \beta > 0}} \frac{1}{\beta} [p(y + \beta x_0) - L(y)].$$

Note that $-p(-x_0) \leq \text{LHS}$ and $\text{RHS} \leq p(x_0)$, so $\text{LHS}, \text{RHS} < \infty$. Now pick $v \in \mathbb{R}$ such that $\text{LHS} \leq v \leq \text{RHS}$. For $x \in M$ and $0 < t \in \mathbb{R}$ we have

$$L(x) - tv \leq p(x - tv_0), \quad L(x) + tv \leq p(x + tv_0).$$

Now define $\widehat{L} : M \oplus \mathbb{R}x_0 \rightarrow \mathbb{R}$ by $\widehat{L}(x + \alpha x_0) = L(x) + \alpha v$. It follows that $(M \oplus \mathbb{R}x_0, \widehat{L}) \in P$. However, $(M, L) \prec (M \oplus \mathbb{R}, \widehat{L})$, a contradiction. Therefore, $M = X$ and the proof is complete. \square

Theorem (Hahn-Banach theorem in \mathbb{C}). Let X be complex vector space and suppose $p : X \rightarrow \mathbb{R}$ is such that

$$p(\alpha x + \beta y) \leq |\alpha|p(x) + |\beta|p(y)$$

for all $\alpha, \beta \in \mathbb{C}$ such that $|\alpha| + |\beta| = 1$ and $x, y \in X$.

Suppose Y subspace of X and $l : Y \rightarrow \mathbb{C}$ is a linear map such that $|l| \leq p$ on Y . Then there exists linear map $L : X \rightarrow \mathbb{C}$ such that $|L| \leq p$ on X and $L = l$ on Y .

Proof. Define $\lambda : Y \rightarrow \mathbb{R}$ by $\lambda(x) = \text{Re}(l(x))$. Note that

$$\lambda(ix) = \text{Re}(il(x)) = -\text{Im}(l(x)).$$

This implies that $l(x) = \lambda(x) - i\lambda(ix)$. Now treat X and Y as vector space over \mathbb{R} and apply Hahn-Banach theorem in \mathbb{R} to extend λ to $\Lambda : X \rightarrow \mathbb{R}$ that agrees with λ on Y .

Define $L : X \rightarrow \mathbb{C}$ by $L(x) = \Lambda(x) - i\Lambda(ix)$. It remains to show that $|L| \leq p$. For $x \in X$, write $L(x) = |L(x)| e^{i\theta}$ for some $\theta \in \mathbb{R}$. It follows that

$$\begin{aligned} |L(x)| &= L(x)e^{-i\theta} \\ &= \Lambda(e^{-i\theta}x) - i\Lambda(ie^{-i\theta}x) \\ &= \Lambda(e^{-i\theta}x) \\ &\leq p(e^{-i\theta}x) \\ &\leq |e^{-i\theta}| p(x) \\ &= p(x), \end{aligned}$$

as desired. □

Theorem (Hahn-Banach theorem for bounded linear functionals). Let X be a normed vector space over \mathbb{F} and Y a subspace of X . If $\lambda \in Y^*$ then there exists $\Lambda \in X^*$ such that $\Lambda = \lambda$ on Y and the operator norm $\|\lambda\|_{Y^*} = \|\Lambda\|_{X^*}$.

Proof. Consider $p : X \rightarrow \mathbb{R}$ where $p(x) = \|\lambda\|_{Y^*} \|x\|$. Apply Hahn-Banach theorem. □

Next we show some useful implications of Hahn-Banach theorem.

Theorem. Let X be a normed vector space and fix $x \in X$. Then the following holds:

1. There exists $\lambda \in X^*$ such that $\|\lambda\| = \|x\|$ and

$$\lambda(x) = \|\lambda\| \|x\| = \|x\|^2.$$

2. We have

$$\|x\| = \max_{\substack{w \in X^* \\ \|w\|=1}} |w(x)|.$$

3. $x = 0$ if and only if $w(x) = 0$ for all $w \in X^*$.

Proof. 1. Let $Y = \mathbb{F}x$ and define $\lambda \in Y^*$ by $\lambda(ax) = a\|x\|^2$. Apply Hahn-Banach theorem.

2. Suppose $x \neq 0$. Define $w = \frac{\lambda}{\|x\|}$ then it follows that $|w(x)| = \|x\|$.

3. Follows directly from (2). □

Proposition. Let X be normed vector space. Then the mapping $\langle \cdot, \cdot \rangle : X^* \times X \rightarrow \mathbb{F}$ by $(w, x) \mapsto w(x)$ is a bilinear map. That is, $\langle \cdot, \cdot \rangle \in \mathcal{L}(X^*, X; \mathbb{F})$. Moreover, if $X \neq \{0\}$, then $\|\langle \cdot, \cdot \rangle\| = 1$.

Proof. It is easy to see that $\langle \cdot, \cdot \rangle$ is bilinear. For boundedness,

$$|\langle w, x \rangle| = |w(x)| \leq \|w\| \|x\|.$$

Hence, $\|\langle \cdot, \cdot \rangle\| \leq 1$. Meanwhile, pick some $x \in X$ with $\|x\| = 1$. It follows that

$$1 = \|x\| = \max_{\substack{w \in X^* \\ \|w\|=1}} |w(x)| \leq \|\langle \cdot, \cdot \rangle\|.$$

Therefore, $\|\langle \cdot, \cdot \rangle\| = 1$. □

Definition (Norming set). Let X be normed vector space and $E \subset X$, $W \subset X^*$. Say W is a **norming set** for E if

$$\|x\| = \sup_{\substack{w \in W \\ \|w\|=1}} |\langle w, x \rangle|$$

for all $x \in E$.

Proposition. Let X be normed vector space and $S \subset X$ be a separable set. Let W be a norming set for S . Then, there exists $\{w_n\}_{n=0}^\infty \subset W$ such that $\|w_n\| = 1$, and the sequence is norming for S . That is,

$$\|x\| = \sup_{n \in \mathbb{N}} |\langle w_n, x \rangle|.$$

Proof. Let $\{v_n\}_{n=0}^\infty \subset S$ be dense. For any $n, k \in \mathbb{N}$, choose $w_{n,k} \in W$ with $\|w_{n,k}\| = 1$ such that

$$(1 - 2^{-k}) \|v_n\| \leq |w_{n,k}, v_n|.$$

Let $x \in S$ and $0 < \varepsilon < 1$ be arbitrary. Pick $v_n \in S$ such that $\|v_n - x\| < \varepsilon$ and pick $j \in \mathbb{N}$ such that $2^{-j} < \varepsilon$. Then,

$$\begin{aligned} (1 - \varepsilon) \|x\| &\leq (1 - 2^{-j}) \|x\| \\ &\leq (1 - 2^{-j}) \|v_n\| + (1 - 2^{-j}) \|v_n - x\| \\ &\leq |\langle w_{n,j}, v_j \rangle| + \varepsilon \\ &\leq |\langle w_{n,j}, x \rangle| + |\langle w_{n,j}, x - v_n \rangle| + \varepsilon \\ &\leq |\langle w_{n,j}, x \rangle| + 2\varepsilon. \end{aligned}$$

This shows that $\{w_{n,k}\}_{n,k=0}^\infty$ is a norming sequence. □

Theorem. Let X be normed vector space and define $J : X \rightarrow X^{**}$ by $\langle Jx, w \rangle = \langle w, x \rangle = w(x)$. Then the following holds:

1. $J \in \mathcal{L}(X, X^{**})$.
2. J is an isometric embedding. In particular, it is injective.
3. $\text{range}(J) \subset X^{**}$ is a norming set for X^* .
4. X is Banach if and only if $\text{range}(J)$ is closed.

Proof. Note that we have

$$\begin{aligned} \|Jx\|_{X^{**}} &= \sup \{ |\langle Jx, w \rangle| : w \in X^* \text{ and } \|w\| \leq 1 \} \\ &= \sup \{ |\langle w, x \rangle| : w \in X^* \text{ and } \|w\| \leq 1 \} \\ &= \|x\|, \end{aligned}$$

where the last step is by a previous theorem that shows the existence of $w \in X^*$ such that $\|w\| = 1$ and $|w(x)| = \|x\|$. This implies (1) and (2). Now we know X is isometrically isomorphic to $\text{range}(J) \subset X^{**}$. Therefore, X is Banach if and only if $\text{range}(J)$ is Banach. However, $X^{**} = \mathcal{L}(X^*, \mathbb{F})$ is Banach, so $\text{range}(J)$ is Banach if and only if $\text{range}(J)$ is closed. This implies (4).

To show (3), note that we have

$$\begin{aligned} \|w\|_{X^*} &= \sup \{ |\langle w, x \rangle| : x \in X \text{ and } \|x\| \leq 1 \} \\ &= \sup \{ |\langle Jx, w \rangle| : x \in X \text{ and } \|x\| \leq 1 \} \\ &= \sup \{ |\langle v, w \rangle| : v \in \text{range}(J) \text{ and } \|v\|_{X^{**}} \leq 1 \}. \end{aligned}$$

This shows (3), completing the proof. □

2 Differential Calculus

2.1 Inverse and implicit function theorem

Theorem (Local injectivity theorem). Let X and Y be Banach spaces, $z \in U \subset X$ with U open. Let $f : U \rightarrow Y$ differentiable with Df continuous at z . Suppose $Df(z) \in \mathcal{L}(X; Y)$ injective with closed range. Then for any $0 < \varepsilon < 1$, there exists $r > 0$ such that

1. $B[z, r] \subset U$.
2. $Df(x)$ injective with closed range for all $x \in B[z, r]$.
3. If $x, y \in B(z, r)$, then

$$(1 - \varepsilon) \|Df(z)(x - y)\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|Df(z)(x - y)\|.$$

4. The restriction $f : B(z, r) \rightarrow f(B(z, r))$ is bi-Lipschitz homeomorphism.

Proof. Since $Df(z)$ injective with closed range, there exists $\theta > 0$ such that

$$\theta \|h\| \leq \|Df(z)h\|$$

for all $h \in X$. Since the set of bounded linear operator that is injective with closed range is open, there exists $\delta > 0$ such that $\|Df(z) - T\| < \delta$ implies T is injective with closed range.

Now let $0 < \varepsilon < 1$. Note that Df is continuous at z , so we can select $r > 0$ so small that $B[z, r] \subset U$, and $x \in B[z, r]$ implies

$$\|Df(x) - Df(z)\| < \min \{\delta, \theta\varepsilon\}.$$

In particular, $Df(x)$ is injective with closed range for all $x \in B[z, r]$. By the mean value theorem, for any $x, y \in B(x, r)$

$$\begin{aligned} \|f(x) - f(y) - Df(z)(x - y)\| &\leq \sup_{w \in B(z, r)} \|Df(w) - Df(z)\| \|x - y\| \\ &\leq \theta\varepsilon \|x - y\| \\ &\leq \varepsilon \|Df(z)(x - y)\|. \end{aligned}$$

It follows that

$$(1 - \varepsilon) \|Df(z)(x - y)\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|Df(z)(x - y)\|,$$

as desired.

This also implies that

$$(1 - \varepsilon)\theta \|x - y\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|Df(z)\| \|x - y\|,$$

so the restriction of f on $B(z, r)$ is a bi-Lipschitz homeomorphism. □

Theorem (Local surjectivity theorem). Let X and Y be Banach spaces, $z \in U \subset X$ with U open. Let $f : U \rightarrow Y$ differentiable with Df continuous at z . Suppose $Df(z) \in \mathcal{L}(X; Y)$ surjective. Then there exists $r_0, \gamma > 0$ such that

1. $B_X[z, r_0] \subset U$.
2. $Df(x)$ surjective for all $x \in B_X[z, r_0]$.
3. $B_Y[f(z), \gamma r] \subset f(B_X[z, r])$ for all $0 \leq r \leq r_0$.

Proof. *** TO-DO *** □

Definition (diffeomorphism). Let X and Y be normed vector spaces and suppose that $\emptyset \neq U \subset X$ is open. Let $f : U \rightarrow Y$. For $k \geq 1$, say f is a C^k diffeomorphism if

1. $f : U \rightarrow f(U)$ homeomorphism with $f(U) \subset Y$ open.
2. $f \in C^k(U; Y)$.
3. $f^{-1} \in C^k(f(U); X)$.

If f is a C^k diffeomorphism for all $k \geq 1$, say f is a smooth diffeomorphism.

Theorem (Inverse function theorem). Let X and Y be Banach spaces, $U \subset X$ open and $x_0 \in U$. Suppose $f : U \rightarrow Y$ differentiable, Df continuous at x_0 , $Df(x_0)$ linear homeomorphism. Then there exists bounded and open $V \subset U$ with $x_0 \in V$ such that

1. $f : V \rightarrow f(V)$ is bi-Lipschitz homeomorphism, $Df(x)$ linear homeomorphism for all $x \in V$, $f(V) \subset Y$ bounded and open, $f^{-1} : f(V) \rightarrow V$ differentiable with

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$$

for all $y \in f(V)$ and Df^{-1} is continuous at $f(x_0)$. Also, there exists $C_0, C_1 > 0$ such that

$$C_0 \leq \|Df(x)\| \leq C_1$$

for all $x \in V$, and

$$\frac{1}{C_1} \leq \|Df^{-1}(y)\| \leq \frac{1}{C_0}$$

for all $y \in f(V)$.

2. If $f \in C^k(U; Y)$ for some $1 \leq k \leq \infty$, then $f^{-1} \in C^k(f(V); X)$. In particular, f is a local C^k diffeomorphism at x_0 .
3. If $f \in C^k(U; Y)$ for $1 \leq k \in \mathbb{N}$, then there exists open $V_k \subset V$ such that $x_0 \in V_k$, $f \in C_b^k(V_k; Y)$ and $f^{-1} \in C_b^k(f(V_k); X)$.

Proof. *** TO-DO *** □

Theorem (Implicit function theorem). Let X and Y be Banach spaces, $U \subset X \times Y$ be open with $(x_0, y_0) \in U$, and suppose $f : U \rightarrow Z$ is differentiable in U with Df continuous at (x_0, y_0) . Further suppose $z_0 = f(x_0, y_0)$ and $D_2f(x_0, y_0) \in \mathcal{L}(Y; Z)$ is an isomorphism. Then there exists open sets $x_0 \in V \subset X$, $z_0 \in W \subset Z$, $y_0 \in S \subset Y$, and $g \in C_b^{0,1}(V \times W; Y)$ such that the following holds:

1. $g(x_0, z_0) = y_0$ and $(x, g(x, z)) \in V \times S \subset U$ for all $(x, z) \in V \times W$. Also, g is differentiable on $V \times W$ and Dg continuous at (x_0, z_0) .
2. $f(x, g(x, z)) = z$ for all $(x, z) \in V \times W$. Moreover, if $(x, y) \in V \times S$ such that $f(x, y) = z$ for some $z \in W$, then $y = g(x, z)$.
3. $D_2f(x, g(x, z))$ is an isomorphism for all $(x, z) \in V \times W$, and

$$\begin{aligned} D_1g(x, z) &= -[D_2f(x, g(x, z))]^{-1} D_1f(x, g(x, z)), \\ D_2g(x, z) &= [D_2f(x, g(x, z))]^{-1}. \end{aligned}$$

4. If $f \in C^k$ then $g \in C^k$ too for $1 \leq k \leq \infty$. If k finite and $f \in C_b^k$ then the sets can be picked such that $g \in C_b^k$.

