Mathematical Studies Analysis

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Contents

1 Advanced topics in metric space theory

1.1 Baire category

Definition. Let X be a metric space.

1. We say that $E \subset X$ is nowhere dense if $(\overline{E})^\circ = \emptyset$.
2. We say that $E \subset X$ is meager in X if
$$E = \bigcup_{\alpha \in A} E_\alpha,$$
where A is a countable set and $E_\alpha \subset X$ is nowhere dense for every $\alpha \in A$.

Theorem. Prove that the following are equivalent for $E \subset X$:

1. E is nowhere dense
2. $\overline{E}$ is nowhere dense
3. $(\overline{E})^c$ is open and dense in X.

Proof. (1) $\implies$ (2). Suppose E is nowhere dense, then $(\overline{E})^\circ = \emptyset$. Note that the closure of $\overline{E}$ is just $\overline{E}$ itself. It follows that $\overline{E}$ is also nowhere dense.

(2) $\implies$ (3). Suppose $\overline{E}$ is nowhere dense. Note that $\overline{E}$ is closed, so $(\overline{E})^c$ is open. Let $x \in X$ be arbitrary. Since $\overline{E}$ is nowhere dense, $x \notin (\overline{E})^\circ$. This implies that for arbitrary $\varepsilon > 0$, we have $B(x,\varepsilon) \not\subset \overline{E}$. This is equivalent to $B(x,\varepsilon) \cap (\overline{E})^c \neq \emptyset$. Hence, $(\overline{E})^c$ is dense in X.

(3) $\implies$ (1). Suppose $(\overline{E})^c$ is dense in X. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. It follows that $B(x,\varepsilon) \cap (\overline{E})^c \neq \emptyset$. This is equivalent to $B(x,\varepsilon) \not\subset \overline{E}$. Therefore, $(\overline{E})^\circ = \emptyset$ and E is nowhere dense. □

Theorem (Baire category thorem)**.** Let X be a complete metric space. Suppose that for each $n \in \mathbb{N}$, $U_n \subset X$ is open and dense in X. Prove that $\bigcap_{n=0}^\infty U_n$ is dense in X. Hint: use the shrinking closed set property.

Proof. Consider any $x \in X$ and arbitrary $\varepsilon > 0$, it suffices to show that $U_n \cap B(x,\varepsilon) \neq \emptyset$ for each $n \in \mathbb{N}$. Now inductively choosing a sequence $x_i \in X$ and $\varepsilon_i > 0$ such that for each $i \in \mathbb{N}$, $B[x_i,\varepsilon_i] \subset U_i$, $B[x_{i+1},\varepsilon_i] \subset B[x_i,\varepsilon_i] \subset B(x,\varepsilon)$, and $\varepsilon_i < 2^{-i}\varepsilon$.

Since U_0 is dense in X, $B(x,\varepsilon) \cap U_0 \neq \emptyset$. Note that both U_0 and $B(x,\varepsilon)$ are open, so we can choose $x_0 \in B(x,\varepsilon) \cap U_0$ and $\varepsilon_0 > 0$ so small that $B[x_0,\varepsilon_0] \subset B(x,\varepsilon) \cap U_0$ and $\varepsilon_0 < \varepsilon$. Now suppose for $0 \leq i \leq n$, we have chosen $x_i \in X$ and $\varepsilon_i > 0$ such that $B[x_i,\varepsilon_i] \subset U_i$ and $\varepsilon_i < 2^{-i}\varepsilon$ for all $0 \leq i \leq n$, $B[x_{i+1},\varepsilon_{i+1}] \subset B[x_i,\varepsilon_i]$ for all $0 \leq i < n$. Since U_{n+1} is dense in X, $B(x_n,\varepsilon_n) \cap U_{n+1} \neq \emptyset$. Note also both U_{n+1} and $B(x_n,\varepsilon_n)$ are open. Therefore, choose $x_{n+1} \in B(x_n,\varepsilon_n) \cap U_{n+1}$ and $\varepsilon_{n+1} > 0$ so small that $B[x_{n+1},\varepsilon_{n+1}] \subset B(x_n,\varepsilon_n) \cap U_{n+1}$ and $\varepsilon_{n+1} < \frac{\varepsilon_n}{2}$. It follows that $B[x_{n+1},\varepsilon_{n+1}] \subset U_{n+1}$ and $B[x_{n+1},\varepsilon_{n+1}] \subset B[x_n,\varepsilon_n] \subset B(x,\varepsilon)$. Also, $\varepsilon < \frac{\varepsilon_n}{2} < 2^{-n-1}\varepsilon$. Now we have successfully constructing the desired sequence.

Since X is complete, $\bigcap_{n=0}^\infty B[x_n,\varepsilon_n] = \{z\}$ for some $z \in X$. Note that for each n, we have $z \in B[x_n,\varepsilon_n] \subset U_n$. Also, $z \in B[x_n,\varepsilon_n] \subset B(x,\varepsilon)$. Therefore, $z \in U_n \cap B(x,\varepsilon)$ for each $n \in \mathbb{N}$ and $\bigcap_{n=0}^\infty U_n$ is dense in X. □

Remark. An equivalent statement of the theorem is the following:
Let X be a complete metric space and $\{C_n\}$ a countable collection of closed subsets of X such that $X = \bigcup_{n\in\mathbb{N}} C_n$. Then at least one of the C_n contains an open ball.

1.2 Open mapping theorem

Linear surjections

Theorem (Open mapping theorem)**.** Let X, Y be Banach spaces over a common field and assume that $T \in \mathcal{L}(X;Y)$. Prove that the following are equivalent.

1. T is surjective.
2. There exists $\delta > 0$ such that $B_Y(0,\delta) \subset \overline{T(B_X(0,1))}$.
3. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $B_Y(0,\delta) \subset T(B_X(0,\varepsilon))$.
4. T is an open map: if $U \subset X$ is open, then $T(U) \subset Y$ is open.
5. There exists $C \geq 0$ such that for each $y \in Y$ there exists $x \in X$ such that $Tx = y$ and
$$\|x\|_X \leq C \|y\|_Y .$$

HINT: Prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (1), keeping in mind the following suggestions.

