Introduction to Functional Analysis

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1 Banach space theory

1.1 Quotient spaces, Baire category and uniform boundedness

Theorem. Let $\|\cdot\|$ be a **seminorm** on a vector space V. If we define $E = \{v \in V : \|v\| = 0\}$, then E is a subspace of V, and the function on V/E defined by

$$\|v + E\| = \|v\|$$

for any $v + E \in V/E$ defines a **norm**.

Theorem (Baire Category Theorem)**.** Let M be a complete metric space, and let $\{C_n\}_{n=0}^{\infty}$ be a collection of closed subsets of M such that $M = \bigcup_{n=0}^{\infty} C_n$. Then at least one of the C_n contains an open ball $B(x, r) = \{y \in M : d(x, y) < r\}$.

Theorem (Uniform Boundedness Theorem)**.** Let B be Banach space and V a normed vector space. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{B}(B, V)$. Then if for all $b \in B$ we have $\sup_n \|T_n b\| < \infty$ (that is, this sequence is pointwise boudned), then $\sup_n \|T_n\| < \infty$ (the operator norms are bounded).

Proof. For each $k \in \mathbb{N}$, define

$$C_k = \left\{ b \in B : \|b\| \leq 1, \sup_{n \in \mathbb{N}} \|T_n b\| \leq k \right\}.$$

This set is closed for each $k \in \mathbb{N}$, but by assumption, we have

$$\{b \in B : \|b\| \leq 1\} = \bigcup_{k=0}^{\infty} C_k.$$

The left hand side is a closed subset of B, and is thus a complete metric space. By Baire Category Theorem, there exists $k \in \mathbb{N}$ such that C_k contains an open ball $B(b_0, \delta_0)$. Then, if $b \in B(0, \delta_0)$, we have $b_0 + b \in B(b_0, \delta_0)$ and thus

$$\sup_{n \in \mathbb{N}} \|T_n(b_0 + b)\| \leq k.$$

It follows that

$$\sup_{n \in \mathbb{N}} \|T_n b\| \leq \sup_{n \in \mathbb{N}} \|T_n(b_0 + b)\| + \sup_{n \in \mathbb{N}} \|T_n b_0\| \leq 2k.$$

Suppose $\|b\| = 1$, then $\frac{\delta_0}{2} b \in B(0, \delta_0)$ and thus for all $n \in \mathbb{N}$, we have

$$\left\| T_n \left(\frac{\delta_0}{2} b \right) \right\| \leq 2k.$$

Therefore,

$$\sup_{n \in \mathbb{N}} \|T_n\| \leq \frac{4k}{\delta_0}.$$

□

2 Hilbert space theory

2.1 Basic Hilbert space theory

Definition (Pre-Hilbert space)**.** A **pre-Hilbert** space H is a vector space over $\mathbb{C}$ with a **Hermitian inner product**, which is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ satisfying the following properties.

1. For all $\lambda_1, \lambda_2 \in C$ and $v_1, v_2, w \in H$, we have

$$\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, 2 \rangle + \lambda_2 \langle v_2, w \rangle .$$

2. For all $v, w \in H$, we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
3. For all $v \in H$, we have $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$.

Definition. Let H be a pre-Hilbert space. For all $v \in H$, we define

$$\|v\| = \langle v, v \rangle^{\frac{1}{2}} .$$

Theorem (Cauchy-Schwarz inequality)**.** Let H be a pre-Hilbert space. For all $u, v \in H$, we have

$$|\langle u, v \rangle| \leq \|u\| \, \|v\| .$$

Proof. Define $f(t) = \|u + tv\|^2$. Notice that

$$\begin{aligned} f(t) &= \langle u + tv, u + tv \rangle \\ &= \langle u, u \rangle + t^2 \langle v, v \rangle + t \langle u, v \rangle + t \langle v, u \rangle \\ &= \|u\|^2 + t^2 \|v\|^2 + 2t \operatorname{Re}(\langle u, v \rangle). \end{aligned}$$