Proof. *** TO-DO *** □

3 Measure and integration

3.1 Introduction to abstract measure theory

3.1.1 Basic definitions

Definition. Let X be a set.

1. An **algebra** on X is $\mathfrak{A} \subset \mathcal{P}(X)$ such that
 - (a) $\emptyset \in \mathfrak{A}$.
 - (b) $E \in \mathfrak{A}$ implies $E^c \in \mathfrak{A}$.
 - (c) $E, F \in \mathfrak{A}$ implies $E \cup F \in \mathfrak{A}$.
2. A **σ -algebra** is an algebra $\mathfrak{M} \subset \mathcal{P}(X)$ such that if $E_k \in \mathfrak{M}$ for all $k \in \mathbb{N}$, then $\bigcup_{k=0}^{\infty} E_k \in \mathfrak{M}$.
3. A pair (X, \mathfrak{M}) with \mathfrak{M} a σ -algebra on X is called a **measurable space**.

Theorem. Let X be a set.

1. Suppose $A \neq \emptyset$ is a set and \mathfrak{M}_α is σ -algebra for each $\alpha \in A$, then $\mathfrak{M} = \bigcap_{\alpha \in A} \mathfrak{M}_\alpha$ is a σ -algebra on X .
2. Suppose $F \subset \mathcal{P}(X)$, there is unique smallest σ -algebra \mathfrak{M} on X such that $F \subset \mathfrak{M}$. Write $\mathfrak{M} = \sigma(F)$ and call this the σ -algebra generated by F .

Theorem. Let X and Y be sets and $f : X \rightarrow Y$.

1. Suppose \mathfrak{M} is a σ -algebra on X and set

$$\mathfrak{N} = \{E \subset Y : f^{-1}(E) \in \mathfrak{M}\}.$$

Then, \mathfrak{N} is a σ -algebra on Y . Call this the **push-forward** of \mathfrak{M} by f .

2. Suppose \mathfrak{N} is a σ -algebra on Y and set

$$\mathfrak{M} = \{f^{-1}(E) : E \in \mathfrak{N}\}.$$

Then, \mathfrak{M} is a σ -algebra on X . Call this the **pull-back** of \mathfrak{N} by f .

Definition. Let $A \neq \emptyset$ be a set.

1. Let Y be a set and X_α be sets with σ -algebra \mathfrak{M}_α for all $\alpha \in A$. Suppose $g_\alpha : X_\alpha \rightarrow Y$ for all $\alpha \in A$. Define

$$\sigma(\{E \subset Y : g_\alpha^{-1}(E) \in \mathfrak{M}_\alpha \text{ for all } \alpha \in A\})$$

to be the **push-forward** of $\{g_\alpha\}_{\alpha \in A}$.

2. Let X be a set and Y_α be sets with σ -algebra \mathfrak{N}_α for all $\alpha \in A$. Suppose $f_\alpha : X \rightarrow Y_\alpha$ for all $\alpha \in A$. Define

$$\sigma(\{f_\alpha^{-1}(E) : E \in \mathfrak{N}_\alpha \text{ for some } \alpha \in A\})$$

to be the **pull-back** of $\{f_\alpha\}_{\alpha \in A}$.

Definition. Let $A \neq \emptyset$ be a set and X_α be sets with σ -algebra \mathfrak{M}_α for all $\alpha \in A$. Then on the set $X = \prod_{\alpha} X_\alpha$ we define the **product σ -algebra** $\bigoplus_{\alpha} \mathfrak{M}_\alpha$ to be the pull-back of projection maps $\pi_\alpha : X \rightarrow X_\alpha$.

Theorem. Let $A \neq \emptyset$ be a set and X_α with σ -algebra \mathfrak{M}_α for all $\alpha \in A$. Let $X = \prod_{\alpha} X_\alpha$ and define

$$\mathcal{R} = \left\{ \prod_{\alpha} M_{\alpha} : M_{\alpha} \in \mathfrak{M}_{\alpha} \right\}.$$

Then,

1. $\bigoplus_{\alpha} \mathfrak{M}_{\alpha} \subset \sigma(\mathcal{R})$. If A countable then $\sigma(\mathcal{R}) = \bigoplus_{\alpha} \mathfrak{M}_{\alpha}$.

2. Suppose $\mathfrak{M}_\alpha = \sigma(\mathcal{E}_\alpha)$ for all $\alpha \in A$ and let

$$\mathcal{E} = \{\pi_\alpha^{-1}(E) : E \in \mathcal{E}_\alpha \text{ for some } \mathcal{E}_\alpha\}.$$

Then $\bigoplus_\alpha \mathfrak{M}_\alpha = \sigma(\mathcal{E})$. Moreover, if A is countable and $X_\alpha \in \mathcal{E}_\alpha$ for all $\alpha \in A$, then $\bigoplus_\alpha \mathfrak{M}_\alpha$ is generated by $\mathcal{F} = \{\prod_\alpha E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}$

Proof. 1. For $E \in \mathfrak{M}_\alpha$, we have $\pi_\alpha^{-1}(E) = \prod_\beta S_\beta$, where

$$S_\beta = \begin{cases} E & (\beta = \alpha), \\ X_\beta & (\beta \neq \alpha). \end{cases}$$

Then,

$$\{\pi_\alpha^{-1}(M_\alpha) : M_\alpha \in \mathfrak{M}_\alpha\} \subset \left\{ \prod_\beta M_\beta : M_\beta \in \mathfrak{M}_\beta \right\} = \mathcal{R}.$$

This implies that $\bigoplus_\alpha \mathfrak{M}_\alpha \subset \sigma(\mathcal{R})$.

On the other hand, if A is countable, then

$$\prod_\alpha M_\alpha = \bigcap_\alpha \pi_\alpha^{-1}(M_\alpha) \in \bigoplus_\alpha \mathfrak{M}_\alpha$$

whenever $M_\alpha \in \mathfrak{M}_\alpha$ for all $\alpha \in A$. This implies that $\sigma(\mathcal{R}) \subset \bigoplus_\alpha \mathfrak{M}_\alpha$.

2. It is clear that $\sigma(\mathcal{E}) \subset \bigoplus_\alpha \mathfrak{M}_\alpha$. On the other hand, for each $\alpha \in A$, let

$$\mathfrak{N}_\alpha = \{E \subset X_\alpha : \pi_\alpha^{-1}(E) \in \sigma(\mathcal{E})\}$$

be the push-forward of $\sigma(\mathcal{E})$ to X_α by π_α . It is clear that $\mathcal{E}_\alpha \subset \mathfrak{N}_\alpha$. This implies $\mathfrak{M}_\alpha = \sigma(\mathcal{E}) \subset \mathfrak{N}_\alpha$. In particular, $\pi_\alpha^{-1}(E) \in \sigma(\mathcal{E})$ for all $E \in \mathfrak{M}_\alpha$. This implies that $\bigoplus_\alpha \mathfrak{M}_\alpha \subset \sigma(\mathcal{E})$.

Now, assume A countable and $X_\alpha \in \mathcal{E}_\alpha$ for all $\alpha \in A$. Then let $E \in \mathfrak{M}_\alpha$ for some $\alpha \in A$. We have $\pi_\alpha^{-1}(E) = \prod_\beta S_\beta$, where

$$S_\beta = \begin{cases} E & (\beta = \alpha), \\ X_\beta & (\beta \neq \alpha). \end{cases}$$

Therefore, $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F})$.

On the other hand, since A is countable, we have

$$\prod_\alpha E_\alpha = \bigcap_\alpha \pi_\alpha^{-1}(E_\alpha) \in \sigma(\mathcal{E}).$$

This implies that $\sigma(\mathcal{F}) \subset \sigma(\mathcal{E})$ and the proof is complete. □

Corollary. If \mathfrak{M}_i is σ -algebra for $i = 1, 2, 3$, then

$$\mathfrak{M}_1 \oplus (\mathfrak{M}_2 \oplus \mathfrak{M}_3) = (\mathfrak{M}_1 \oplus \mathfrak{M}_2) \oplus \mathfrak{M}_3 = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3,$$

since they are all generated by

$$\{M_1 \times (M_2 \times M_3)\} = \{(M_1 \times M_2) \times M_3\} = \{M_1 \times M_2 \times M_3\}.$$

Theorem. Let X_1, \dots, X_n be metric spaces and $X = \prod_{i=1}^n X_i$ be equipped with the usual metric. Then, $\bigoplus_{i=1}^n \mathfrak{B}_{X_i} \subset \mathfrak{B}_X$. However, if each X_i is separable, then $\mathfrak{B}_X = \bigoplus_{i=1}^n \mathfrak{B}_{X_i}$.

Proof. We know by the previous theorem that $\bigoplus_{i=1}^n \mathfrak{B}_{X_i}$ is generated by $\{\prod_i U_i : U_i \subset X_i \text{ open}\}$. However, $\prod_i U_i$ is open in X . Therefore, $\bigoplus_{i=1}^n \mathfrak{B}_{X_i} \subset \mathfrak{B}_X$.

Suppose now each X_i is separable and let $D_i \subset X_i$ be countable and dense. Consider

$$\mathcal{E}_i = \{B(x_i, r) : X_i \in D_i, r = \infty \text{ or } r \in \mathbb{Q}^+\},$$

which is countable and $\sigma(\mathcal{E}_i) = \mathfrak{B}_{X_i}$ since every open set in X_i is countable union of elements in \mathcal{E}_i . Similarly, \mathfrak{B}_X is generated by $\{\prod_i E_i : E_i \in \mathcal{E}_i\}$. But item 2 from the previous theorem implies that $\bigoplus_{i=1}^n \mathfrak{B}_{X_i}$ is generated by the same set. Therefore, $\bigoplus_{i=1}^n \mathfrak{B}_{X_i} = \mathfrak{B}_X$. □

Remark. The above theorem is not true in general if X_i is not separable for some i .

Definition. Let X be a metric space. Define

$$F_\sigma(X) = \left\{ \bigcup_{k=0}^{\infty} C_k : C_k \subset X \text{ closed} \right\},$$

$$G_\delta(X) = \left\{ \bigcap_{k=0}^{\infty} U_k : U_k \subset X \text{ open} \right\}.$$

Note that $F_\sigma(X) \subset \mathfrak{B}_X$ and $G_\delta(X) \subset \mathfrak{B}_X$.

Theorem. Let X be a metric space. Then the following holds:

1. F_σ and G_δ are both closed under finite union and intersection.
2. If $C \subset X$ is closed, then $C \in G_\delta$. If $U \subset X$ is open, then $U \in F_\sigma$.
3. Suppose X is σ -compact, that is, $X = \bigcup_{n=0}^{\infty} K_n$ for $K_n \subset X$ compact, then each $F \in F_\sigma$ is also σ -compact. In particular, all open sets are σ -compact.

Theorem. Let X and Y be metric spaces and $f : X \rightarrow Y$ be continuous. Then the following holds:

1. $E \in F_\sigma(Y)$ implies that $f^{-1}(E) \in F_\sigma(X)$, and $E \in G_\delta(Y)$ implies that $f^{-1}(E) \in G_\delta(X)$.
2. If $E \in \mathfrak{B}(Y)$, then $f^{-1}(E) \in \mathfrak{B}(X)$.

Theorem. Let X and Y be metric spaces with X σ -compact. Then,

1. If $E \in F_\sigma(X)$ and $f : E \rightarrow Y$ is continuous, then $f(E) \in F_\sigma(Y)$ and σ -compact.
2. If $f : X \rightarrow Y$ is a continuous injection, then $E \in \mathfrak{B}(X)$ implies $f(E) \in \mathfrak{B}(Y)$.

Corollary. Let $\emptyset \neq X \subset Y$ for Y a metric space. Then $\mathfrak{B}(X) = \mathfrak{B}(Y)_X := \{X \cap E : E \in \mathfrak{B}(Y)\}$.

Proof. We know $V \subset X$ open if and only if $V = X \cap U$ for some U open in Y . Therefore,

$$\{V \subset X : V \text{ open in } X\} \subset \mathfrak{B}(Y)_X.$$

This implies that $\mathfrak{B}(X) \subset \mathfrak{B}(Y)_X$.

On the other hand, the inclusion map $I : X \rightarrow Y$ is a continuous injection, so if $E \in \mathfrak{B}(Y)$, then $I^{-1}(E) \in \mathfrak{B}(X)$. However, $I^{-1}(E) = E \cap X$. Therefore, $\mathfrak{B}(Y)_X \subset \mathfrak{B}(X)$. □

3.1.2 Measures

Definition (Measure). Let X be a set with \mathfrak{M} a σ -algebra on X . A **measure** is a map $\mu : \mathfrak{M} \rightarrow [0, \infty]$ such that

1. $\mu(\emptyset) = 0$.
2. If $\{E_k\}_{k=0}^{\infty} \subset \mathfrak{M}$ pairwise disjoint, then $\mu(\bigcup_{k=0}^{\infty} E_k) = \sum_{k=0}^{\infty} \mu(E_k)$.

Such a triple (X, \mathfrak{M}, μ) is a **measure space**.

Definition. We say (X, \mathfrak{M}, μ) is **finite** if $\mu(X) < \infty$. We say (X, \mathfrak{M}, μ) is **σ -finite** if $X = \bigcup_{n=0}^{\infty} X_n$ for $X_n \in \mathfrak{M}$ and $\mu(X_n) < \infty$.

Theorem. Let (X, \mathfrak{M}, μ) be a measure space. Then the following holds:

1. If E and F is measurable and $E \subset F$, then $\mu(E) \leq \mu(F)$.
2. If $E_k \in \mathfrak{M}$ for all $k \in \mathbb{N}$, then $\mu(\bigcup_{k=0}^{\infty} E_k) \leq \sum_{k=0}^{\infty} \mu(E_k)$.

3.1.3 Outer measures and Carathéodory construction

Definition (Outer measure). Let X be a set. An **outer measure** is a map $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ such that

1. $\mu^*(\emptyset) = 0$.
2. $E \subset F$ implies $\mu^*(E) \leq \mu^*(F)$.
3. If $E_k \subset X$ for all $k \in \mathbb{N}$, then $\mu^*(\bigcup_{k=0}^{\infty} E_k) \leq \sum_{k=0}^{\infty} \mu^*(E_k)$.

Proposition. Let $\mu_{\alpha}^* : \mathcal{P}(X) \rightarrow [0, \infty]$ be an outer measure for all $\alpha \in A \neq \emptyset$. Then $\lambda : \mathcal{P}(X) \rightarrow [0, \infty]$ defined by $\lambda(E) = \sup_{\alpha \in A} \mu_{\alpha}^*(E)$ is an outer measure.