1. For (1) $\implies$ (2): Study the sets $C_n = \overline{T(B_X(0,n))} \subset Y$ for $n \geq 1$.
2. For (2) $\implies$ (3): Prove that $\overline{T(B_X(0,1))} \subset T(B_X(0,3))$ by considering $y \in \overline{T(B_X(0,1))}$ and inductively constructing $\{x_j\}_{j=0}^\infty \subset X$ such that $\|x_j\|_X < 2^{-j}$ and $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$ for all $m \in \mathbb{N}$.

Proof. (1) $\implies$ (2). Following the hint, for $n \geq 1$ let $C_n = \overline{T(B_X(0,n))}$. Then each of the C_n are closed. Since T is surjective, $Y = \bigcup_{n=1}^\infty C_n$. Suppose for contradiction that each C_n are nowhere dense. It then follows that C_n^c are dense in Y. By Baire Category Theorem, $\bigcap_{n=1}^\infty C_n^c$ is dense in Y. However, $\bigcap_{n=1}^\infty C_n^c = \left(\bigcup_{n=1}^\infty C_n\right)^c = \emptyset$, a contradiction. Therefore, at least one C_n is not nowhere dense. That is, there exists some $n \geq 1$, $\overline{T(B_X(0,n))}$ contains an open ball. However, this is the same set as $n\overline{T(B_X(0,1))}$. Therefore, $\overline{T(B_X(0,1))}$ contains an open ball $B_Y(y_0, 4r)$ for some $y_0 \in Y$ and $r > 0$.

Let $y_1 = Tx_1$ for some $x_1 \in B_Y(0,1)$ such that $\|y_0 - y_1\| < 2r$. It follows that $B_Y(y_1, 2r) \subset B_Y(y_0, 4r) \subset \overline{T(B_X(0,1))}$. For any $y \in Y$ such that $\|y\| < r$, we have
$$y = -\frac{1}{2}y_1 + \frac{1}{2}(2y + y_1) = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1).$$
However, notice that
$$\frac{1}{2}(2y + y_1) \subset \frac{1}{2}B_Y(y_1, 2r) \subset \frac{1}{2}\overline{T(B_X(0,1))} = \overline{T(B_X(0,\tfrac{1}{2}))}.$$
It follows that
$$y = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1) \in -T\left(\frac{x_1}{2}\right) + \overline{T(B_X(0,\tfrac{1}{2}))}.$$
Note that $-T(\frac{x_1}{2}) \in T(B_X(0,\frac{1}{2}))$. Therefore, $y \in \overline{T(B_X(0,1))}$. Since y is arbitrary with $\|y\| < r$, we have $B_Y(0,r) \subset \overline{T(B_X(0,1))}$.

(2) $\implies$ (3). Following the hint, we first show $\overline{T(B_X(0,1))} \subset T(B_X(0,3))$. By assumption, we have $B_Y(0,R) \subset \overline{T(B_X(0,1))}$ for some $R > 0$. It follows from homogeneity that for each $m \in \mathbb{N}$, we have
$$2^{-m}B_Y(0,R) = B_Y(0, 2^{-m}R) \subset 2^{-m}\overline{T(B_X(0,1))} = \overline{T(B_X(0,2^{-m}))}.$$
Let $y \in \overline{T(B_X(0,1))}$ and pick $x_0 \in X$ with $\|x\| < 1$ such that $\|y - Tx\| < 2^{-1}R$. Now suppose we have chosen x_j for $0 \leq j \leq m$ such that $\|x_j\| < 2^{-j}$ and $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$ for all $m \in \mathbb{N}$. By the inclusion above, we can pick $x_{m+1} \in X$ with $\|x_{m+1}\| < 2^{-m-1}$ such that
$$\left\| y - \sum_{j=0}^m Tx_j - Tx_{m+1} \right\| = \left\| y - \sum_{j=0}^{m+1} Tx_j \right\| < 2^{-m-2}R.$$

Therefore, $y - \sum_{j=0}^{m+1} Tx_j \in B_Y(0, 2^{-m-2})R$. This completes the inductive construction, and we have found a sequence $\{x_j\}$ such that $\|x_j\| < 2^{-j}$ and $y - \sum_{j=0}^{m} Tx_j \in B_Y(0, 2^{-m-1}R)$ for each $m \in \mathbb{N}$. Note that

$$\sum_{j=0}^{\infty} \|x_j\| \leq \sum_{j=0}^{\infty} 2^{-j} = 2,$$

so $\sum_{j=0}^{\infty} x_j$ converges absolutely. Since X is Banach, $\sum_{j=0}^{\infty} x_j$ converges to some $x \in X$ with $\|x\| \leq 2$. Also, since $y - \sum_{j=0}^{m} Tx_j \in B_Y(0, 2^{-m-1}R)$, taking the limit where m approaches infinity we obtain

$$y = \sum_{j=0}^{\infty} Tx_j = T\left(\sum_{j=0}^{\infty} x_j\right) = Tx.$$

Therefore, $y \in T(B_X(0,3))$ and thus $\overline{T(B_X(0,1))} \subset T(B_X(0,3))$.

Now for every $\varepsilon > 0$, we have $\frac{\varepsilon}{3}\overline{T(B_X(0,1))} \subset \frac{\varepsilon}{3}T(B_X(0,3)) = T(B_X(0,\varepsilon))$. By assumption, there exists $\delta > 0$ such that $B_Y(0,\delta) \subset \overline{T(B_X(0,1))}$. Therefore,

$$B_Y\left(0, \frac{\delta\varepsilon}{3}\right) = \frac{\varepsilon}{3}B_Y(0,\delta) \subset \frac{\varepsilon}{3}\overline{T(B_X(0,1))} \subset T(B_X(0,\varepsilon)).$$

(3) $\implies$ (4). Let $U \subset X$ be open and $y \in T(U)$. There exists $x \in U$ such that $Tx = y$. Since U is open, there exists $\varepsilon > 0$ such that $B_X(x,\varepsilon) \subset U$. By assumption, there exists $\delta > 0$ such that $B_Y(0,\delta) \subset T(B_X(0,\varepsilon))$. It follows that

$$B_Y(y,\delta) = y + B_Y(0,\delta) \subset Tx + T(B_X(0,\varepsilon)) = T(x + B_X(0,\varepsilon)) \subset T(U).$$

Therefore, $T(U)$ is open and T is an open map.