This implies that

$$0 \leq f(t_{\min}) = \|u\|^2 - \frac{\operatorname{Re}(\langle u, v \rangle)^2}{\|v\|^2}.$$

It follows that

$$|\operatorname{Re}(\langle u, v \rangle)| \leq \|u\| \, \|v\| .$$

This is almost what we want. To finish up, first note that if $\langle u, v \rangle = 0$ then there is nothing to prove, so suppose $\langle u, v \rangle \neq 0$, and define

$$\lambda = \frac{\overline{\langle u, v \rangle}}{|\langle u, v \rangle|}.$$

Note that we have $|\lambda| = 1$ and we have the chain of equalities of real numbers:

$$|\langle u, v \rangle| = \lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \operatorname{Re} \langle \lambda u, v \rangle \leq \|\lambda u\| \, \|v\| .$$

However, $\|\lambda u\| = \|u\|$, so the proof is complete.

□

Theorem. If H is a pre-Hilbert space, then $\|\cdot\|$ is a norm on H.

Proof. Note that

$$\|v\| = 0 \iff \langle v, v \rangle = 0 \iff v = 0.$$

Now if $\lambda \in \mathbb{C}$ and $v \in H$, then

$$\langle \lambda v, \lambda v \rangle = \lambda \overline{\lambda} \langle v, v \rangle = |\lambda|^2 \|v\|^2 .$$

Therefore, $\|\lambda v\| = |\lambda| \, \|v\|$.

Finally, let $u, v \in H$, then

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v\rangle \\ &\le \|u\|^2 + \|v\|^2 + 2\left|\langle u, v\rangle\right| \\ &\le \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \\ &= (\|u\| + \|v\|)^2.\end{aligned}$$

This completes the proof. □

Theorem. If $u_n \to u$ and $v_n \to v$ in a pre-Hilbert space H, then $\langle u_n, v_n\rangle \to \langle u, v\rangle$.

Proof. If $u_n \to u$ and $v_n \to v$, then $\|u_n - u\| \to 0$ and $\|v_n - v\| \to 0$. It follows that

$$\begin{aligned}\left|\langle u_n, v_n\rangle - \langle u, v\rangle\right| &= \left|\langle u_n - u, v_n\rangle - \langle u, v - v_n\rangle\right| \\ &\le \left|\langle u_n - u, v_n\rangle\right| + \left|\langle u, v - v_n\rangle\right| \\ &\le \|u_n - u\|\|v_n\| + \|u\|\|v - v_n\| \\ &\le \|u_n - u\| \sup_{k\in\mathbb{N}} \|v_k\| + \|u\|\|v - v_n\| \\ &\to 0\end{aligned}$$

as $n \to \infty$. This completes the proof. □

Definition (Hilbert space)**.** A **Hilbert space** is a pre-Hilbert space that is complete with repsect to the norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.

Example. Some examples of Hilbert spaces:

- $\mathbb{C}^n = \{z = (z_1, \dots, z_n) : z_j \in \mathbb{C}\}$ with $\langle z, w\rangle = \sum_j z_j \overline{w_j}$ is a Hilbert space.
- $\ell^2 = \left\{a = \{a_k\}_{k=0}^{\infty} : a_k \in \mathbb{C},\ \sum_{k=0}^{\infty} |a_k|^2 < \infty\right\}$ with $\langle a, b\rangle = \sum_{k=0}^{\infty} a_k \overline{b_k}$ is a Hilbert space.
- If $E \subset \mathbb{R}$ is measurable, then $L^2(E) = \left\{f : E \to \mathbb{C},\ \int_E |f|^2 < \infty\right\}$ with $\langle f, g\rangle = \int_E f\overline{g}$ is a Hilbert space.

We will show that each separable Hilbert spaces is isometrically isomorphic to either $\mathbb{C}^n$ or ℓ^2.

Now we have seen that ℓ^2 and L^2 spaces are Hilbert spaces. A natural question is whether other ℓ^p or L^p spaces are also Hilbert spaces with respect to some inner product? It turns out there is a simple way to decide whether a norm come from a inner-product, and thus whether a Banach space is a Hilbert space.