Proof. 1. $\mu_{\alpha}^*(\emptyset) = 0$ for all $\alpha \in A$ implies that $\lambda(\emptyset) = 0$.

2. Suppose $E \subset F$, then $\mu_{\alpha}^*(E) \leq \mu_{\alpha}^*(F) \leq \lambda(F)$ for all $\alpha \in A$. Take the sup and we obtain $\lambda(E) \leq \lambda(F)$.

3. Let $E_k \subset X$ for each $k \in \mathbb{N}$. Then,

$$\mu_{\alpha}^*\left(\bigcup_{k=0}^{\infty} E_k\right) \leq \sum_{k=0}^{\infty} \mu_{\alpha}^*(E_k) \leq \sum_{k=0}^{\infty} \lambda(E_k)$$

This implies that $\lambda(\bigcup_{k=0}^{\infty} E_k) \leq \sum_{k=0}^{\infty} \lambda(E_k)$.

□

Definition. Let X be a set with outer measure μ^* . Say a set $E \subset X$ is measurable with respect to μ^* if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for all $A \subset X$.

Theorem (Carathéodory construction). Let X be a set with outer measure μ^* , the following holds.

1. The collection $\mathfrak{M} = \{E \subset X : E \text{ measurable}\}$ is a σ -algebra.
2. If $E \subset X$ is such that $\mu^*(E) = 0$, then $E \in \mathfrak{M}$.
3. The restriction $\mu = \mu^*|_{\mathfrak{M}}$ is a measure, and (X, \mathfrak{M}, μ) is a complete measure space.

Definition (Cover regular). Let μ^* be an outer measure on X . Say μ^* is cover-regular if for any $A \subset X$, there exists $E \in \mathfrak{M}$ such that $A \subset E$ and $\mu^*(A) = \mu(E)$.

Proposition. Let μ^* be an outer measure on X . Then μ^* is outer-regular if and only if for any $A \subset X$, $\mu^*(A) = \inf \{\mu(E) : A \subset E \in \mathfrak{M}\}$. In either case, the inf is a min.

Proposition. Let X be a set with cover-regular outer measure μ^* . Suppose for $n \in \mathbb{N}$, we have $A_n \subset A_{n+1}$. Then,

$$\mu^* \left(\bigcup_{n=0}^{\infty} A_n \right) = \lim_{n \rightarrow \infty} \mu^*(A_n).$$

Proof. First note that $\mu^*(A_n) \leq \mu^*(A_{n+1}) \leq \mu^*(A)$, where $A = \bigcup_{n=0}^{\infty} A_n$. Therefore,

$$\lim_{n \rightarrow \infty} \mu^*(A_n) \leq \mu^*(A).$$

On the other hand, by cover regularity, there exists $A_n \subset E_n \in \mathfrak{M}$ such that $\mu^*(A_n) = \mu(E_n)$. In particular, $\lim_{n \rightarrow \infty} \mu^*(A_n) = \lim_{n \rightarrow \infty} \mu(E_n)$. Then,

$$A = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} A_k \subset \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} E_k \in \mathfrak{M},$$

and

$$\mu^*(A) \leq \mu \left(\bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} E_k \right) = \lim_{n \rightarrow \infty} \mu \left(\bigcap_{k=n}^{\infty} E_k \right) \leq \lim_{n \rightarrow \infty} \mu(E_n) = \lim_{n \rightarrow \infty} \mu(A_n),$$

where we have used monotone continuity of **measure**. Therefore, $\lim_{n \rightarrow \infty} \mu^*(A_n) = \mu^* \left(\bigcup_{n=0}^{\infty} A_n \right)$. \square

3.1.4 Constructing outer measures

Definition. Let X be a set. A gauge on X is a pair (\mathcal{E}, γ) where $\mathcal{E} \subset \mathcal{P}(X)$ is such that $\emptyset \in \mathcal{E}$ and $\gamma : \mathcal{E} \rightarrow [0, \infty]$ is such that $\gamma(\emptyset) = 0$.

Theorem. Let X be a set and (\mathcal{E}, γ) be a gauge on X . Define $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ via

$$\mu^*(E) = \inf \left\{ \sum_{n=0}^{\infty} \gamma(E_n) : E \subset \bigcup_{n=0}^{\infty} E_n \text{ and } \{E_n\}_{n=0}^{\infty} \subset \mathcal{E} \right\}.$$

Then μ^* is an outer measure on X and hence generates (X, \mathfrak{M}, μ) , a complete measure space thorough Carathéodory construction.

Proof. *** TO-DO *** \square

Theorem. Let (X, d) be a metric space with gauge (\mathcal{E}, γ) and outer measures $\mu_\delta^* : \mathcal{P}(X) \rightarrow [0, \infty]$ produced by $(\mathcal{E}_\delta, \gamma_\delta)$ for $\delta > 0$. Define $\mu_d^* : \mathcal{P}(X) \rightarrow [0, \infty]$ by

$$\mu_d^*(A) = \sup_{\delta > 0} \mu_\delta^*(A).$$

Then μ_d^* is a metric outer measure. Moreover, $\mu_d^*(A) = \lim_{\delta \rightarrow 0} \mu_\delta^*(A)$ for $A \subset X$.

Proof. *** TO-DO *** \square

Definition. We call μ_d^* the metric outer measure generated by (\mathcal{E}, γ) .

Lemma. Let X be a set with gauge (\mathcal{E}, γ) that covers X . Let $A \subset X$, then the following holds:

1. Let μ^* be the outer measure generated by (\mathcal{E}, γ) . Then there exists collection $\{E_{m,n}\}_{m,n=0}^{\infty} \subset \mathcal{E}$ such that $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$ such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$.
2. Suppose (X, d) is metric space and the gauge is fine. Let μ_d^* be the metric outer measure. Then there exists collection $\{E_{m,n}\}_{m,n=0}^{\infty} \subset \mathcal{E}$ such that $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$ such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$.

Proof. The proof for (1) is very similar to the proof for (2), so we only show (2) as follows. Since the gauge is fine, $(\mathcal{E}_\delta, \gamma_\delta)$ covers X for all $\delta > 0$. Then, for any $m \in \mathbb{N}$, there exists $\{E_{m,n}\}_n \subset \mathcal{E}_{2^{-m}}$ such that $A \subset \bigcup_{n=0}^{\infty} E_{m,n}$ and $\sum_{n=0}^{\infty} \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}$. Now let $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$. Note that $A \subset E$ and for any $m \in \mathbb{N}$, we have

$$\mu_{2^{-m}}^*(E) \leq \mu_{2^{-m}}^* \left(\bigcup_{n=0}^{\infty} E_{m,n} \right) \leq \sum_{n=0}^{\infty} \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}.$$

Taking the limit as $m \rightarrow \infty$, we have

$$\mu_d^*(E) \leq \mu_d^*(A) \leq \mu_d^*(E),$$

as desired. □

Theorem. Let (X, d) be metric space with (\mathcal{E}, γ) such that all sets in \mathcal{E} are open. Assume that μ^* is a metric outer measure on X such that either

1. μ^* is generated by (\mathcal{E}, γ) , or
2. $\mu^* = \mu_d^*$ is generated by $(\mathcal{E}_\delta, \gamma_\delta)$.

Further suppose that $X = \bigcup_{n=0}^{\infty} A_n$ where $A_n \subset X$ is such that $\mu^*(A_n) < \infty$. Then the following holds:

1. The gauge covers X in case 1 and is fine in case 2.
2. In both cases, μ^* is cover-regular. More precisely, for each $A \subset X$, there is $G \in G_\delta(X) \subset \mathfrak{B}(X) \subset \mathfrak{M}$ such that $A \subset G$ and $\mu^*(A) = \mu^*(G)$.
3. In both cases, the following are equivalent for $E \subset X$:
 - (a) $E \in \mathfrak{M}$, i.e. E is measurable.
 - (b) there exists $G \in G_\delta(X)$ such that $E \subset G$ and $\mu^*(G \setminus E) = 0$.
 - (c) there exists $F \in F_\sigma(X)$ such that $F \subset E$ and $\mu^*(E \setminus F) = 0$.

Proof. Step 0: proof for (1) and (2).

We know $X = \bigcup_{n=0}^{\infty} A_n$ for some $\mu^*(A_n) < \infty$. For case (1), we can pick $\{E_{n,m}\} \subset \mathcal{E}$ such that $A_n \subset \bigcup_{m=0}^{\infty} E_{n,m}$. Then $X = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n,m} E_{n,m}$. Therefore, \mathcal{E} covers X . For case (2), note that $\mu_d^*(A_n) < \infty$ and $\mu_d^*(A_n) \geq \mu_\delta^*(A_n)$ for each $\delta > 0$ and $n \in \mathbb{N}$. Then for each $\delta > 0$, there exists $\{E_{n,m}\} \subset \mathcal{E}_\delta$ such that $A_n \subset \bigcup_{m=0}^{\infty} E_{n,m}$. It follows that $X = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n,m} E_{n,m}$. Therefore, (\mathcal{E}, γ) is fine.

We have the following observations:

1. μ^* is a metric outer measure. This implies that $\mathfrak{B}(X) \subset \mathfrak{M}$.
2. $G_\delta(X) \cup F_\sigma(X) \subset \mathfrak{B}(X) \subset \mathfrak{M}$ and $\mu^*(A) = 0$ implies $A \in \mathfrak{M}$.
3. By previous lemma and all sets in \mathcal{E} are open, we know for each $A \subset X$ there is $E \in G_\delta(X)$ such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$. In particular, μ^* is cover regular.

Step 1: starting on (3).

For (b) \implies (a), suppose (b) holds for $E \subset X$. Then $E = G \setminus (G \setminus E) \in \mathfrak{M}$ since $\mu^*(G \setminus E) = 0$.

For (c) \implies (a), suppose (c) holds for $E \subset X$. Then $E = F \cup (E \setminus F) \in \mathfrak{M}$ since $\mu^*(E \setminus F) = 0$.

Next we show “(a) \implies (c)” implies “(a) \implies (b)”. Suppose $E \in \mathfrak{M}$, then $E^c \in \mathfrak{M}$. By (a) \implies (b) we know there exists $F \in F_\sigma$ such that $F \subset E^c$ and $\mu^*(E^c \setminus F) = 0$. Let $G = F^c \in G_\delta$ then $E \subset G$ and $G \setminus E = E^c \setminus F$.

Therefore, it remains to show (a) \implies (c) to complete the proof for the theorem.

Step 2: reduction for (a) \implies (c).

Claim it suffices to show it for E such that $\mu^*(E) < \infty$. Suppose we did this and $\mu^*(E) = \infty$. Using observation there exists $B_n \in \mathfrak{M}$ such that $A_n \subset B_n$ and $\mu^*(B_n) = \mu^*(A_n) < \infty$. Then $E_n = E \cap B_n \in \mathfrak{M}$ and $\mu^*(E_n) < \infty$. Then by special case there is $F_n \in F_\sigma(X)$ such that $F_n \subset E_n$ and $\mu^*(F_n \setminus E_n) = 0$. Let $F = \bigcup_{n=0}^\infty F_n \in F_\sigma$ then $F \subset \bigcup_{n=0}^\infty E_n = E$ and

$$\mu^*(E \setminus F) \leq \sum_{n=0}^\infty \mu^*(E_n \setminus F_n) = 0.$$

Step 3: further reduction.

Claim it suffices to show it for the case where $\mu^*(E) < \infty$ and $E \in G_\delta(X)$. Suppose we have proved this and consider $E \subset X$ such that $\mu^*(E) < \infty$. Observation 3 allows us to pick $G \in G_\delta(X)$ such that $E \subset G$ and $\mu^*(E) = \mu^*(G)$. Now pick $H \in G_\delta$ such that $G \setminus E \subset H$ and $\mu^*(H) = \mu^*(G \setminus E)$.

Now apply special case. This gives $F \in F_\sigma$ such that $F \subset G$ and $\mu^*(G \setminus F) = 0$. Let $K = F \setminus H = F \cap H^c \in F_\sigma$ and $K = F \cap H^c \subset G \cap (G \setminus E)^c \subset E$.

Note that $E, F, G, H, K \in \mathfrak{M}$, so

$$\begin{aligned} \mu^*(E \setminus K) &= \mu^*(E) - \mu^*(K) \\ &= \mu^*(G) - \mu^*(F \setminus H) \\ &= \mu^*(G) - \mu^*(F) + \mu^*(F \cap H) \\ &\leq \mu^*(G) - \mu^*(F) + \mu^*(H) \\ &= \mu^*(G \setminus F) + \mu^*(H) \\ &= \mu^*(G \setminus E) \\ &= \mu^*(G) - \mu^*(E) \\ &= 0. \end{aligned}$$

Therefore, K is the desired F_σ set.

Step 4: finishing (a) \implies (c).

Suppose $E \in G_\delta(X)$ and $\mu^*(E) < \infty$. Write $E = \bigcup_{n=0}^\infty V_n$ where $V_n \subset X$ open. For $m, n \in \mathbb{N}$, let

$$C_{n,m} = \{x \in V_n : \text{dist}(x, V_n^c) \geq 2^{-m}\} \subset V_n.$$

Note that $C_{n,m}$ is closed, $C_{n,m} \subset C_{n,m+1}$, $V_n = \bigcup_m C_{n,m}$. Since $E, C_{n,m}, V_n \in \mathfrak{M}$, we have

$$\mu^*(E) = \mu^*(E \cap V_n) = \lim_{m \rightarrow \infty} \mu^*(E \cap C_{n,m}).$$

Thus, there exists $M(n, k)$ such that $\mu^*(E \setminus C_{n, M(n, k)}) < 2^{-n-k}$. Now let $D_k = \bigcup_{n=0}^\infty C_{n, M(n, k)}$ closed. Also, $D_k \subset \bigcup_{n=0}^\infty V_n = E$ and

$$\mu^*(E) - \mu^*(D_k) = \mu^*(E \setminus D_k) \leq \sum_{n=0}^\infty \mu^*(E \setminus C_{n, M(n, k)}) \leq 2^{-k+1}.$$

Let $F = \bigcup_{k=0}^\infty D_k \subset E$ and note that $F \in F_\sigma$. Then

$$\mu^*(E \setminus F) = \mu^*(E) - \mu^*(F) \leq \mu^*(E) - \mu^*(D_k) < 2^{-k+1}$$

for all $k \in \mathbb{N}$. Therefore, $\mu^*(E \setminus F) = 0$.

□

Lemma. Suppose (X, d) metric space with metric outer measure μ^* . Suppose $X = \bigcup_{n=0}^\infty V_n$ for $V_n \subset X$ open and $\mu^*(V_n) < \infty$. Suppose $E \subset G \in G_\delta(X)$ such that $\mu^*(G \setminus E) = 0$. Then for each $\varepsilon > 0$, there exists open $U \subset X$ such that $E \subset U$ and $\mu^*(U \setminus E) < \varepsilon$.