(4) $\implies$ (5). Since T is an open map, $T(B_X(0,1))$ is open. Also, $T(0) = 0$ so there exists $r > 0$ such that $B_Y(0,r) \subset T(B_X(0,1))$. Now let $y \in Y$. Then, $\frac{r}{2\|y\|}y \in B_Y(0,r)$ and there exists $x \in B_X(0,1)$ such that $Tx = \frac{r}{2\|y\|}y$. It follows that

$$T\left(\frac{2\|y\|}{r}x\right) = y,$$

and since $x \in B_X(0,1)$,

$$\left\|\frac{2\|y\|}{r}x\right\| = \frac{2\|y\|\|x\|}{r} < \frac{2}{r}\|y\|.$$

Letting $C = \frac{2}{r}$ completes the proof.

(5) $\implies$ (1). Since for each $y \in Y$ there exists $x \in X$ such that $Tx = y$, T is surjective. □

Linear homeomorphisms, norm equivalence, and closed graphs

Theorem. Let X and Y be Banach spaces and suppose that $T \in \mathcal{L}(X,Y)$ is a bijection. Prove that $T^{-1} \in \mathcal{L}(Y,X)$, and in particular T is a linear (and thus bi-Lipschitz) homeomorphism.

Proof. Since $T \in \mathcal{L}(X,Y)$ is a bijection, T is a surjection. It follows that T is an open map. In particular, for any $U \subset X$ open, $T(U) = (T^{-1})^{-1}(U)$ is open. Therfore, T^{-1} is continuous and thus T is a linear homeomorphism. □

Theorem. Let X be a vector space that is complete when equipped with both of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that if there exists a constant $C_1 > 0$ such that $\|x\|_2 \leq C_1\|x\|_1$ for all $x \in X$, then there exists a constant $C_0 > 0$ such that $C_0\|x\|_1 \leq \|x\|_2 \leq C_1\|x\|_1$ for all $x \in X$.

Proof. Let $T : X_1 \to X_2$, where X_1 and X_2 are X equipped with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, be the identity map. Then for any $x \in X$ with $\|x\|_1 = 1$, we have

$$\|Tx\|_2 = \|x\|_2 \leq C_1 \|x\|_1 = C_1.$$

Therefore, $T \in \mathcal{L}(X_1, X_2)$. T is also surjective. Therefore, there exists a constant $C \geq 0$ such that each $\|x\|_1 \leq C \|x\|_2$. Hence, for each $x \in X$

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C_1 \|x\|_1 .$$

Letting $C_0 = \frac{1}{C}$ completes the proof. □

Theorem. Let X and Y be Banach spaces and let $T : X \to Y$ be linear (just the algebraic condition). Prove that the following are equivalent

1. T is continuous, i.e. $T \in \mathcal{L}(X; Y)$.
2. The graph of T, $\Gamma(T) = \{(x, Tx) : x \in X\} \subset X \times Y$, is closed in $X \times Y$, where $X \times Y$ is endowed with any of the usual p-norms.

Proof. (a) $\implies$ (b). Let $\{(x_n, Tx_n)\}$ be a convergent sequence in $\Gamma(T)$. Since X is Banach, $x_n \to x$ for some $x \in X$. Since $T \in \mathcal{L}(X; Y)$, we have

$$\lim_{n\to\infty} Tx_n = T\left(\lim_{n\to\infty} x_n\right) = Tx.$$

Therefore, $(x_n, Tx_n) \to (x, Tx) \in \Gamma(T)$, and thus $\Gamma(T)$ is closed.

(b) $\implies$ (a). Let $\pi_1 : \Gamma(T) \to X$ and $\pi_2 : \Gamma(T) \to Y$ by $\pi_1(x, Tx) = x$ and $\pi_2(x, Tx) = Tx$. Since $\Gamma(T)$ is a closed in Banach space Y, $\Gamma(T)$ is Banach space. It is clear that both π_1 and π_2 are bounded linear maps. Moreover, π_1 is a bijection. It follows that $S = \pi_1^{-1}$ is a bounded linear map. Therefore, $T = \pi_2 \circ S$ is a bounded linear map. □

Linear injections with closed range

Theorem. Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Prove the following are equivalent.

1. T is injective and range(T) is closed.
2. $T : X \to \text{range}(T)$ is a linear homeomorphism.
3. There exists $C \geq 0$ such that $\|x\|_X \leq C \|Tx\|_Y$ for all $x \in X$.

HINT: Prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

Proof. (1) $\implies$ (2). If T is injective and range(T) is closed, then $\Gamma(T) = \{(x, Tx) : x \in X\}$ is closed in $X \times Y$. Therefore, $T : X \to \text{range}(T)$ is a bounded linear map. Since T is injective, this map is actually bijective from X to range(T). Therefore, T is a linaer homeomorphism.

(2) $\implies$ (3). Since T is a bijective bounded linear map, from X to range(T). There exists a contant $C \geq 0$ such that for each $y \in \text{range}(T)$ there exists a unique $x \in X$ such that $Tx = y$ and $\|x\| \leq C \|y\| = C \|Tx\|$. Since T is a bijection, $\|x\| \leq C \|Tx\|$ for all $x \in X$.