Theorem (Parallelogram Law)**.** If H is a pre-Hilbert space, then for all $u, v \in H$, we have

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right).$$

In addition, if H is a normed vector space satisfying this equality, then H is a pre-Hilbert space.

Using the previous theorem, we can verify that ℓ^p and L^p with $p \neq 2$ are **not** Hilbert spaces.

Definition (Orthogonal)**.** If H is a pre-Hilbert space, $u, v \in H$ are **orthogonal** if $\langle u, v\rangle = 0$. We denote this as $u \perp v$.

Definition (Orthonormal sets)**.** If H is a pre-Hilbert space, a subset $\{e_\lambda\}_{\lambda\in\Lambda} \subset H$ is **orthonormal** if for all $\lambda \in \Lambda$, we have $\|e_\lambda\| = 1$ and $\lambda_1 \neq \lambda_2$ implies $e_{\lambda_1} \perp e_{\lambda_2}$.

Remark. we will mainly be interested in the case where we have a countable orthonormal set.

Example. The set $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{Z}}$ as elements in $L^2([-\pi,\pi])$ is a orthonormal subset of $L^2([-\pi,\pi])$. Indeed, for any $m,n\in\mathbb{Z}$, we have

$$\int_{-\pi}^{\pi} e^{imx}\overline{e^{inx}} = \int_{-\pi}^{\pi} e^{i(m-n)x}$$

This evaluates to 2π if $m=n$ and 0 if $m\neq n$.

Theorem (Bessel)**.** If $\{e_n\}_{n=0}^{\infty}$ is countable orthonormal subset of a pre-Hilbert space H, then for all $u\in H$, we have

$$\sum_{n=0}^{\infty}|\langle u,e_n\rangle|^2\leq\|u\|^2\,.$$

Proof. We first do the finite case. Suppose $\{e_n\}_{n=1}^{N}$ is an orthonormal subset of H. Then,

$$\begin{aligned}\left\|\sum_{n=1}^{N}\langle u,e_n\rangle e_n\right\|^2 &= \left\langle\sum_{n=1}^{N}\langle u,e_n\rangle e_n,\sum_{n=1}^{N}\langle u,e_n\rangle e_n\right\rangle\\ &=\sum_{n=1}^{N}\sum_{m=1}^{N}\langle u,e_n\rangle\overline{\langle u,e_m\rangle}\langle e_n,e_m\rangle\\ &=\sum_{n=1}^{N}|\langle u,e_n\rangle|^2\,.\end{aligned}$$

Also,

$$\begin{aligned}\left\langle u,\sum_{n=1}^{N}\langle u,e_n\rangle e_n\right\rangle &=\sum_{n=1}^{N}\overline{\langle u,e_n\rangle}\langle u,e_n\rangle\\ &=\sum_{n=1}^{N}|\langle u,e_n\rangle|^2\,.\end{aligned}$$

Therefore,

$$\begin{aligned}0&\leq\left\|u-\sum_{n=1}^{N}\langle u,e_n\rangle e_n\right\|^2\\ &=\|u\|^2+\left\|\sum_{n=1}^{N}\langle u,e_n\rangle e_n\right\|^2-2\operatorname{Re}\left\langle u,\sum_{n=1}^{N}\langle u,e_n\rangle e_n\right\rangle\\ &=\|u\|^2-\sum_{n=1}^{N}|\langle u,e_n\rangle|^2\,,\end{aligned}$$

as desired.

For the infinite case, just take the limit as $N\to\infty$. □

Definition (Maximal orthonormal subset)**.** An orthonormal subset $\{e_\lambda\}_\lambda$ of a pre-Hilbert space is **maximal** if $u\in H$ and $\langle u,e_\lambda\rangle=0$ for all $\lambda\in\Lambda$ implies that $u=0$.

Theorem. Every non-trivial pre-Hilbert space has a maximal orthonormal subset.