Proof. Let $E_n = E \cap V_n$ and $G = G \cap V_n$. Write $G = \bigcap_{j=0}^{\infty} W_j$ where W_j open. Now set

$$Z_{n,m} = V_n \cap \bigcap_{j=0}^m W_j,$$

which are open for all $n, m \in \mathbb{N}$. Now notice that $G_n \subset Z_{n,m+1} \subset Z_{n,m} \subset V_n$. Note that $\mu^*(V_n) < \infty$, so $\mu^*(G_n) = \lim_{m \rightarrow \infty} \mu^*(Z_{n,m})$. Therefore, for all $\varepsilon > 0$, there exists $M(n)$ such that

$$\mu^*(Z_{n,M(n)} \setminus G_n) < \varepsilon 2^{-n-2}.$$

Then set $U = \bigcup_{n=0}^{\infty} Z_{n,M(n)} \supset \bigcup_{n=0}^{\infty} G_n = G \supset E$ open, then we have

$$\begin{aligned} \mu^*(U \setminus E) &= \mu^*(U \setminus G) + \mu^*(G \setminus E) \\ &= \mu^*\left(\bigcup_{n=0}^{\infty} Z_{n,M(n)} \cap \bigcap_{n=0}^{\infty} G_n^c\right) \\ &\leq \sum_{n=0}^{\infty} \mu^*(Z_{n,M(n)} \setminus G_n) \\ &< \varepsilon, \end{aligned}$$

as desired. □

Definition (Outer-regular). Let X be a metric space, \mathfrak{M} a σ -algebra with $\mathfrak{B}(X) \subset \mathfrak{M}$ and suppose $\mu : \mathfrak{M} \rightarrow [0, \infty]$ is a measure. Say μ is outer-regular if

$$\mu(E) = \inf \{ \mu(U) : E \subset U \text{ open} \}.$$

3.2 Lebesgue and Hausdorff measure

*** TO-DO ***

3.3 Measurable and μ -measurable functions

Definition (Measurable functions). Let (X, \mathfrak{M}) and (Y, \mathfrak{N}) be measurable sets. A map $f : X \rightarrow Y$ is called $(\mathfrak{M}, \mathfrak{N})$ measurable if $f^{-1}(E) \in \mathfrak{M}$ for all $E \in \mathfrak{N}$.

*** TO-DO ***

Definition (Simple functions). Let (X, \mathfrak{M}) and (Y, \mathfrak{N}) be measurable sets. A map $f : X \rightarrow Y$ is called simple if it is measurable and $f(X)$ is finite. Write the set of all simple functions from X to Y as $S(X, Y)$.

Theorem (Characterization of $\overline{\mathbb{R}}$ measurability). Let (X, \mathfrak{M}) be measure space and $f : X \rightarrow \overline{\mathbb{R}}$. The following are equivalent:

1. f is measurable.
2. There exists $\{\varphi_k\}_{k=0}^{\infty} \subset S(X; \overline{\mathbb{R}})$ such that $\varphi_k \rightarrow f$ pointwise as $k \rightarrow \infty$.

Moreover, if f is measurable, the sequence can be built such that

- On the set $\{f \geq 0\}$, we have $0 \leq \varphi_k \leq \varphi_{k+1} \leq f$.
- On the set $\{f < 0\}$, we have $f \leq \varphi_{k+1} \leq \varphi_k \leq 0$.
- If f is actually from X to \mathbb{R} and is bounded, then $\varphi_k \rightarrow f$ uniformly.

Proof. (2) \implies (1). Pointwise limit of measurable functions are measurable.

(1) \implies (2). Suppose $f : X \rightarrow [0, \infty]$ is measurable. For $k \in \mathbb{N}$, define $\varphi_k : [0, \infty)$ by

$$\varphi_k(x) = \begin{cases} (j-1)2^{-k} & \text{if } (j-1)2^{-k} \leq f(x) < j2^{-k} \text{ for } 1 \leq j \leq k2^k, \\ k & \text{if } f(x) > k. \end{cases}$$

Because f is measurable, φ_k is simple for each $k \in \mathbb{N}$.

Note that $0 \leq \varphi_k \leq \varphi_{k+1} \leq f$. Also, if $f(x) < \infty$, then $0 \leq f(x) - \varphi_k(x) \leq 2^{-k}$. If $f(x) = \infty$, then $\varphi_k(x) = k$. This shows that $\varphi_k \rightarrow f$. Moreover, if f is bounded then $\varphi_k \rightarrow f$ uniformly.

In the general case, apply the special case to f on $\{f \geq 0\}$ and $-f$ on $\{f < 0\}$.

□

Definition (Separably-valued). Let X be a set and Y a metric space. A map $f : X \rightarrow Y$ is **separably-valued** if $f(X) \subset Y$ is separable.

Theorem. Let (X, \mathfrak{M}) be measure space and Y be metric space, $f : X \rightarrow Y$. The following are equivalent for $f : X \rightarrow Y$:

1. f is $(\mathfrak{M}, \mathfrak{B}(Y))$ measurable and separably valued.
2. There exists $\{\varphi_k\}_{k=0}^\infty \in S(X; Y)$ such that $\varphi_k \rightarrow f$ pointwise.

Proof. (2) \implies (1). The pointwise limit of measurable function is measurable. On the other hand, $f(X) = \bigcup_{k=0}^\infty \varphi_k(X)$, which is separable since $\varphi_k(X)$ finite for any $k \in \mathbb{N}$.

(1) \implies (2). Assume initially that Y is totally bounded. Then for each $n \in \mathbb{N}$ there exists $y_0^n, \dots, y_{K(n)}^n \in Y$ such that $Y = \bigcup_{k=0}^{K(n)} B(y_k^n, 2^{-n})$. Let $V_0^n = B(y_0^n, 2^{-n})$ and for $k \geq 1$ define $V_k^n = B(y_k^n, 2^{-n}) \setminus \bigcup_{j=0}^{k-1} B(y_j^n, 2^{-n})$. Then, $Y = \bigcup_{k=0}^{M(n)} V_k^n$ where $V_k^n = \emptyset$ for $M(n) < k \leq K(n)$.

Define $\varphi_n : Y \rightarrow \{y_0^n, \dots, y_{M(n)}^n\}$ via $\varphi_n(y) = y_k^n$ if $y \in V_k^n$. Clearly φ_n is simple and $d(\varphi_n(y), y) < 2^{-n}$ for all $n \in \mathbb{N}$ and $y \in Y$. Therefore, $\varphi_n(y) \rightarrow (y)$ pointwise. Then $f_n = \varphi_n \circ f$ are simple functions from X to Y . Also, since $\varphi_n \rightarrow \text{id}$ pointwise, $f_n \rightarrow f$ pointwise.

Now consider the general case in which $f(X)$ is a separable subset of Y . Then there exists a homeomorphism $h : f(X) \rightarrow Z$ for Z a totally bounded metric space, for example take Z a subset of Hilbert cube H^∞ since all separable metric space is homeomorphism to a subset of the Hilbert cube. Thus $h \circ f : X \rightarrow Z$ is measurable with Z totally bounded, so the special case provides a sequence $\{\varphi_n\}_{n=0}^\infty \subset S(X; Z)$ such that $\varphi_n \rightarrow h \circ f$ pointwise. Then, $h^{-1} \circ \varphi_n \in S(X; Y)$ is such that $h^{-1} \circ \varphi_n \rightarrow h^{-1} \circ h \circ f = f$ pointwise, using continuity of h and h^{-1} .

□

Definition (Almost everywhere). Let (X, \mathfrak{M}, μ) be a measure space and let $P(x)$ be a proposition for every $x \in X$. Say P is true **almost everywhere** (a.e.) if there exists a set $N \in \mathfrak{M}$ such that $\mu(N) = 0$ and $P(x)$ is true for all $x \in N^c$.

Theorem. Let (X, \mathfrak{M}, μ) be a measure space. Let Y be a metric space, $f : X \rightarrow Y$. The following are equivalent:

1. There exists $\{\psi_n\}_{n=0}^\infty \subset S(X; Y)$ such that $\psi_n \rightarrow f$ pointwise a.e. in X .
2. There exists a measurable and separably valued $F : X \rightarrow Y$ such that $f = F$ a.e.
3. There exists a null set $N \in \mathfrak{M}$ and a measurable $F : X \rightarrow Y$ such that $f = F$ on N^c and $f(N^c)$ is separable in Y .

Proof. (1) \implies (2). There exists $N \in \mathfrak{M}$ null such that $\psi_n \rightarrow f$ pointwise in N^c . Thus, $f : N^c \rightarrow Y$ is measurable and separably valued by the previous theorem. Note the constant map $N \ni x \mapsto y \in Y$ for $y \in Y$ fixed is measurable. Thus we can define $F : X \rightarrow Y$ by

$$F(x) = \begin{cases} f(x) & (x \in N^c), \\ y & (x \in N). \end{cases}$$

Then F is measurable. It is also separably valued since $F(X) = f(N^c) \cup \{y\}$.

(2) \implies (3). Trivial.

(3) \implies (1). Note that $F : N^c \rightarrow Y$ is measurable and $F(N^c) = f(N^c)$ is separable. By previous theorem, there exists $\{\varphi_n\}_{n=0}^\infty \in S(N^c; Y)$ such that $\varphi_n \rightarrow F = f$ pointwise on N^c . Now let $\psi_n \in S(X; Y)$ be φ_n in N^c and $y \in Y$ fixed in N . Then $\psi_n \rightarrow f$ pointwise in N^c . □

Definition. Let (X, \mathfrak{M}) be measurable, Y be either a normed vector space or $\overline{\mathbb{R}}$. Let $\psi \in S(X; Y)$.

1. A **representation** of ψ is a finite and well-defined sum $\psi = \sum_{k=1}^K v_k \chi_{E_k}$ for $v_k \in Y$ and $E_k \in \mathfrak{M}$.
2. A **canonical representation** is $\psi = \sum_{v \in \psi(X)} v \chi_{\psi^{-1}(\{v\})}$
3. Now suppose μ is a measure. We say a representation $\psi = \sum_{k=1}^K v_k \chi_{E_k}$ is **finite** if $\mu(E_k) < \infty$ for all k such that $v_k \neq 0$. We say ψ is a **finite simple function** if it has a finite representation.

We write $S_{\text{fin}}(X; Y) = \{f \in S(X; Y) : f \text{ is finite}\}$. Note that it is clear ψ is finite if and only if the canonical representation is finite if and only if $\mu(\text{supp}(\psi)) < \infty$ where $\text{supp}(\psi) = \{x \in X : \psi(x) \neq 0\}$ is the support of ψ .

Definition. Let (X, \mathfrak{M}, μ) be a measure space and Y be a metric space.

1. We say $f : X \rightarrow Y$ is **almost measurable** if $f = F$ a.e. with $F : X \rightarrow Y$ is measurable.
2. We say $f : X \rightarrow Y$ is **almost separably valued** if there exists a null set $N \in \mathfrak{M}$ such that $f(N^c)$ is separable.
3. We say $f : X \rightarrow Y$ is **μ -measurable** if it is almost measurable and almost separably valued. Equivalently, f is the a.e. limit of simple functions.
4. Suppose Y is a normed vector space or $\overline{\mathbb{R}}$. We say $f : X \rightarrow Y$ is **strongly μ -measurable** if there exists $\{\psi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; Y)$ such that $\psi_n \rightarrow f$ a.e. as $n \rightarrow \infty$.

Example. Let $X = \{1, 2, 3\}$ and $\mathfrak{M} = \{\emptyset, \{1, 2\}, \{3\}, \{1, 2, 3\}\}$. Let $f, g : X \rightarrow \mathbb{R}$ via $f(x) = x$ and $g(x) = 3$. Then f is not measure since $f^{-1}(\{1\}) = \{1\} \notin \mathfrak{M}$ but g is measurable.

Now equip (X, \mathfrak{M}) with the measure δ_3 . Then, $f = g$ a.e. This shows that equality almost everywhere does not preserve measurability. The problem is that $(X, \mathfrak{M}, \delta_3)$ is not **complete**.

This brings us to the next theorem.

Theorem. Let (X, \mathfrak{M}, μ) be a measure space. Then the following are equivalent:

1. (X, \mathfrak{M}, μ) is complete.
2. If (Y, \mathfrak{N}) is a measure space, $f, g : X \rightarrow Y$, f is measurable and $f = g$ a.e., then g is measurable.
3. If Y is a metric space with $\text{card } Y = 2$, $f, g : X \rightarrow Y$, f measurable, $f = g$ a.e., then g is measurable.

Proof. (1) \implies (2). Suppose $f, g : X \rightarrow Y$, f is measurable, $f = g$ a.e. Pick null set $N \in \mathfrak{M}$ such that $f = g$ on N^c . Take $E \in \mathfrak{N}$, then

$$\begin{aligned} g^{-1}(E) &= (g^{-1}(E) \cap N) \cup (g^{-1}(E) \cap N^c) \\ &= (g^{-1}(E) \cap N) \cup (f^{-1}(E) \cap N^c). \end{aligned}$$

Note that $f^{-1}(E) \cap N^c$ is measurable, and $g^{-1}(E) \cap N \subset N$ null, so it is also measurable. Therefore, $g^{-1}(E)$ is measurable and g is measurable.

(2) \implies (3). Clear.

(3) \implies (1). Prove the contrapositive. Suppose (X, \mathfrak{M}, μ) is not complete and $Y = \{y, z\}$ a metric space. Find $\emptyset \neq A \subsetneq B$ such that $\mu(B) = 0$ and $A \notin \mathfrak{M}$. Define $f, g : X \rightarrow Y$ by

$$g(x) = \begin{cases} y & (x \notin A), \\ z & (x \in A). \end{cases}$$

and $f(x) = y$ be constant. Then $f = g$ a.e., f is measurable, and g is not measurable. □

Corollary. Let (X, \mathfrak{M}, μ) be a complete measurable space, Y a separable metric space, and $f : X \rightarrow Y$. Then, f is μ -measurable if and only if f is measurable.

Proposition. Let (X, \mathfrak{M}, μ) be a measure space and Y be a metric space. The following holds:

1. Let $f, g : X \rightarrow Y$. If f is μ -measurable and $f = g$ a.e., then g is μ -measurable.
2. Suppose Y is a normed vector space or \mathbb{R} . If $f, g : X \rightarrow Y$, f is strongly μ -measurable, $f = g$ a.e., then g is strong μ -measurable.

Proof. 1. Let $\{\varphi_n\}_{n=0}^\infty \subset S(X; Y)$ be such that $\varphi_n \rightarrow g$ pointwise a.e. Pick null set $N \in \mathfrak{M}$ such that $f = g$ on N^c . Pick null set $Z \in \mathfrak{M}$ such that $f = \lim_{n \rightarrow \infty} \varphi_n$. This implies that $g = \lim_{n \rightarrow \infty} \varphi_n$ on $(N \cup Z)^c$.