(3) $\implies$ (1). Let $x \in X$ be such that $Tx = 0$. It follows that $\|x\| \leq C \|Tx\| = 0$. Therefore, $x = 0$ and T is injective. To show that range(T) is closed, consider a convergent sequence $\{y_n\} \subset \text{range}(T)$ with $y_n = Tx_n$. Since for any $n, m \in \mathbb{N}$ we have

$$\|x_n - x_m\| \leq C \|T(x_n - x_m)\| = C \|y_n - y_m\| ,$$

$\{x_n\}$ is Cauchy. Since X is Banach, $x_n \to x$ for some $x \in X$. Therefore, for all $n \in \mathbb{N}$ we have

$$\|y_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \|x_n - x\| ,$$

and $y_n \to Tx$. Hence, range(T) is closed and the proof is complete. □

Theorem. Let X and Y be Banach spaces over a common field. Then, the following subsets of $\mathcal{L}(X;Y)$ are open:

1. $\{T \in \mathcal{L}(X;Y) : T \text{ is surjective}\}$,
2. $\{T \in \mathcal{L}(X;Y) : T \text{ is injective with closed range}\}$,
3. $\mathcal{H}(X;Y) = \{T \in \mathcal{L}(X;Y) : T \text{ is a homeomorphism}\}$.

Proof. 1. Let $T \in \mathcal{L}(X;Y)$ be surjective. By open mapping theorem, there is $\delta > 0$ such that $B_Y(0,\delta) \subset TB_X(0,1)$. By homogeneity we have $B_Y(0,r) \subset TB_X(0,\alpha r)$ for all $r > 0$ where $\alpha = \delta^{-1}$. Now let $S \in \mathcal{L}(X;Y)$ be such that $\|T - S\| < \beta < (2\alpha)^{-1}$. Claim S is surjective.

Let $y \in Y$, inductively construct sequences $\{x_n\}$ and $\{y_n\}$. First let $y_0 = y$. Then, $\|y_0\| \in B(0, 2\|y_0\|)$. Select $x_0 \in X$ be such that $Tx_0 = y_0$ and $\|x_0\| \leq 2\alpha \|y_0\|$. Suppose we have selected y_i, x_i for $0 \leq i \leq n$. Set $y_{n+1} = y_n - Sx_n$ and select x_{n+1} be such that $Tx_{n+1} = y_{n+1}$ and $\|x_{n+1}\| \leq 2\alpha \|y_{n+1}\|$. Then, we have

$$\|y_{n+1}\| = \|Tx_n - Sx_n\| \leq \|T - S\| \|x_n\| < 2\alpha\beta \|y_n\|$$

and

$$\|x_{n+1}\| = 2\alpha \|y_{n+1}\| \leq 2\alpha \|T - S\| \|x_n\| < 2\alpha\beta \|x_n\|.$$

Note that $2\alpha\beta < 1$ and X is Banach, define

$$x = \sum_{n=0}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=0}^{N} x_n.$$

Also note that $\lim_{n\to\infty} y_n = 0$. It follows that

$$Sx = \sum_{n=0}^{\infty} Sx_n = \sum_{n=0}^{\infty} (y_n - y_{n+1}) = y_0 - \lim_{n\to\infty} y_{n+1} = y.$$

Therefore S is surjective and the set of surjective bounded linear maps are open.

2. Suppose $T \in \mathcal{L}(X;Y)$ is injective with closed range. Then, closed range theorem gives $C > 0$ such that $\|x\| \leq C\|Tx\|$ for all $x \in X$. Now supose $S \in \mathcal{L}(X;Y)$ is such that $\|T - S\| < (2C)^{-1}$. Claim that S is also injective with closed range. Indeed,

$$\begin{aligned}\|x\| &\leq C\|Tx\| \leq C\|Sx\| + C\|(T-S)x\| \\ &\leq C\|Sx\| + \frac{1}{2}\|x\|.\end{aligned}$$

This shows that $\|x\| \leq 2C\|Sx\|$ for all $x \in X$. By closed range theorem, S is injective with closed range. This implies that the set of injective bounded linear operator with closed range is open.

3. This directly follows from

$$\mathcal{H}(X;Y) = \{T \in \mathcal{L}(X;Y) : T \text{ is surjective}\} \cap \{T \in \mathcal{L}(X;Y) : T \text{ is injective with closed range}\}.$$

□

Theorem. Let X and Y be Banach spaces over a common field. Then the following holds.

1. The sets

$$\mathcal{L}_R(X;Y) = \{T \in \mathcal{L}(X;Y) : \text{there exists } S \in \mathcal{L}(Y;X) \text{ such that } ST = I_X\}$$

and

$$\mathcal{L}_L(X;Y) = \{T \in \mathcal{L}(X;Y) : \text{there exists } S \in \mathcal{L}(Y;X) \text{ such that } TS = I_Y\}$$

are open.

2. The following inclusion holds:

$$\mathcal{L}_L(X;Y) \subset \{T \in \mathcal{L}(X;Y) : T \text{ is surjective}\}$$

and

$$\mathcal{L}_R(X;Y) \subset \{T \in \mathcal{L}(X;Y) : T \text{ is injective with closed range}\}.$$

3. The sets $\mathcal{L}_L(X;Y) \setminus \mathcal{L}_R(X;Y)$ and $\mathcal{L}_R(X;Y) \setminus \mathcal{L}_L(X;Y)$ are open.

Proof. 1. Let $T_0 \in \mathcal{L}_R$ and $S_0 \in \mathcal{L}(Y;X)$ be such that $T_0S_0 = I_Y$. Note that $I_X \in \mathcal{H}(X)$ and when $\|P\| < 1$ for $P \in \mathcal{L}(X)$, we have $I_X + P \in \mathcal{H}(X)$. Suppose now $T \in \mathcal{L}(X;Y)$ and $\|T\| < \|S_0\|^{-1}$. It follows that $I_X + S_0T \in \mathcal{H}(X)$. For such T, we then have

$$T_0 + T = T_0(I_X + S_0T).$$

Also,

$$(T_0 + T)(I_X + S_0T)^{-1}S_0 = T_0(I_X + S_0T)(I_X + S_0T)^{-1}S_0 = T_0S_0 = I_Y.$$

Therefore, $T_0 + T \in \mathcal{L}_R$ for $T \in B(T_0, \|S_0\|^{-1})$ and $\mathcal{L}_R$ is open.