This can be proved using Zorn's Lemma. We will prove something less strong but often equally useful by hand, without applying Zorn's Lemma.

Theorem. Every non-trivial separable pre-Hilbert space has a countable maximal orthonormal subset.

Proof. Use the Gram-Schimdt process. Let $\{v_j\}_{j=0}^{\infty}$ be a countable dense subset of H where $v_0\neq 0$. Claim that for any $n\in\mathbb{N}$, there exists $m(n)\leq n$ and an orthonormal subset $\left\{e_1,\dots,e_{m(n)}\right\}$ such that

1. span $\{e_1, \ldots, e_{m(n)}\} = \text{span}\{v_1, \ldots, v_n\}$.
2. If $v_n \in \text{span}\{v_1, \ldots, v_{n-1}\}$, we have
$$\{e_1, \ldots, e_{m(n)}\} = \{e_1, \ldots, e_{m(n-1)}\} \cup \emptyset.$$
Otherwise, we have
$$\{e_1, \ldots, e_{m(n)}\} = \{e_1, \ldots, e_{m(n-1)}\} \cup e_{m(n)}$$
for some $e_{m(n)} \in H$.

Prove this by induction. For the base case, let $e_1 = \frac{v_1}{\|v_1\|}$. For the inductive step, suppose the claim holds for $n = k$. If $v_{k+1} \in \text{span}\{v_1, \ldots, v_k\}$, then
$$\text{span}\{e_1, \ldots, e_{n(k)}\} = \text{span}\{v_1, \ldots, v_k\} = \text{span}\{v_1, \ldots, v_{k+1}\}.$$
Now suppose $v_{k+1} \notin \text{span}\{v_1, \ldots, v_k\}$. Define
$$w_{k+1} = v_{k+1} - \sum_{j=1}^{m(k)} \langle v_{k+1}, e_j \rangle e_j.$$
Note that $w_{k+1} \neq 0$ and define $e_{m(k+1)} = \frac{w_{k+1}}{\|w_{k+1}\|}$. Then, $\|e_{m(k+1)}\| = 1$ and for all $1 \leq l \leq m(k)$,
$$\begin{aligned} \langle e_{m(k+1)}, e_l \rangle &= \frac{1}{\|w_{k+1}\|} \left\langle v_{k+1} - \sum_{j=1}^{m(k)} \langle v_{k+1}, e_j \rangle, e_l \right\rangle \\ &= \frac{1}{\|w_{k+1}\|} \left(\langle v_{k+1}, e_l \rangle - \langle v_{k+1}, e_l \rangle \right) \\ &= 0. \end{aligned}$$
Therefore, $e_{m(k+1)}$ is the desired vector we want and we have completed the proof for the claim.

Now let
$$S = \bigcup_{n=0}^{\infty} \{e_1, \ldots, e_{m(n)}\}.$$
Then S is a countable orthonormal subset of H. Now we show S is maximal. Suppose $u \in H$ and $\langle u, e_l \rangle = 0$. Since $\{v_j\}_{j=0}^{\infty}$ is dense in H, there exists $\{v_{j(k)}\}_{k=0}^{\infty}$ such that $v_{j(k)} \to u$ as $k \to \infty$. By our claim, we know $v_{j(k)} \in \text{span}\{e_1, \ldots, e_{m(j(k))}\}$. By Bessel's inequality,
$$\|v_{j(k)}\|^2 = \sum_{l=1}^{m(j(k))} |\langle v_{j(k)}, e_l \rangle|^2 = \sum_{l=1}^{m(j(k))} |\langle v_{j(k)} - u, e_l \rangle|^2 \leq \|v_{j(k)} - u\|^2,$$
Since $v_{j(k)} \to u$ as $k \to \infty$, this implies that $\|v_{j(k)}\| \to 0$ as $k \to \infty$ and thus $\|u\| = 0$, completing the proof that S is a maximal orthonormal subset of H. □

2.2 Orthonormal bases and Fourier Series