2. Same proof as the first item but let $\{\varphi_n\}_{n=0}^\infty \in S_{\text{fin}}(X; Y)$. □

Theorem. Let (X, \mathfrak{M}, μ) be a measure space and Y be a normed vector space with $V \neq \{0\}$. Then the following are equivalent:

1. (X, \mathfrak{M}, μ) is σ -finite.
2. If $f : X \rightarrow Y$ is μ -measurable, then f is strongly μ -measurable.
3. Let $f : X \rightarrow Y$, then f is μ -measurable if and only if f is strongly μ -measurable.
4. If $y \in Y \setminus \{0\}$, then $f : X \rightarrow Y$ via $f(x) = y$ strongly μ -measurable.

Proof. (1) \implies (2). Suppose (X, \mathfrak{M}, μ) is σ -finite. We can find $\{X_n\}_{n=0}^\infty \subset \mathfrak{M}$ such that $X_n \subset X_{n+1}$, $\mu(X_n) < \infty$ and $\bigcup_{n=0}^\infty X_n = X$. Let $f : X \rightarrow Y$ be μ -measurable. Pick $\{\psi_n\}_{n=0}^\infty \subset S(X; Y)$ such that $\psi_n \rightarrow f$ pointwise a.e. Define $\varphi_n = \chi_{X_n} \psi_n$. This shows that f is strongly μ -measurable.

(2) \iff (3). Trivial since strongly μ -measurability implies μ -measurability.

(2) \implies (4). Constant function are μ -measurable.

(4) \implies (1). Let $y \in Y \setminus \{0\}$ and define $f : X \rightarrow Y$ via $f(x) = y$. This is strongly μ -measurable by assumption. Then there exists $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; Y)$ such that $\varphi_n \rightarrow f$ pointwise on N^c where N is null.

Pick $\varepsilon > 0$ such that $\{0\} \cap B(y, \varepsilon) = \emptyset$. Set $X_n = \varphi_n^{-1}(B(y, \varepsilon))$. Then we have $\mu(X_n) < \infty$. For any $x \in N^c$ and n sufficiently large, $\varphi_n(x) \in B(y, \varepsilon)$. Therefore, $N^c \subset \bigcup_{n=0}^\infty X_n$ and the proof we are complete. □

Finally, we present a useful characterization of μ -measurability of Banach-valued maps.

Theorem (Pettis). Let (X, \mathfrak{M}, μ) be a measure space and V be a Banach space over \mathbb{F} . Suppose $W \subset V^*$ is a norming subspace. Let $f : X \rightarrow V$. Then the following are equivalent:

1. f is μ -measurable.
2. f is almost separably valued, and $w \circ f : X \rightarrow \mathbb{F}$ is μ -measurable for each $w \in V^*$.
3. f is almost separably valued, and $w \circ f : X \rightarrow \mathbb{F}$ is μ -measurable for each $w \in W$.

In any case, there exists $\{\varphi_n\}_{n=0}^\infty \subset S(X; V)$ such that $\|\varphi_n\| \leq 2\|f\|$ on X such that $\varphi_n \rightarrow f$ pointwise a.e. as $n \rightarrow \infty$. Moreover, the same equivalence holds with μ -measurability replaced by strongly μ -measurability and $\{\varphi_n\}_{n=0}^\infty$ replaced by $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; V)$.

Proof. (1) \implies (2). Suppose f is μ -measurable, which means it is almost separably valued. Each $w \in V^*$ is also continuous so $w \circ f$ is μ -measurable.

(2) \implies (3). Trivial since $W \subset V^*$.

(3) \implies (1). Suppose f is almost separably valued. Then there exists null set $N_* \subset X$ such that $f(X \setminus N_*) \subset V$ separable. Define the subspace

$$M = \text{span}(f(X \setminus N_*)) \subset V,$$

which is separable by construction. Pick a dense set $\{v_n\}_{n=0}^\infty \subset M$ such that $v_0 = 0$. Then by a previous theorem, we know there exists a norming sequence $\{w_n\}_{n=0}^\infty \subset W$ for M .

Now, given any $v \in V$ and $n \in \mathbb{N}$, define the function $\Phi_{n,v} : X \rightarrow [0, \infty)$ by

$$\Phi_{n,v}(x) = |\langle w_n, f(x) - v \rangle| = |w_n(f(x) - v)|.$$

Note that $X \ni x \mapsto \langle w_n, v \rangle \in \mathbb{F}$ is μ -measurable and the map $X \ni x \mapsto \langle w_n, f(x) \rangle \in \mathbb{F}$ is also μ -measurable by assumption. It follows that $\Phi_{n,v}$ is μ -measurable. Therefore, there exists null set $N_{n,v} \subset X$ and a measurable map $\Psi_{n,v} : X \rightarrow [0, \infty)$ such that $\Psi_{n,v} = \Phi_{n,v}$ on $X \setminus N_{n,v}$. For each $v \in V$ define null set

$$N(v) = N_* \cup \bigcup_{n=0}^\infty N_{n,v} \subset X,$$

with $\Psi_{n,v} = \Phi_{n,v}$ on $X \setminus N(v)$ for all $n \in \mathbb{N}$.

For $v \in M$ define the map $\Phi_v : X \rightarrow [0, \infty]$ by $\Phi_v(x) = \|f(x) - v\|$ and note that $\{w_n\}_{n=0}^\infty$ is norming sequence for M . This implies that

$$\Phi_v(x) = \sup_{n \in \mathbb{N}} |\langle w_n, f(x) - v \rangle|$$

for all $x \in X \setminus N_*$. We also have that

$$\Phi_v(x) = \sup_{n \in \mathbb{N}} \Phi_{n,v}(x) = \sup_{n \in \mathbb{N}} \Psi_{n,v}(x)$$

for all $x \in X \setminus N(v)$, so Φ_v is measurable when restricted to $X \setminus N(v)$. We can then define the set

$$N = \bigcup_{m=0}^\infty N(v_m) \subset X,$$

which is null. By construction, each Φ_{v_m} is measurable when restricted to N^c . In particular, $\Phi_0 = \Phi_{v_0} = \|f\|$ is measurable when restricted to N^c .

For $u \in M$ and $n \in \mathbb{N}$, define

$$k(n, u) = \min \left\{ 0 \leq k \leq n : \|u - v_k\| = \min_{0 \leq j \leq n} \|u - v_j\| \right\}.$$

By construction,

$$\|v_{k(n,u)}\| \leq \|u - v_{k(n,u)}\| + \|u\| \leq \|u - v_0\| + \|u\| = 2\|u\|.$$

We then define $S_n : M \rightarrow \{v_0, \dots, v_n\}$ via $S_n(u) = v_{k(n,u)}$. Note that $\|S_n(u)\| \leq 2\|u\|$. Also, $\{v_m\}_{m=0}^\infty$ dense in M implies $S_n(u) \rightarrow u$ as $n \rightarrow \infty$.

Finally, for $n \in \mathbb{N}$, define $\psi_n : N^c \rightarrow \{v_0, \dots, v_n\} \subset V$ via $\psi_n = S_n \circ f$. For $0 \leq k \leq n$, we compute

$$\begin{aligned} & \{x \in N^c : \psi_n(x) = v_k\} \\ &= \left\{x \in N^c : \|f(x) - v_k\| = \min_j \|f(x) - v_j\|\right\} \cap \bigcap_{j=0}^{k-1} \{x \in N^c : \|f(x) - v_k\| < \|f(x) - v_j\|\} \end{aligned}$$

This set is measurable since Φ_{v_m} measurable on N^c for each $m \in \mathbb{N}$. It follows that ψ_n is measurable on N^c . Let $\varphi_n \in S(X; V)$ by

$$\varphi_n(x) = \begin{cases} \psi_n(x) & (x \in N^c), \\ 0 & (x \in N). \end{cases}$$

Then, $\|\varphi_n\| \leq 2\|f\|$ and $\varphi_n(x) = \psi_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for $x \in N^c$. Therefore, $\varphi_n \rightarrow f$ a.e. and thus f is μ -measurable. □

3.4 Lebesgue-Bochner Integral

Lemma. Let (X, \mathfrak{M}, μ) be a measure space and $Y \in \{V, [0, \infty]\}$. Let $\psi : X \rightarrow Y$ be simple such that

$$\psi = \sum_{i=1}^I \alpha_i \chi_{E_i} = \sum_{j=1}^J \beta_j \chi_{F_j}.$$

Additionally, if $Y = V$ suppose both representation are finite. Then,

$$\sum_{i=1}^I \alpha_i \mu(E_i) = \sum_{j=1}^J \beta_j \mu(F_j).$$

Based on this lemma, we can define

$$\int_X \psi \, d\mu = \sum_{i=1}^I \alpha_i \mu(E_i).$$

This induces maps $\int_X \cdot \, d\mu : S(X; [0, \infty]) \rightarrow [0, \infty]$ and $\int_X \cdot \, d\mu : S_{\text{fin}}(X; V) \rightarrow V$.

Proposition. Let (X, \mathfrak{M}, μ) be a measure space and $Y \in \{V, [0, \infty]\}$. Then the following holds:

1. If $Y = V$, then

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu$$

for all $\alpha, \beta \in \mathbb{F}$ and $f, g \in S_{\text{fin}}(X; V)$. If $Y = [0, \infty]$, the same equality holds for any $\alpha, \beta > 0$ and $f, g \in S(X; V)$.

2. If $Y = V$, then $\|f\| \in S_{\text{fin}}(X; [0, \infty))$ and

$$\left\| \int_X f \, d\mu \right\| \leq \int_X \|f\| \, d\mu.$$

3. If $E \in \mathfrak{M}$, then

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu.$$

4. If $N \in \mathfrak{M}$ is a null set, then

$$\int_N f \, d\mu = 0.$$

5. If $A, B \in \mathfrak{M}$ is such that $A \cap B = \emptyset$, then

$$\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu.$$

6. Suppose $\{X_n\}_{n=0}^\infty \subset \mathfrak{M}$ is such that $X_n \subset X_{n+1}$ and $\mu(X_n) < \infty$. Then

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_{X_n} f \, d\mu.$$

Proof. Write $f = \sum_k f_k \chi_{E_k}$ be the canonical representation. We then have

$$\int_{X_n} f \, d\mu = \sum_k f_k \mu(X_n \cap E_k).$$

For each k , we have $X_n \cap E_k \subset X_{n+1} \cap E_k$ and $\bigcup_{n=0}^\infty (X_n \cap E_k) = E_k$. It follows that

$$\lim_{n \rightarrow \infty} \mu(X_n \cap E_k) = \mu(E_k).$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{X_n} f \, d\mu = \sum_k f_k \mu(E_k) = \int_X f \, d\mu.$$

□

7. If $Y = \mathbb{R}$ or $Y = [0, \infty]$ and $f \leq g$ a.e., then

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

3.4.1 Integration of $\overline{\mathbb{R}}$ -valued functions

Note that if (X, \mathfrak{M}, μ) is a measure space and $\varphi \in S(X; [0, \infty])$, then

$$\int_X \varphi \, d\mu = \sup \left\{ \int_X \psi \, d\mu : \psi \in S(X; [0, \infty]) \text{ and } \psi \leq \varphi \text{ a.e.} \right\}.$$

Definition. Let (X, \mathfrak{M}, μ) be a measure space. Let $f : X \rightarrow [0, \infty]$ be μ -measurable. We define

$$\int_X f \, d\mu = \sup \left\{ \int_X \psi \, d\mu : \psi \in S(X; [0, \infty]) \text{ and } \psi \leq f \text{ a.e.} \right\} \in [0, \infty].$$

We say f is **integrable** if $\int_X f \, d\mu < \infty$.

Remark. There are two remarks with regard to the definition above.

1. In principle we do not need f to be μ -measurable here. We build this into the definition because the resulting integral is more-or-less useless without this assumption.
2. $[0, \infty]$ is a separable metric space, so for $f : X \rightarrow [0, \infty]$, f is measurable implies f is μ -measurable, and f almost measurable implies f is μ -measurable.

Theorem. Let (X, \mathfrak{M}, μ) be a measure space, $f, g : X \rightarrow [0, \infty]$ be μ -measurable functions. The following holds:

1. For $\alpha \in [0, \infty)$, we have

$$\int_X \alpha f \, d\mu = \alpha \int_X f \, d\mu.$$

2. If $f \leq g$ a.e., then

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

3. If $f = g$ a.e., then

$$\int_X f \, d\mu = \int_X g \, d\mu.$$

4. For $E \in \mathfrak{M}$, we have

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu.$$

5. If $N \in \mathfrak{M}$ is null, then

$$\int_N f \, d\mu = 0.$$

Proof. Follow directly from corresponding results in $S(X; [0, \infty])$ and the definition of $\int_X f \, d\mu$. \square

Theorem (Monotone convergence theorem, basic version). Let (X, \mathfrak{M}, μ) be a measure space and suppose for each $n \in \mathbb{N}$, we have $f_n : X \rightarrow [0, \infty]$ **measurable**. Further suppose that $f_n \leq f_{n+1}$ on X and $f : X \rightarrow [0, \infty]$ is given by $f = \lim_{n \rightarrow \infty} f_n$. Then f is measurable and

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu = \sup_{n \in \mathbb{N}} \int_X f_n \, d\mu.$$

Proof. We already know f is measurable. Also, $f_n \leq f_{n+1} \leq f$ on X , so

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f \, d\mu.$$

It follows that

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu.$$

To show the opposite inequality, let $\varphi \in S(X; [0, \infty])$ such that $\varphi \leq f$ a.e. and $\alpha \in (0, 1)$. Let $N \in \mathfrak{M}$ be a null set and $\varphi \leq f$ on N^c . Also, for each $n \in \mathbb{N}$, let $E_n = \{x \in X : f_n(x) \geq \alpha\varphi(x)\}$. Note the following:

1. Since $f_n \leq f_{n+1}$, we have $E_n \subset E_{n+1}$.
2. Since $f_n \rightarrow f$ pointwise, we have $X = N \cup \bigcup_{n=0}^{\infty} E_n$.
3. We have

$$\alpha \int_{N \cup E_n} \varphi \, d\mu = \int_{E_n} \alpha\varphi \, d\mu \leq \int_{E_n} f_n \, d\mu \leq \int_X f_n \, d\mu$$

4. We have

$$\int_X \varphi \, d\mu = \lim_{n \rightarrow \infty} \int_{N \cup E_n} \varphi \, d\mu.$$

Therefore,

$$\alpha \int_X \varphi \, d\mu = \lim_{n \rightarrow \infty} \alpha \int_{N \cup E_n} \varphi \, d\mu \leq \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

Since the above inequality holds for all $\alpha \in (0, 1)$, we know $\int_X \varphi \, d\mu \leq \lim_{n \rightarrow \infty} \int_X f_n \, d\mu$. This is then true for all simple function φ such that $\varphi \leq f$ a.e. Taking the sup gives

$$\int_X f \, d\mu \leq \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

The proof is then complete. \square

Theorem. Let (X, \mathfrak{M}, μ) be measure space, $f, g : X \rightarrow [0, \infty]$ be μ -measurable. Then

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Proof. Recall that μ -measurable functions are almost measurable. Choose measurable functions $F, G : X \rightarrow [0, \infty]$ such that $f = F$ and $g = G$ a.e. We may then choose $\{\varphi_n\}_{n=0}^\infty, \{\psi_n\}_{n=0}^\infty \subset S(X; [0, \infty])$ such that $\lim_{n \rightarrow \infty} \varphi_n = F$ and $\lim_{n \rightarrow \infty} \psi_n = G$, $0 \leq \varphi_n \leq \varphi_{n+1} \leq F$ and $0 \leq \psi_n \leq \psi_{n+1} \leq G$. Then

$$0 \leq \varphi_n + \psi_n \leq \varphi_{n+1} + \psi_{n+1} \leq F + G = \lim_{n \rightarrow \infty} (\varphi_n + \psi_n).$$

It follows then from monotone convergence theorem that

$$\begin{aligned} \int_X (F + G) d\mu &= \lim_{n \rightarrow \infty} \int_X (\varphi_n + \psi_n) d\mu \\ &= \lim_{n \rightarrow \infty} \int_X \varphi_n d\mu + \lim_{n \rightarrow \infty} \int_X \psi_n d\mu \\ &= \int_X F d\mu + \int_X G d\mu. \end{aligned}$$

Since $f = F$ and $g = G$ a.e., we have

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

□

Recall: given $f : X \rightarrow \overline{\mathbb{R}}$, we write $f^\pm : X \rightarrow [0, \infty]$ via

$$f^+ = \max\{0, f\}, \quad f^- = \max\{0, -f\}.$$

Then we have $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Also, if f is measurable or μ -measurable, then f^\pm is also measurable or μ -measurable since they are composition of a continuous function (namely $x \mapsto \max\{0, x\}$) with a measurable or μ -measurable function.