Now let $T_0 \in \mathcal{L}_L$ and $S_0 \in \mathcal{L}(Y;X)$ be such that $S_0T_0 = I_X$. Again, for $T \in \mathcal{L}(X;Y)$ with $\|T\| < \|S_0\|^{-1}$, we have

$$T_0 + T = (I_X + TS_0)T_0.$$

and

$$S_0(I_X + TS_0)^{-1}(T_0 + T) = I_X.$$

Therefore, $\mathcal{L}_R$ is also open.

2. Let $T \in \mathcal{L}_R$ and $S \in \mathcal{L}(Y;X)$ be such that $TS = I_Y$. Then for any $y \in Y$ let $x = Sy$. It follows that $Tx = TSy = y$. Also, $\|x\| \leq \|S\| \|y\|$ so the 4th item in open mapping theorem guarantees that T is surjective. Hence, $\mathcal{L}_L \subset \{T \in \mathcal{L}(X;Y) : T \text{ is surjective}\}$.

Now let $T \in \mathcal{L}_L$ and $S \in \mathcal{L}(Y;X)$ such that $ST = I_X$. Now for any $x \in X$, we have $\|x\| = \|STx\| \leq \|S\| \|Tx\|$. Then the closed range theorem guarantees that T is injective with closed range. Hence, $\mathcal{L}_R \subset \{T \in \mathcal{L}_R(X;Y) : T \text{ is injective with closed range}\}$.

3. ***TO-DO***

□

1.3 Hahn-Banach theorem and duality

Theorem (Hahn-Banach theorem in $\mathbb{R}$)**.** Let X be a real vector space and suppose $p : X \to \mathbb{R}$ is such that

$$p(tx + (1-t)y) \leq tp(x) + (1-t)p(y)$$

for all $t \in [0,1]$ and $x, y \in X$.

Suppose Y subspace of X and $l : Y \to \mathbb{R}$ is a linear map such that $l \leq p$ on Y. Then there exists linear map $L : X \to \mathbb{R}$ such that $L \leq p$ on X and $L = l$ on Y.

Proof. Let

$$P = \{(Z, \lambda) : \ Y \subset Z \subset X, \ \lambda \text{ linear functional on } Z, \ \lambda \leq p \text{ on } Z \text{ and } l = \lambda \text{ on } Y\}$$

Define partial order $(Z_1, \lambda_1) \preceq (Z_2, \lambda_2)$ if and only if $Z_1 \subset Z_2$ and $\lambda_1 = \lambda_2$ on Z_1. It is easy to verify that this is a partial order. Towards using Zorn's Lemma, let $C \subset P$ be a chain and define

$$U = \bigcup_{(Z,\lambda)\in C} Z, \qquad \Lambda = \bigcup_{(Z,\lambda)\in C} \lambda.$$

It is easy to verify that (U, Λ) is an upper bound for the chain. By Zorn's Lemma, P has a maximal element (M, L). It remains to show thta $M = X$.

Suppose for contradiction that $M \neq X$. Pick $x_0 \in X \setminus M$. For any $x, y \in M$, we have

$$\begin{aligned}
\beta L(x) + \alpha L(y) &= L(\beta x + \alpha y) \\
&= \frac{1}{\alpha+\beta} L\left(\frac{\beta}{\alpha+\beta}x + \frac{\alpha}{\alpha+\beta}y\right) \\
&\leq (\alpha+\beta) p\left(\frac{\beta}{\alpha+\beta}x + \frac{\alpha}{\alpha+\beta}y\right) \\
&= (\alpha+\beta) p\left(\frac{\beta}{\alpha+\beta}(x - \alpha x_0) + \frac{\alpha}{\alpha+\beta}(y + \beta x_0)\right) \\
&\leq \beta p(x - \alpha x_0) + \alpha p(y + \beta x_0).
\end{aligned}$$

This implies that

$$\sup_{\substack{x\in M \\ \alpha>0}} \frac{1}{\alpha}\left[L(x) - p(x - \alpha x_0)\right] \leq \inf_{\substack{y\in M \\ \beta>0}} \frac{1}{\beta}\left[p(y+\beta x_0) - L(y)\right].$$

Note that $-p(-x_0) \leq \text{LHS}$ and $\text{RHS} \leq p(x_0)$, so $\text{LHS}, \text{RHS} < \infty$. Now pick $v \in \mathbb{R}$ such that $\text{LHS} \leq v \leq \text{RHS}$. For $x \in M$ and $0 < t \in \mathbb{R}$ we have

$$L(x) - tv \leq p(x - tv_0), \qquad L(x) + tv \leq p(x + tv_0).$$

Now define $\widehat{L} : M \oplus \mathbb{R}x_0 \to \mathbb{R}$ by $\widehat{L}(x + \alpha x_0) = L(x) + \alpha v$. It follows that $(M \oplus \mathbb{R}x_0, \widehat{L}) \in P$. However, $(M, L) \prec (M \oplus \mathbb{R}, \widehat{L})$, a contradiction. Therefore, $M = X$ and the proof is complete.

□

Theorem (Hahn-Banach theorem in $\mathbb{C}$)**.** Let X be complex vector space and suppose $p : X \to \mathbb{R}$ is such that

$$p(\alpha x + \beta y) \leq |\alpha|\, p(x) + |\beta|\, p(y)$$

for all $\alpha, \beta \in \mathbb{C}$ such that $|\alpha| + |\beta| = 1$ and $x, y \in X$.

Suppose Y subspace of X and $l : Y \to \mathbb{C}$ is a linear map such that $|l| \leq p$ on Y. Then there exsits linear map $L : X \to \mathbb{C}$ such that $|L| \leq p$ on X and $L = l$ on Y.

Proof. Define $\lambda : Y \to \mathbb{R}$ by $\lambda(x) = \operatorname{Re}(l(x))$. Note that

$$\lambda(ix) = \operatorname{Re}(il(x)) = -\operatorname{Im}(l(x)).$$

This implies that $l(x) = \lambda(x) - i\lambda(ix)$. Now treat X and Y as vector space over $\mathbb{R}$ and apply Hahn-Banach theorem in $\mathbb{R}$ to extend λ to $\Lambda : X \to \mathbb{R}$ that agrees with λ on Y.