Definition. Let (X, \mathfrak{M}, μ) be measure space and $f : X \rightarrow \overline{\mathbb{R}}$ be μ -measurable. If either f^+ or f^- is **integrable**, we say f is **extended integrable** and set

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu \in \overline{\mathbb{R}}.$$

We say f is **integrable** if f^\pm are both integrable.

Proposition (absolute integrability). Let (X, \mathfrak{M}, μ) be a measure space, $f : X \rightarrow \overline{\mathbb{R}}$ be μ -measurable. Then f is integrable if and only if $|f|$ is integrable.

Proof. We know f is integrable if and only if f^\pm are both integrable, but $|f| = f^+ + f^-$. Therefore, f integrable implies $|f|$ is integrable. Conversely, if $|f|$ is integrable, then $0 \leq f^\pm \leq |f|$, so f^\pm are both integrable. □

Theorem. Let (X, \mathfrak{M}, μ) be a measure space, $f, g : X \rightarrow \overline{\mathbb{R}}$ are extended integrable. The following holds:

1. For all $E \in \mathfrak{M}$, we have $\int_E f d\mu = \int_X f \chi_E d\mu$.
2. For all $\alpha \in \mathbb{R}$, we have $\alpha \int_X f d\mu = \int_X \alpha f d\mu$.
3. $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$, provided that all operations are well-defined.
4. $\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$ for all $A, B \in \mathfrak{M}$ such that $A \cap B = \emptyset$.
5. If $f \leq g$ a.e. then $\int_X f d\mu \leq \int_X g d\mu$.

6. $|\int_X f \, d\mu| \leq \int_X |f| \, d\mu$.
7. If $|f| \leq g$ a.e. and g integrable, then f is integrable.

Theorem (Chebyshev inequality). If f is measurable, then

$$\mu(\{x \in X : |f(x)| \geq \alpha\}) \leq \frac{1}{\alpha} \int_X |f| \, d\mu$$

for all $\alpha \in (0, \infty)$.

Proof.

$$\text{LHS} = \int_{\{|f| \geq \alpha\}} 1 \, d\mu = \int_{\{|f| \geq \alpha\}} \frac{|f|}{\alpha} \, d\mu = \frac{1}{\alpha} \int_X |f| \, d\mu = \text{RHS}.$$

□

Corollary. Let (X, \mathfrak{M}, μ) be a measure space and $f : X \rightarrow \overline{\mathbb{R}}$.

1. If f is integrable, then there exists a null set $N \in \mathfrak{M}$ and a σ -finite set $E \in \mathfrak{M}$ such that $\{|f| = \infty\} \subset N$ and $\text{supp}(f) \subset E$.
2. If f is extended integrable, then there exists a null set $N \in \mathfrak{M}$ such that either $\{f = \infty\} \subset N$ or $\{f = -\infty\} \subset N$.

Proof. 1. Suppose initially that f is measurable and integrable, then Chebyshev inequality implies that

$$\mu(\{|f| = \infty\}) \leq \mu(\{|f| > 2^k\}) \leq 2^{-k} \int_X |f| \, d\mu$$

for all $k \in \mathbb{N}$. It follows that $\mu(\{|f| = \infty\})$ is null.

On the other hand, $\text{supp}(f) = \bigcup_{k=0}^{\infty} \{|f| > 2^{-k}\}$, but

$$\mu(\{|f| > 2^{-k}\}) \leq 2^k \int_X |f| \, d\mu < \infty.$$

It follows that $\text{supp}(f)$ is σ -finite.

In general, if f is integrable and μ -measurable, pick $F = f$ a.e. for F measurable and integrable and apply the argument above.

2. Next, if f is extended integrable but not integrable, then either f^+ is integrable or f^- is integrable. If f^+ is integrable, then $\{f = +\infty\}$ is contained in some null set. If f^- is integrable, $\{f = -\infty\}$ is contained in a null set.

□

To prove the more general form of monotone convergence theorem, we first need a useful lemma.

Lemma. Let (X, \mathfrak{M}, μ) be a measure space and suppose that $f : X \rightarrow \overline{\mathbb{R}}$ is μ -measurable and $g : X \rightarrow \mathbb{R}$ is integrable. Further suppose $g \leq f$ a.e. Then, f and $f - g$ are extended integrable, and

$$\int_X (f - g) \, d\mu = \int_X f \, d\mu - \int_X g \, d\mu.$$

Proof. Since $g \leq f$ a.e., we have $f^- \leq g^-$ a.e. Since g is integrable, f^- is integrable and thus f is extended-integrable. We also have $f - g$ well defined on all of E and $f - g \geq 0$ a.e. Therefore, $f - g$ is extended-integrable.

If f is integrable, then we immediately have the desired equality. Suppose not f is not integrable but only extended-integrable. This implies f^+ is not integrable. We must then have $f - g$ not integrable, otherwise $f = (f - g) + g$ is integrable. Therefore, $\int_X (f - g) \, d\mu = \int_X f \, d\mu = \infty$, and the desired equality holds. □

Theorem (Monotone convergence theorem, general form). Let (X, \mathfrak{M}, μ) be a measure space and suppose $f_k : X \rightarrow \overline{\mathbb{R}}$ is μ -measurable for all $k \in \mathbb{N}$. Suppose that $f : X \rightarrow \overline{\mathbb{R}}$ is such that $f_k \rightarrow f$ a.e. Then, f is μ -measurable and the following holds:

1. Suppose that $\{f_k\}_{k=0}^\infty$ is almost everywhere nondecreasing, that is, $f_k \leq f_{k+1}$ a.e. Suppose also that there exists an integrable function $g : X \rightarrow \overline{\mathbb{R}}$ such that $g \leq f_k$ for all $k \in \mathbb{N}$. Then, f and f_k are extended integrable for all $k \in \mathbb{N}$, and

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu.$$

2. Suppose that $\{f_k\}_{k=0}^\infty$ is almost everywhere nonincreasing, that is, $f_k \geq f_{k+1}$ a.e. Suppose also that there exists an integrable function $g : X \rightarrow \overline{\mathbb{R}}$ such that $g \geq f_k$ for all $k \in \mathbb{N}$. Then, f and f_k are extended integrable for all $k \in \mathbb{N}$, and

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu.$$

Proof. Since g is integrable, there exists a null set $\tilde{N} \in \mathfrak{M}$ such that $\{|g| = \infty\} \subset \tilde{N}$. Now g is \mathbb{R} -valued in N^c . We can also select a null set $N \supset \tilde{N}$ such that the following holds:

- g is measurable on N^c .
- $f_k \rightarrow f$ as $k \rightarrow \infty$ on N^c .
- For each $k \in \mathbb{N}$, f_k is measurable on N^c , $f_k \leq f_{k+1} \leq f$ on N^c , and $g \leq f_k \leq f$ on N^c .

By Lemma 10.3.22, we know f , $f - g$ are extended integrable on N^c and f_k , $f_k - g$ are extended integrable on N^c for each $k \in \mathbb{N}$. Additionally, we have

$$\int_{N^c} (f - g) d\mu = \int_{N^c} f d\mu - \int_{N^c} g d\mu,$$

and for each $k \in \mathbb{N}$

$$\int_{N^c} (f_k - g) d\mu = \int_{N^c} f_k d\mu - \int_{N^c} g d\mu.$$

Note now $f_k - g$ is measurable function on N^c taking values in $[0, \infty]$. Also, $f_k - g \leq f_{k+1} - g$ on N^c and $f_k - g \rightarrow f - g$ pointwise as $k \rightarrow \infty$ on N^c . By the basic version of monotone convergence theorem, we have

$$\lim_{k \rightarrow \infty} \int_{N^c} (f_k - g) d\mu = \int_{N^c} (f - g) d\mu.$$

Therefore,

$$\lim_{k \rightarrow \infty} \int_{N^c} f_k d\mu - \int_{N^c} g d\mu = \int_{N^c} f d\mu - \int_{N^c} g d\mu.$$

However, note that $\int_{N^c} g d\mu \in \mathbb{R}$ and it then follows that

$$\lim_{k \rightarrow \infty} \int_{N^c} f_k d\mu = \int_{N^c} f d\mu.$$

Since both f_k and f are extended integrable and N is null, we have

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu,$$

as desired. □

Corollary. 1. Let (X, \mathfrak{M}, μ) be a measure space, $f_k : X \rightarrow (-\infty, \infty]$ be μ -measurable for all $k \in \mathbb{N}$ and $f_k \geq 0$ a.e. Then,

$$\int_X \sum_{k=0}^{\infty} f_k d\mu = \sum_{k=0}^{\infty} \int_X f_k d\mu.$$

2. Suppose (X, \mathfrak{M}, μ) is a measure space, $X = \bigcup_{k=0}^{\infty} E_k$ such that $\{E_k\}_{k=0}^{\infty} \subset \mathfrak{M}$ and $\mu(E_k \cap E_j) = 0$ for all $k \neq j$. Given $f : X \rightarrow [0, \infty]$ μ -measurable, we then have

$$\int_X f d\mu = \sum_{k=0}^{\infty} \int_{E_k} f d\mu.$$

Proof. 1. Note that $\text{supp}(f_k^-)$ is in a null set, so each f_k is extended integrable. The same holds for $\sum_{k=0}^{\infty} f_k : X \rightarrow [-\infty, \infty]$. On the other hand, the partial sums $\sum_{k=0}^m f_k \leq \sum_{k=0}^{m+1} f_k$ a.e. Apply monotone convergence theorem gives the desired equality.

2. Use the first claim on $f_k = f\chi_{E_k}$.

□

Theorem (Fatou's lemma). Let (X, \mathfrak{M}, μ) be a measure space, and suppose that $f_k : X \rightarrow \overline{\mathbb{R}}$ are μ -measurable for all $k \in \mathbb{N}$. Suppose that $g : X \rightarrow \overline{\mathbb{R}}$ is extended integrable, $\int_X g d\mu > -\infty$, and $g \leq f_k$ a.e. for all $k \in \mathbb{N}$. Then the following holds:

1. For each $k \in \mathbb{N}$, f_k is extended integrable.
2. The function $\liminf_{k \rightarrow \infty} f_k$ is extended integrable.
3. We have

$$\int_X g d\mu \leq \int_X \liminf_{k \rightarrow \infty} f_k d\mu \leq \liminf_{k \rightarrow \infty} \int_X f_k d\mu.$$

Proof. Note that $\int_X g d\mu > -\infty$ implies g^- is integrable. Write

$$f = \liminf_{k \rightarrow \infty} f_k,$$

which is a μ -measurable function. Then, $g \leq f_k$ a.e. implies $g \leq f$ a.e. as well. It follows that $-f_k \leq -g$ and $-f \leq -g$. Therefore, $f_k^- \leq g^-$ and $f^- \leq g^-$. This shows that f_k and f are extended-integrable. Next, note that

$$\int_X g d\mu \leq \int_X \inf_{j \geq k} f_j d\mu \leq \int_X f_k d\mu.$$

By monotone convergence theorem, we know the middle term converges when k approaches infinity. Taking the \liminf , we have

$$\int_X g d\mu \leq \liminf_{k \rightarrow \infty} \int_X f_k d\mu = \lim_{k \rightarrow \infty} \int_X \inf_{j \geq k} f_j d\mu \leq \liminf_{k \rightarrow \infty} \int_X f_k d\mu.$$

□

Theorem (Dominated convergence theorem). Let (X, \mathfrak{M}, μ) be a measure space and suppose $f_k, g_k : X \rightarrow \overline{\mathbb{R}}$ μ -measurable for each $k \in \mathbb{N}$. Suppose that $f, g : X \rightarrow \overline{\mathbb{R}}$ are such that $f_k \rightarrow f$ a.e. and $g_k \rightarrow g$ a.e. Suppose further that g_k is integrable and $|f_k| \leq g_k$ a.e. for each $k \in \mathbb{N}$. Suppose also g is integrable and that

$$\lim_{k \rightarrow \infty} \int_X g_k d\mu = \int_X g d\mu.$$

Then, f_k is integrable for each $k \in \mathbb{N}$, f is integrable, and

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu.$$

Moreover, $f_k - f$ is well-defined for all $k \in \mathbb{N}$ outside a null set $N \subset X$, and

$$\lim_{k \rightarrow \infty} \int_{N^c} |f_k - f| d\mu = 0$$

Proof. We know $|f_k| \leq g_k$ a.e., $g_k \rightarrow g$ a.e., and $f_k \rightarrow f$ a.e. Then, $|f| \leq g$ a.e., so f_k and f are integrable. In turn, we can use a previous corollary to pick $N \in \mathfrak{M}$ null such that f_k, f, g_k, g are all \mathbb{R} -valued and all assumed inequalities hold on N^c . Then, $|f - f_k| \leq g + g_k$ on N^c , and so

$$0 \leq g + g_k - |f - f_k|.$$

Apply Fatou's lemma, we then have

$$\begin{aligned} \int_{N^c} 2g d\mu &= \int_{N^c} \liminf_{k \rightarrow \infty} (g + g_k - |f - f_k|) d\mu \\ &\leq \liminf_{k \rightarrow \infty} \int_{N^c} (g + g_k - |f - f_k|) d\mu \\ &= \liminf_{k \rightarrow \infty} \int_{N^c} (g + g_k - |f - f_k|) d\mu + \liminf_{k \rightarrow \infty} \int_{N^c} -(g + g_k) d\mu + \int_{N^c} 2g d\mu \\ &\leq \liminf_{k \rightarrow \infty} \int_{N^c} -|f - f_k| d\mu + \int_{N^c} 2g d\mu. \end{aligned}$$

It follows that

$$0 \leq \limsup_{k \rightarrow \infty} \int_{N^c} |f - f_k| d\mu = -\liminf_{k \rightarrow \infty} \int_{N^c} -|f - f_k| d\mu \leq 0.$$

Therefore,

$$\lim_{k \rightarrow \infty} \int_{N^c} |f - f_k| d\mu = 0.$$

Note that f_k and f are integrable, so

$$\left| \int_X f d\mu - \int_X f_k d\mu \right| = \left| \int_{N^c} f d\mu - \int_{N^c} f_k d\mu \right| \leq \int_{N^c} |f - f_k| d\mu.$$

This then implies

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu,$$

and the proof is complete. \square

Remark. Usually, dominated convergence theorem is applied with $g_k = g$, in which case the assumption $\int_X g_k d\mu \rightarrow \int_X g d\mu$ becomes trivial.