Define $L : X \to \mathbb{C}$ by $L(x) = \Lambda(x) - i\Lambda(ix)$. It remains to show that $|L| \leq p$. For $x \in X$, write $L(x) = |L(x)|\, e^{i\theta}$ for some $\theta \in \mathbb{R}$. It follows that

$$\begin{aligned}
|L(x)| &= L(x)e^{-i\theta} \\
&= \Lambda(e^{-i\theta}) - i\Lambda(ie^{-i\theta}x) \\
&= \Lambda(e^{-i\theta}x) \\
&\leq p(e^{-i\theta x}) \\
&\leq \left|e^{-i\theta}\right| p(x) \\
&= p(x),
\end{aligned}$$

as desired.

□

Theorem (Hahn-Banach theorem for bounded linear functionals)**.** Let X be a normed vector space over $\mathbb{F}$ and Y a subspace of X. If $\lambda \in Y^*$ then there exists $\Lambda \in X^*$ such that $\Lambda = \lambda$ on Y and the operator norm $\|\lambda\|_{Y^*} = \|\Lambda\|_{X^*}$.

Proof. Consider $p : X \to \mathbb{R}$ where $p(x) = \|\lambda\|_{Y^*} \|x\|$. Apply Hahn-Banach theoerm. □

Next we show some useful implications of Hahn-Banach theorem.

Theorem. Let X be a normed vector space and fix $x \in X$. Then the following holds:

1. There exists $\lambda \in X^*$ such that $\|\lambda\| = \|x\|$ and
$$\lambda(x) = \|\lambda\| \|x\| = \|x\|^2 .$$
2. We have
$$\|x\| = \max_{\substack{w \in X^* \\ \|w\|=1}} |w(x)| .$$
3. $x = 0$ if and only if $w(x) = 0$ for all $w \in X^*$.

Proof.
1. Let $Y = \mathbb{F}x$ and define $\lambda \in Y^*$ by $\lambda(ax) = a\|x\|^2$. Apply Hahn-Banach theorem.
2. Suppose $x \neq 0$. Define $w = \frac{\lambda}{\|x\|}$ then it follows that $|w(x)| = \|x\|$.
3. Follows directly from (2).

□

Proposition. Let X be normed vector space. Then the mapping $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{F}$ by $(w, x) \mapsto w(x)$ is a bilinear map. That is, $\langle \cdot, \cdot \rangle \in \mathcal{L}(X^*, X; \mathbb{F})$. Moreover, if $X \neq \{0\}$, then $\|\langle \cdot, \cdot \rangle\| = 1$.

Proof. It is easy to see that $\langle \cdot, \cdot \rangle$ is bilinear. For boundedness,
$$|\langle w, x \rangle| = |w(x)| \leq \|w\| \|x\| .$$
Hence, $\|\langle \cdot, \cdot \rangle\| \leq 1$. Meanwhile, pick some $x \in X$ with $\|x\| = 1$. It follows that
$$1 = \|x\| = \max_{\substack{w \in X^* \\ \|w\|=1}} |w(x)| \leq \|\langle \cdot, \cdot \rangle\| .$$
Therefore, $\|\langle \cdot, \cdot \rangle\| = 1$. □

Definition. Let X be normed vector space and $E \subset X$, $W \subset X^*$. Say W is a *norming set* for E if
$$\|x\| = \sup_{\substack{w \in W \\ \|w\|=1}} |\langle w, x \rangle|$$
for all $x \in E$.

Proposition. Let X be normed vector space and $S \subset X$ be a separable set. Let W be a norming set for S. Then, there exists $\{w_n\}_{n=0}^{\infty} \subset W$ such that $\|w_n\| = 1$, and the sequence is norming for S. That is,
$$\|x\| = \sup_{n \in \mathbb{N}} |\langle w_n, x \rangle| .$$

Proof. Let $\{v_n\}_{n=0}^{\infty} \subset S$ be dense. For any $n, k \in \mathbb{N}$, choose $w_{n,k} \in W$ with $\|w_{n,k}\| = 1$ such that

$$(1 - 2^{-k}) \|v_n\| \leq |w_{n,k}, v_n| .$$

Let $x \in S$ and $0 < \varepsilon < 1$ be arbitrary. Pick $v_n \in S$ such that $\|v_n - x\| < \varepsilon$ and pick $j \in \mathbb{N}$ such that $2^{-j} < \varepsilon$. Then,

$$\begin{aligned}(1-\varepsilon)\|x\| &\leq (1-2^{-j})\|x\| \\ &\leq (1-2^{-j})\|v_n\| + (1-2^{-j})\|v_n - x\| \\ &\leq |\langle w_{n,j}, v_j\rangle| + \varepsilon \\ &\leq |\langle w_{n,j}, x\rangle| + |\langle w_{n,j}, x - v_n\rangle| + \varepsilon \\ &\leq |\langle w_{n,j}, x\rangle| + 2\varepsilon.\end{aligned}$$

This shows that $\{w_{n,k}\}_{n,k=0}^{\infty}$ is a norming sequence.

□

Theorem. Let X be normed vector space and define $J : X \to X^{**}$ by $\langle Jx, w\rangle = \langle w, x\rangle = w(x)$. Then the following holds:

1. $J \in \mathcal{L}(X, X^{**})$.
2. J is an isometric embedding. In particular, it is injective.
3. range$(J) \subset X^{**}$ is a norming set for X^*.
4. X is Banach if and only if range(J) is closed.

Proof. Note that we have

$$\begin{aligned}\|Jx\|_{X^{**}} &= \sup\{|\langle Jx, w\rangle| : w \in X^* \text{ and } \|w\| \leq 1\} \\ &= \sup\{|\langle w, x\rangle| : w \in X^* \text{ and } \|w\| \leq 1\} \\ &= \|x\|,\end{aligned}$$

where the last step is by a previous theorem that shows the existence of $w \in X^*$ such that $\|w\| = 1$ and $|w(x)| = \|x\|$. This implies (1) and (2). Now we know X is isometrically isomophic to range$(J) \subset X^{**}$. Therefore, X is Banach if and only if range(J) is Banach. However, $X^{**} = \mathcal{L}(X^*, \mathbb{F})$ is Banach, so range(J) is Banach if and only if range(J) is closed. This implies (4).