3.4.2 Bochner integration

Lemma. Suppose (X, \mathfrak{M}, μ) is a measure space and V a normed vector space, and $\varphi : X \rightarrow V$ simple. Note then $\|\varphi\| : X \rightarrow [0, \infty)$ is a simple function now. Then, φ is a **finite** simple function if and only if $\|\varphi\|$ is integrable.

Proof. (\implies) Suppose φ is finite, then $\|\varphi\|$ is finite. Then, $\|\varphi\|$ is integrable.

(\impliedby) Suppose $\|\varphi\|$ is integrable. We know φ is simple, so $\varphi(X) \setminus \{0\}$ is a finite set in V . Then, there exists $0 < m \in \mathbb{R}$ such that $\|v\| \geq m$ for all $v \in \varphi(X) \setminus \{0\}$. Then,

$$\mu(\text{supp}(\varphi)) = \mu(\{x \in X : \|\varphi(x)\| > 0\}) = \mu(\{\|\varphi\| \geq m\}).$$

By Chebyshev inequality, we have

$$\mu(\text{supp}(\varphi)) \leq \frac{1}{m} \int_X \|\varphi\| d\mu < \infty.$$

This completes the proof. \square

Lemma. Let (X, \mathfrak{M}, μ) be a measure space, V be a Banach space, $f : X \rightarrow V$ μ -strongly measurable. Suppose that for $j \in \{0, 1\}$, we have $\{\varphi_k^j\}_{k=0}^\infty \subset S_{\text{fin}}(X; V)$ such that

$$\lim_{k \rightarrow \infty} \int_X \|f - \varphi_k^j\| d\mu = 0.$$

Then, $\{\int_X \varphi_k^j\}_{k=0}^\infty$ is convergent in V for both $j \in \{0, 1\}$ and

$$\lim_{k \rightarrow \infty} \int_X \varphi_k^0 d\mu = \lim_{k \rightarrow \infty} \int_X \varphi_k^1 d\mu.$$

Proof. For $k, m \in \mathbb{N}$, we have

$$\begin{aligned} \left\| \int_X \varphi_m^j d\mu - \int_X \varphi_k^j d\mu \right\| &= \left\| \int_X (\varphi_m^j - \varphi_k^j) d\mu \right\| \\ &\leq \int_X \|\varphi_m^j - \varphi_k^j\| d\mu \\ &\leq \int_X \|f - \varphi_m^j\| d\mu + \int_X \|f - \varphi_k^j\| d\mu. \end{aligned}$$

This shows that $\{\int_X \varphi_k^j\}_{k=0}^\infty$ is Cauchy and hence convergent.

On the other hand,

$$\begin{aligned} \left\| \int_X \varphi_k^0 d\mu - \int_X \varphi_k^1 d\mu \right\| &\leq \int_X \|\varphi_k^0 - \varphi_k^1\| d\mu \\ &\leq \int_X \|f - \varphi_k^0\| d\mu + \int_X \|f - \varphi_k^1\| d\mu \\ &\rightarrow 0, \end{aligned}$$

completing the proof. □

This leads to the following definition for Bochner integration.

Definition. Let (X, \mathfrak{M}, μ) be a measure space and V a Banach space. A map $f : X \rightarrow V$ is (Bochner) integrable if it is strongly μ -measurable and there exists a sequence $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; V)$ such that $\varphi_n \rightarrow f$ a.e. and

$$\lim_{n \rightarrow \infty} \int_X \|f - \varphi_n\| d\mu = 0,$$

in which case we define

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X \varphi_n d\mu \in V.$$

Note that this is well-defined by the previous lemmas.

Theorem (absolute integrability). Let (X, \mathfrak{M}, μ) be a measure space, V a Banach space, $f : X \rightarrow V$. Then, f is integrable if and only if μ -measurable and $\|f\| : X \rightarrow [0, \infty]$ is integrable. In either case,

$$\left\| \int_X f d\mu \right\| \leq \int_X \|f\| d\mu.$$

Proof. (\implies) Suppose f is integrable. This implies that f is strongly μ -measurable and in particular μ -measurable. Also, $\|f\| : X \rightarrow [0, \infty)$ is μ -measurable. Suppose $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; V)$ is such that $\varphi_n \rightarrow f$ a.e. and $\int_X \|f - \varphi_n\| d\mu \rightarrow 0$. Then,

$$\int_X \|f\| d\mu \leq \int_X \|f - \varphi_n\| d\mu + \int_X \|\varphi_n\| d\mu < \infty$$

for n sufficiently large. This implies that $\|f\|$ is integrable.

(\Leftarrow) Suppose f is μ -measurable and $\int_X \|f\| d\mu < \infty$. Then, Pettis theorem gives a sequence $\{\varphi_n\}_{n=0}^\infty \in S(X; V)$ such that $\varphi_n \rightarrow f$ a.e. and $\|\varphi_n\| \leq 2\|f\|$. Then,

$$\int_X \|\varphi_n\| d\mu \leq 2 \int_X \|f\| d\mu < \infty.$$

Therefore, $\{\varphi_n\}_{n=0}^\infty$ is actually a sequence of finite simple functions. This implies that f is actually strongly μ -measurable. On the other hand, $\|f - \varphi_n\| \leq 3\|f\|$, so dominated convergence theorem implies

$$\int_X \|f - \varphi_n\| d\mu \rightarrow 0$$

as $n \rightarrow \infty$. By definition, f is now integrable. Moreover,

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X \varphi_n d\mu.$$

It follows then from the dominated convergence theorem that

$$\left\| \int_X f d\mu \right\| = \lim_{n \rightarrow \infty} \left\| \int_X \varphi_n d\mu \right\| \leq \lim_{n \rightarrow \infty} \int_X \|\varphi_n\| d\mu = \int_X \|f\| d\mu.$$

□

Theorem (dominated convergence theorem for Bochner). Let (X, \mathfrak{M}, μ) be a measure space, V a Banach space, and suppose $f_n : X \rightarrow V$, $g_n : X \rightarrow \mathbb{R}$ are μ -measurable $n \in \mathbb{N}$. Further suppose $f : X \rightarrow V$ and $g : X \rightarrow \mathbb{R}$ are such that $f_n \rightarrow f$ a.e. and $g_n \rightarrow g$ a.e. Also, suppose g_n, g are integrable. Finally suppose $\|f_n\| \leq g_n$ a.e. and

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X g d\mu.$$

Then, f_n, f are integrable and

$$\lim_{n \rightarrow \infty} \int_X \|f_n - f\| d\mu = 0,$$

so we also have

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

Proof. Since $\|f_n\| \leq g_n$ and $\|f\| \leq g$, we have f_n and f integrable. Note that $\|f - f_n\| \leq g + g_n$ and $g + g_n \rightarrow 2g$ as $n \rightarrow \infty$. Dominated convergence theorem then implies

$$\lim_{n \rightarrow \infty} \int_X \|f - f_n\| d\mu = 0,$$

completing the proof. □

Proposition. Let (X, \mathfrak{M}, μ) be a measure space and V a Banach space over \mathbb{F} . Let $f : X \rightarrow V$ integrable. The following holds:

1. If W is a Banach space over F and $T \in \mathcal{L}(V, W)$, then $T \circ f : X \rightarrow W$ is integrable and

$$\int_X T \circ f d\mu = T \int_X f d\mu.$$

2. Suppose $g : X \rightarrow V$ is integrable, then $\int_X f d\mu = \int_X g d\mu$ if and only if $\int_X w \circ f d\mu = \int_X w \circ g d\mu$ for every $w \in V^*$.

Proof. 1. Let $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; V)$ such that $\varphi_n \rightarrow f$ a.e. and $\int_X \|f - \varphi_n\| \rightarrow 0$. Then we have $T \circ \varphi_n \rightarrow T \circ f$ a.e. and

$$\int_X \|T \circ f - T \circ \varphi_n\| d\mu \leq \|T\| \int_X \|f - \varphi_n\| d\mu \rightarrow 0.$$

Therefore, $T \circ f$ is integrable and

$$\int_X T \circ f d\mu = \lim_{n \rightarrow \infty} \int_X T \circ \varphi_n d\mu = \lim_{n \rightarrow \infty} T \int_X \varphi_n d\mu = T \int_X f d\mu.$$

2. Let $w \in V^*$, then $\int_X f d\mu = \int_X g d\mu$ clearly implies $\int_X w \circ f d\mu = \int_X w \circ g d\mu$. On the other hand, if $\int_X w \circ f d\mu = \int_X w \circ g d\mu$ for all $w \in V^*$, then

$$w \left[\int_X f d\mu - \int_X g d\mu \right] = 0$$

for all $w \in V^*$. By Hahn-Banach theorem, this implies $\int_X f d\mu = \int_X g d\mu$.

□

3.5 Constructing product measures

Definition (Pre-measure). Let X be a set and \mathfrak{A} be an algebra on X . A map $\gamma : \mathfrak{A} \rightarrow [0, \infty]$ is a **pre-measure** if the following is satisfied:

1. $\gamma(\emptyset) = 0$.
2. If $\{A_i\}_{i=0}^\infty \subset \mathfrak{A}$ is disjoint and $\bigcup_{i=0}^\infty A_i$, then $\gamma(\bigcup_{i=0}^\infty A_i) = \sum_{i=0}^\infty \gamma(A_i)$.

Theorem (Pre-measure extension theorem). Let X be a set, \mathfrak{A} is an algebra on X , and γ a pre-measure. Let $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ be the outer measure constructed from (X, γ) . Denote \mathfrak{M} as the measurable space and $\mu : \mathfrak{M} \rightarrow [0, \infty]$ the corresponding measure. Then the following holds:

1. $\mathfrak{A} \subset \mathfrak{M}$ and $\mu = \gamma$ on \mathfrak{A} .
2. Suppose \mathfrak{N} is a σ -algebra on X such that $\mathfrak{A} \subset \mathfrak{N} \subset \mathfrak{M}$, and $\nu : \mathfrak{N} \rightarrow [0, \infty]$ is a measure such that $\nu = \gamma$ on \mathfrak{A} . Then $\nu \leq \mu$ on \mathfrak{N} and $\nu(E) = \mu(E)$ whenever E is σ -finite w.r.t. μ .

In particular, if X is “ γ σ -finite”, then $\mu = \nu$ on \mathfrak{N} .

Proof. First show $\mu = \gamma$ on \mathfrak{A} . It suffices to show that $\mu^* = \gamma$ on \mathfrak{A} .

For any $E \in \mathfrak{A}$, we know E is covered by E , so $\mu^* = \gamma$. On the other hand, let $E \subset \mathfrak{A}$ and $\{A_k\}_{k=0}^\infty \subset \mathfrak{A}$ be a cover of E . Define $B_0 = E \cap A_0 \in \mathfrak{A}$ and $B_k = E \cap (A_k \setminus \bigcup_{i=0}^{k-1} A_i) \in \mathfrak{A}$. Then $\{B_k\}_{k=0}^\infty$ is pairwise disjoint and $\bigcup_{k=0}^\infty B_k = E$. It follows that

$$\gamma(E) = \gamma\left(\bigcup_{k=0}^\infty B_k\right) = \sum_{k=0}^\infty \gamma(B_k) \leq \sum_{k=0}^\infty \gamma(A_k).$$

Therefore, $\mu^* = \gamma$ on \mathfrak{A} .

Next we show $\mathfrak{A} \subset \mathfrak{M}$. Let $E \in \mathfrak{A}$ be arbitrary and we want to show $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all $A \subset X$. Fix arbitrary $A \subset X$ and $\varepsilon > 0$. Pick $\{A_k\}_{k=0}^\infty \subset \mathfrak{A}$ covering A such that

$$\sum_{k=0}^\infty \gamma(A_k) < \mu^*(A) + \varepsilon.$$

It follows that

$$\begin{aligned}
\mu^*(A \cap E) + \mu^*(A \cap E^c) &\leq \mu^*\left(\bigcup_{k=0}^{\infty} A_k \cap E\right) + \mu^*\left(\bigcup_{k=0}^{\infty} A_k \cap E^c\right) \\
&\leq \sum_{k=0}^{\infty} \mu^*(A_k \cap E) + \mu^*(A_k \cap E^c) \\
&= \sum_{k=0}^{\infty} \gamma(A_k \cap E) + \gamma(A_k \cap E^c) \\
&= \sum_{k=0}^{\infty} \gamma(A_k).
\end{aligned}$$

This implies that E is measurable, completing the proof for the first item.

For the second item, we first show that $\nu \leq \mu$. Let $E \in \mathfrak{N} \subset \mathfrak{M}$ and $\{A_k\}_{k=0}^{\infty} \subset \mathfrak{A}$ that covers E . It follows that

$$\nu(E) \leq \nu\left(\bigcup_{k=0}^{\infty} A_k\right) = \lim_{n \rightarrow \infty} \nu\left(\bigcup_{i=0}^n A_i\right).$$

Note that $\bigcup_{i=0}^n A_i \in \mathfrak{A}$, so $\nu(\bigcup_{i=0}^n A_i) = \mu(\bigcup_{i=0}^n A_i)$. This implies that

$$\nu(E) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=0}^n A_i\right) = \mu\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} \gamma(A_k).$$

Therefore, $\nu \leq \mu$.