To show (3), note that we have

$$\begin{aligned}\|w\|_{X^*} &= \sup\{|\langle w, x\rangle| : x \in X \text{ and } \|x\| \leq 1\} \\ &= \sup\{|\langle Jx, w\rangle| : x \in X \text{ and } \|x\| \leq 1\} \\ &= \sup\{|\langle v, w\rangle| : v \in \text{range}(J) \text{ and } \|v\|_{X^{**}} \leq 1\}.\end{aligned}$$

This shows (3), completing the proof. □

2 Differential Calculus

Theorem (Local injectivity theorem)**.** Let X and Y be Banach spaces, $z \in U \subset X$ with U open. Let $f : U \to Y$ differentiable with Df continuous at z. Suppose $Df(z) \in \mathcal{L}(X;Y)$ injective with closed range. Then for any $0 < \varepsilon < 1$, there exists $r > 0$ such that

1. $B[z,r] \subset U$.
2. $Df(x)$ injective with closed range for all $x \in B[z,r]$.
3. If $x, y \in B(z,r)$, then
$$(1-\varepsilon)\left\|Df(z)(x-y)\right\| \leq \left\|f(x)-f(y)\right\| \leq (1+\varepsilon)\left\|Df(z)(x-y)\right\|.$$
4. The restriction $f : B(z,r) \to f(B(z,r))$ is bi-Lipschitz homeomorphism.

Theorem (Local surjectivity thorem)**.** Let X and Y be Banach spaces, $z \in U \subset X$ with U open. Let $f : U \to Y$ differentiable with Df continuous at z. Suppose $Df(z) \in \mathcal{L}(X;Y)$ surjective. Then there exists $r_0,\ \gamma > 0$ such that

1. $B_X[z, r_0] \subset U$.
2. $Df(x)$ surjective for all $x \in B_X[z, r_0]$.
3. $B_Y[f(z), \gamma r] \subset f(B_X[z,r])$ for all $0 \leq r \leq r_0$.

Definition. Let X and Y be normed vector spaces and suppose that $\emptyset \neq U \subset X$ is open. Let $f : U \to Y$. For $k \geq 1$, say f is a C^k diffeomorphism if

1. $f : U \to f(U)$ homeomorphism with $f(U) \subset Y$ open.
2. $f \in C^k(U;Y)$.
3. $f^{-1} \in C^k(f(U);X)$.

If f is a C^k diffeomorphism for all $k \geq 1$, say f is a smooth diffeomorphism.

Theorem (Inverse function theorem)**.** Let X and Y be Banach spaces, $U \subset X$ open and $x_0 \in U$. Suppose $f : U \to Y$ differentiable, Df continuous at x_0, $Df(x_0)$ linear homeomorphism. Then there exists bounded and open $V \subset U$ with $x_0 \in V$ such that

1. $f : V \to f(V)$ is bi-Lipschitz homeomorphism, $Df(x)$ linear homeomorphism for all $x \in V$, $f(V) \subset Y$ bounded and open, $f^{-1} : f(V) \to V$ differentiable with
$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$$
for all $y \in f(V)$ and Df^{-1} is continuous at $f(x_0)$. Also, there exists $C_0,\ C_1 > 0$ such that
$$C_0 \leq \left\|Df(x)\right\| \leq C_1$$
for all $x \in V$, and
$$\frac{1}{C_1} \leq \left\|Df^{-1}(y)\right\| \leq \frac{1}{C_0}$$
for all $y \in f(V)$.
2. If $f \in C^k(U;Y)$ for some $1 \leq k \leq \infty$, then $f^{-1} \in C^k(f(V);X)$.
3. If $f \in C^k(U;Y)$ for $1 \leq k \in \mathbb{N}$, then there exists open $V_k \subset V$ such that $x_0 \in V_k$, $f \in C^k_b(V_k;Y)$ and $f^{-1} \in C^k_b(f(V_k);X)$.

Theorem (Implicit function theorem)**.** Let X and Y be Banach spaces, $U \subset X \times Y$ be open with $(x_0, y_0) \in U$, and suppose $f : U \to Z$ is differentiable in U with Df continuous at (x_0, y_0). Further suppose $z_0 = f(x_0, y_0)$ and $D_2 f(x_0, y_0) \in \mathcal{L}(Y;Z)$ is an isomorphism. Then there exists open sets $x_0 \in V \subset X$, $z_0 \in W \subset Z$, $y_0 \in S \subset Y$, and $g \in C^{0,1}_b(V \times W; Y)$ such that the following holds:

1. $g(x_0, z_0) = y_0$ and $(x, g(x, z)) \in V \times S \subset U$ for all $(x, z) \in V \times W$. Also, g is differentiable on $V \times W$ and Dg continuous at (x_0, z_0).
2. $f(x, g(x, z)) = z$ for all $(x, z) \in V \times W$. Moreover, if $(x, y) \in V \times S$ such that $f(x, y) = z$ for some $z \in W$, then $y = g(x, z)$.
3. $D_2 f(x, g(x, z))$ is an isomorphism for all $(x, z) \in V \times W$, and

$$\begin{aligned} D_1 g(x, z) &= -\left[D_2 f(x, g(x, z))\right]^{-1} D_1 f(x, g(x, z)), \\ D_2 g(x, z) &= \left[D_2 f(x, g(x, z))\right]^{-1}. \end{aligned}$$

4. If $f \in C^k$ then $g \in C^k$ too for $1 \leq k \leq \infty$. If k finite and $f \in C_b^k$ then the sets can be picked such that $g \in C_b^k$.