Next we show $\nu(E) = \mu(E)$ for $\mu(E) < \infty$. Let $\varepsilon > 0$ and select $\{A_k\}_{k=0}^{\infty} \subset \mathfrak{A}$ covering E such that

$$\sum_{k=0}^{\infty} \gamma(A_k) < \mu^*(E) + \varepsilon = \mu(E) + \varepsilon.$$

Then,

$$\mu\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} \gamma(A_k) < \mu(E) + \varepsilon.$$

It follows that $\mu(\bigcup_{k=0}^{\infty} A_k \setminus E) < \varepsilon$ and thus

$$\mu(E) \leq \mu\left(\bigcup_{k=0}^{\infty} A_k\right) = \nu\left(\bigcup_{k=0}^{\infty} A_k\right) = \nu(E) + \nu\left(\bigcup_{k=0}^{\infty} A_k \setminus E\right) \leq \nu(E) + \varepsilon,$$

where for $\mu(\bigcup_{k=0}^{\infty} A_k) = \nu(\bigcup_{k=0}^{\infty} A_k)$ we used the same limit argument as the previous part.

For the case where E is σ -finite, it follows from a similar argument.

□

Theorem (Product measures). Let $2 \leq n \in \mathbb{N}$ and suppose $(X_i, \mathfrak{M}_i, \mu_i)$ is measure space for $1 \leq i \leq n$. Let $X = \prod_i X_i$ and

$$\mathcal{E} = \left\{ E = \prod_i E_i : E_i \in \mathfrak{M}_i \text{ for } 1 \leq i \leq n \right\}.$$

The following holds:

1. $\mathfrak{A} = \left\{ \bigcup_{k=0}^K A^k : \{A^k\}_k \subset \mathcal{E} \text{ and disjoint} \right\}$ is an algebra.
2. Suppose $\{E^k\}_{k=0}^{\infty} \subset \mathcal{E}$ and $\{F^k\}_{k=0}^{\infty} \subset \mathcal{E}$ are both pairwise disjoint sequences of sets and $\bigcup_{k=0}^{\infty} E^k = \bigcup_{k=0}^{\infty} F^k$, then

$$\sum_{k=0}^{\infty} \prod_{i=1}^n \mu_i(E_i^k) = \sum_{k=0}^{\infty} \prod_{i=1}^n \mu_i(F_i^k).$$

3. The map $\gamma : \mathfrak{A} \rightarrow [0, \infty]$ defined by

$$\gamma \left(\bigcup_{k=0}^K \prod_{i=1}^n E_i^k \right) = \sum_{k=0}^K \prod_{i=1}^n \mu_i(E_i^k)$$

is a well-defined pre-measure.

4. If $(X_i, \mathfrak{M}_i, \mu_i)$ is σ -finite, then X is γ σ -finite.

Proof. 1. It is easy to check that \mathfrak{A} is indeed an algebra.

2. Suppose $\bigcup_{k=0}^{\infty} E^k = \bigcup_{k=0}^{\infty} F^k$, then we have

$$\sum_{k=0}^{\infty} \prod_{i=1}^n \chi_{E_i^k}(x_i) = \sum_{k=0}^{\infty} \prod_{i=1}^n \chi_{F_i^k}(x_i)$$

for all $x = (x_1, \dots, x_n) \in X$. Now fix (x_2, \dots, x_n) , we then have

$$\sum_{k=0}^{\infty} \chi_{E_1^k}(x_1) \alpha_1^k = \sum_{k=0}^{\infty} \chi_{F_1^k}(x_1) \beta_1^k,$$

where $\alpha_1^k = \prod_{i=2}^n \chi_{E_i^k}(x_i)$ and $\beta_1^k = \prod_{i=2}^n \chi_{F_i^k}(x_i)$. Using the monotone convergence theorem and integrate both sides, we have

$$\sum_{k=0}^{\infty} \mu_1(E_1) \alpha_1^k = \sum_{k=0}^{\infty} \mu_1(F_1) \beta_1^k.$$

Iterate this argument gives the desired equality.

3. Suppose $\{A_i\}_{i=0}^{\infty} \subset \mathfrak{A}$ disjoint such that $\bigcup_{i=0}^{\infty} A_i \in \mathfrak{A}$. By construction, there exists sequence $\{F^j\}_{j=0}^J \subset \mathfrak{A}$ with $J < \infty$ such that $\{A_i\}_{i=0}^{\infty} = \bigcup_{j=0}^J F_j$. Also, $A_i \in \mathfrak{A}$ for each $i \in \mathbb{N}$, so $\bigcup_{i=0}^{\infty} A_i = \bigcup_{k=0}^{\infty} E^k$ where $\{E^k\}_{k=0}^{\infty} \subset \mathcal{E}$ disjoint. It follows that

$$\gamma \left(\bigcup_{i=0}^{\infty} A_i \right) = \gamma \left(\bigcup_{j=0}^J F_j \right) = \sum_{j=0}^J \prod_{i=1}^n \mu_i(F_i^j) = \sum_{k=0}^{\infty} \prod_{i=1}^n \mu_i(E_i^k),$$

where the last equality is by item 2. However,

$$\gamma \left(\bigcup_{i=0}^{\infty} A_i \right) = \sum_{k=0}^{\infty} \prod_{i=1}^n \mu_i(E_i^k) = \sum_{i=0}^{\infty} \gamma(A_i).$$

This shows that γ is a pre-measure.

4. For each $1 \leq i \leq n$, there exists $\{S_i\}_{k=0}^{\infty} \subset \mathfrak{M}_i$ such that $\bigcup_{k=0}^{\infty} S_i^k = X_i$ and $\mu_i(S_i^k) < \infty$. Consider $\{\prod_{i=1}^n S_i^k\}_{k=0}^{\infty}$.

□

Corollary. Suppose that $\{(X_i, \mathfrak{M}_i, \mu_i)\}_{i=1}^n$ be a sequence of σ -finite measure space. Let $X = \prod_{i=1}^n X_i$ be endowed with the product σ -algebra $\bigoplus_{i=1}^n \mathfrak{M}_i$. Let \mathfrak{A} and $\gamma : \mathfrak{A} \rightarrow [0, \infty]$ be the algebra and pre-measure from the previous theorem. Then, there exists a unique measure $\nu : \bigoplus_{i=1}^n \mathfrak{M}_i \rightarrow [0, \infty]$ such that $\nu = \gamma$ on \mathfrak{A} . Moreover, ν is σ -finite.

Proof. Use the previous theorem and extend the pre-measure.

□

3.6 Area formula and change of variable formula

We first need to develop a few facts in linear algebra.

Proposition. Let V_1, \dots, V_n, W be vector space over \mathbb{F} and $T \in L(V_1, \dots, V_n; W)$. Suppose $x_i^j \in V_i$ for $j = 0, 1$ and $1 \leq i \leq n$. Then,

$$\begin{aligned} T(x_1^0 + x_1^1, \dots, x_n^0 + x_n^1) &= \sum_{\beta \in B(n)} T(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}) \\ &= \sum_{m=0}^n \sum_{\beta \in B_m(n)} T(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}), \end{aligned}$$

where

$$\begin{aligned} B(n) &= \{\beta : \{1, \dots, n\} \rightarrow \{0, 1\}\}, \\ B_m(n) &= \left\{ \beta \in B(n) : \sum \beta(k) = m \right\}. \end{aligned}$$

Proof. Induction on $n \geq 1$. □

Definition. 1. For $1 \leq k \leq n$ we set

$$\mathcal{A}(n, k) = \left\{ (\alpha_1, \dots, \alpha_k) \in \{1, \dots, n\}^k : \alpha_1 < \alpha_2 < \dots < \alpha_k \right\}.$$

We also set $\mathcal{A}(n, 0) = \{0\}$.

2. For $1 \leq k \leq n$, let $M \in \mathbb{F}^{n \times k}$, $N \in \mathbb{F}^{k \times n}$, $P \in \mathbb{F}^{n \times n}$. For $\alpha \in \mathcal{A}(n, k)$, we set $M_\alpha, N^\alpha, P_\alpha \in \mathbb{F}^{k \times k}$ via

$$(M_\alpha)_{i,j} = M_{\alpha_i,j}, \quad (N_\alpha)_{i,j} = N_{i,\alpha_j}, \quad (P_\alpha)_{i,j} = P_{\alpha_i,\alpha_j}.$$

Theorem. Let $M \in \mathbb{F}^{n \times n}$ and $Z \in \mathbb{F}$. Then,

$$\det(zI + M) = z^n + \sum_{k=0}^{n-1} z^k \sum_{\alpha \in \mathcal{A}(n, n-k)} \det(M_\alpha^\alpha).$$

Proof. Fix $z \in \mathbb{F}$. Let $x_i^0 = ze_i \in \mathbb{F}^n$ and $x_i^1 = M_i \in \mathbb{F}^n$ be the i -th column of M . Recall that $\det \in L^n(\mathbb{F}^n; \mathbb{F})$. Therefore,

$$\begin{aligned} \det(zI + M) &= \det(x_1^0 + x_1^1, \dots, x_n^0 + x_n^1) \\ &= \sum_{k=0}^n \sum_{\beta \in B_k(n)} \det(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}) \\ &= z^n + \sum_{k=1}^n \sum_{\beta \in B_k(n)} \det(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}). \end{aligned}$$

Now given $1 \leq k \leq n$ and $\beta \in B_k(n)$, we set $\alpha \in \mathcal{A}(n, k)$ to be an increasing enumeration of $\{1 \leq i \leq n : \beta(i) = 1\}$. This gives a bijection from $\mathcal{A}(n, k)$ to $B_k(n)$. On the other hand, if $\beta \in B_k(n)$, then

$$\det(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}) = z^{n-k} \det(M_\alpha^\alpha),$$

for the $\alpha \in \mathcal{A}(n, k)$ that corresponds to the $\beta \in B_k(n)$. This completes the proof. □

Theorem. Let $1 \leq n \leq m$, $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times m}$. The following holds:

1. (Sylvester's formula) $\det(I_m + AB) = \det(I_n + BA)$.
2. (Cauchy-Binet formula) $\det(BA) = \sum_{\alpha \in \mathcal{A}(m, n)} \det A_\alpha \det B^\alpha$.

In particular, if $A^* \in \mathbb{F}^{n \times m}$ given by $A_{ij}^* = \overline{A_{ji}}$, then $\det(A^*A) = \sum_{\alpha \in \mathcal{A}(m, n)} |\det A_\alpha|^2$.

Proof. 1. We have

$$\begin{bmatrix} I_m & A \\ 0_{n \times m} & I_n \end{bmatrix} \begin{bmatrix} I_m & -A \\ B & I_n \end{bmatrix} = \begin{bmatrix} I_m + AB & 0_{m \times n} \\ B & I_n \end{bmatrix}$$

and

$$\begin{bmatrix} I_m & -A \\ B & I_n \end{bmatrix} \begin{bmatrix} I_m & A \\ 0_{n \times m} & I_n \end{bmatrix} = \begin{bmatrix} I_m & 0_{m \times n} \\ B & I_n + BA \end{bmatrix}.$$

It follows that $\det(I_m + AB) = \det(I_n + BA)$.

2. Fix $z \in \mathbb{F} \setminus \{0\}$. Then,

$$\begin{aligned} z^{-m} \det(zI_m + AB) &= \det(I_m + z^{-1}AB) \\ &= \det(I_n + B(z^{-1}A)) \\ &= z^{-n} \det(zI_n + BA). \end{aligned}$$

It follows that $z^n \det(I_m + AB) = z^m \det(I_n + BA)$. By our previous propositions, we have

$$z^{n+m} \sum_{k=0}^{m-1} z^{k+n} \sum_{\alpha \in A(m, m-k)} \det(AB)_\alpha^\alpha = z^{n+m} \sum_{k=0}^{n-1} z^{k+m} \sum_{\alpha \in A(n, n-k)} \det(BA)_\alpha^\alpha.$$

Consider the coefficients of degree m , we obtain

$$\sum_{\alpha \in A(n, n)} \det(BA)_\alpha^\alpha = \sum_{\alpha \in A(m, m)} \det(AB)_\alpha^\alpha.$$

Note that $\text{LHS} = \det BA$ and $(AB)_\alpha^\alpha = A_\alpha B^\alpha$. This completes the proof. \square

Definition (Jacobian map). Let $\emptyset \neq U \subset \mathbb{R}^n$ be an open set and $f \in C^1(U; \mathbb{R}^m)$ with $n \leq m$. Define the **Jacobian map** $J_f \in C^0(U; [0, \infty))$ by

$$J_f = \llbracket Df \rrbracket = \sqrt{\det(Df^T Df)}.$$

Lemma. Let $\emptyset \neq U \subset \mathbb{R}^n$, $f \in C^1(U; \mathbb{R}^m)$ for some $n \leq m$. Suppose $z \in U$ is such that $Df(z)$ is injective. Then for $0 < \varepsilon < 1$, there exists $B(z, r) \subset U$ such that

1. $f|_{B(z, r)}$ is a Lipschitz injection.
2. If $E \subset B(z, r)$ is Lebesgue measurable, then $f(E) \in \mathfrak{H}^n(\mathbb{R}^m)$ and

$$(1 - \varepsilon)^{n+1} \int_E J_f d\lambda \leq \mathcal{H}^n(f(E)) \leq (1 + \varepsilon)^{n+1} \int_E J_f d\lambda.$$

Proof. Define the following $M = Df(z)$, $L \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that $LM = I_n$, and $g = f \circ L$, so $f = g \circ M$.

Let $0 < \varepsilon < 1$ and pick $r > 0$ such that

$$(1 - \varepsilon) \|M(x - y)\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|M(y - x)\| \quad (\text{A})$$

for all $x, y \in B(z, r)$ and

$$(1 + \varepsilon)^{-1} J_f(z) \leq J_f(x) \leq (1 - \varepsilon)^{-1} J_f(z) \quad (\text{B})$$

for all $x \in B(z, r)$. Note that

$$\mathcal{H}^n(ME) = J_f(z) \lambda(E).$$

Then equation A gives $[g] \leq 1 + \varepsilon$ and $[M \circ f^{-1}] \leq (1 - \varepsilon)^{-1}$. It follows that

$$\mathcal{H}^n(f(E)) = \mathcal{H}^n(g(ME)) \leq (1 + \varepsilon)^n \mathcal{H}^n(ME) = (1 + \varepsilon)^n J_f(z) \lambda(E).$$

Also,

$$J_f(z)\lambda(E) = \mathcal{H}^n(ME) = \mathcal{H}^n(M \circ f^{-1}(f(E))) \leq (1 - \varepsilon)^{-n} \mathcal{H}^n(f(E)).$$

Now, equation B gives

$$J_f(z)\lambda(E) = \int_E J_f(z) d\lambda \leq (1 + \varepsilon) \int_E J_f d\lambda$$

and

$$J_f(z)\lambda(E) = \int_E J_f(z) d\lambda \geq (1 - \varepsilon) \int_E J_f d\lambda.$$

This completes the proof.

□

Definition. Let X be a set equipped with counting measure $\mathcal{H}^0 : \mathcal{P}(X) \rightarrow [0, \infty]$. Let Y be a set and $f : X \rightarrow Y$. For any $E \subset X$, define $N_f(\cdot, E) : Y \rightarrow [0, \infty]$ by

$$N_f(y, E) = \mathcal{H}^0(E \cap f^{-1}(\{y\})) = \mathcal{H}^0(\{x \in E : f(x) = y\}).$$