3 Measure and integration

3.1 Outer measure and the Carathéodory construction

Definition. Let X be a set with outer measure μ^*. Say a set $E \subset X$ is measurable with respect to μ^* if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for all $A \subset X$.

Theorem (Carathéodory construction)**.** Let X be a set with outer measure μ^*, the following holds.

1. The collection $\mathfrak{M} = \{E \subset X : \ E \text{ measurable}\}$ is a σ-algebra.
2. If $E \subset X$ is such that $\mu^*(E) = 0$, then $E \in \mathfrak{M}$.
3. The restriction $\mu = \mu^*|_{\mathfrak{M}}$ is a measure, and $(X, \mathfrak{M}, \mu)$ is a complete measure space.

Definition. Let μ^* be an outer measure on X. Say μ^* is cover-regular if for any $A \subset X$, there exists $E \in \mathfrak{M}$ such that $A \subset E$ and $\mu^*(A) = \mu(E)$.

3.2 Constructing outer measures

Definition. Let X be a set. A gauge on X is a pair $(\mathcal{E}, \gamma)$ where $\mathcal{E} \subset \mathcal{P}(X)$ is such that $\emptyset \in \mathcal{E}$ and $\gamma : \mathcal{E} \to [0, \infty]$ is such that $\gamma(\emptyset) = 0$.

Theorem. Let X be a set and $(\mathcal{E}, \gamma)$ be a gauge on X. Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ via

$$\mu^*(E) = \inf \left\{ \sum_{n=0}^{\infty} \gamma(E_n) : E \subset \bigcup_{n=0}^{\infty} E_n \text{ and } \{E_n\}_{n=0}^{\infty} \subset \mathcal{E} \right\}.$$

Then μ^* is an outer measure on X and hence generates $(X, \mathfrak{M}, \mu)$, a complete measure space thorugh Carathéodory construction.

Theorem. Let (X, d) be a metric space with gauge $(\mathcal{E}, \gamma)$ and outer measures $\mu_\delta^* : \mathcal{P}(X) \to [0, \infty]$ produced by $(\mathcal{E}_\delta, \gamma_\delta)$ for $\delta > 0$. Define $\mu_d^* : P(X) \to [0, \infty]$ by

$$\mu_d^*(A) = \sup_{\delta > 0} \mu_d^*(A).$$

Then μ_d^* is a metric outer measure. Moreover, $\mu_d^*(A) = \lim_{\delta \to 0} \mu_\delta^*(A)$ for $A \subset X$.

Lemma. Let X be a set with gauge $(\mathcal{E}, \gamma)$ that covers X. Let $A \subset X$, then the following holds:

1. Let μ^* be the outer measure generated by $(\mathcal{E}, \gamma)$. Then there exists collection $\{E_{m,n}\}_{m,n=0}^{\infty} \subset \mathcal{E}$ such that $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$ such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$.
2. Suppose (X, d) is metric space and the gauge is fine. Let μ_d^* be the metric outer measure. Then there exists collection $\{E_{m,n}\}_{m,n=0}^{\infty} \subset \mathcal{E}$ such that $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$ such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$.

Proof. The proof for (1) is very similar to the proof for (2), so we only show (2) as follows. Since the gauge is fine, $(\mathcal{E}_\delta, \gamma_\delta)$ covers X for all $\delta > 0$. Then, for any $m \in \mathbb{N}$, there exists $\{E_{m,n}\}_n \subset \mathcal{E}_{2^{-m}}$ such that $A \subset \bigcup_{n=0}^{\infty} E_{m,n}$ and $\sum_{n=0}^{\infty} \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}$. Now let $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$. Note that $A \subset E$ and for any $m \in \mathbb{N}$, we have

$$\mu_{2^{-m}}^*(E) \leq \mu_{2^{-m}}^* \left(\bigcup_{n=0}^{\infty} E_{m,n} \right) \leq \sum_{n=0}^{\infty} \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}.$$

Taking the limit as $m \to \infty$, we have

$$\mu_d^*(E) \leq \mu_d^*(A) \leq \mu_d^*(E),$$

as desired. □

Theorem. Let (X,d) be metric space with $(\mathcal{E},\gamma)$ such that all sets in $\mathcal{E}$ are open. Assume that μ^* is a metric outer measure on X such that either

1. μ^* is generated by $(\mathcal{E},\gamma)$, or
2. $\mu^* = \mu^*_d$ is generated by $(\mathcal{E}_\delta,\gamma_\delta)$.

Further suppose that $X = \bigcup_{n=0}^\infty A_n$ where $A_n \subset X$ is such that $\mu^*(A_n) < \infty$. Then the following holds:

1. The gauge covers X in case 1 and is fine in case 2.
2. In both cases, μ^* is cover-regular. More precisely, for each $A \subset X$, there is $G \in G_\delta(X) \subset \mathfrak{B}(X) \subset \mathfrak{M}$ such that $A \subset G$ and $\mu^*(A) = \mu^*(G)$.
3. In both cases, the following are equivalent for $E \subset X$:
   (a) $E \in \mathfrak{M}$, i.e. E is measurable.
   (b) there exists $G \in G_\delta(X)$ such that $E \subset G$ and $\mu^*(G \setminus E) = 0$.
   (c) there exists $F \in F_\sigma(X)$ such that $F \subset E$ and $\mu^*(E \setminus F) = 0$.

Proof. **Step 0: proof for (1) and (2).**

We know $X = \bigcup_{n=0}^\infty A_n$ for some $\mu^*(A_n) < \infty$. For case (1), we can pick $\{E_{n,m}\} \subset \mathcal{E}$ such that $A_n \subset \bigcup_{m=0}^\infty E_{n,m}$. Then $X = \bigcup_{n=0}^\infty A_n = \bigcup_{n,m} E_{n,m}$. Therefore, $\mathcal{E}$ covers X. For case (2), note that $\mu^*_d(A_n) < \infty$ and $\mu^*_d(A_n) \geq \mu^*_\delta(A_n)$ for each $\delta > 0$ and $n \in \mathbb{N}$. Then for each $\delta > 0$, there exists $\{E_{n,m}\} \subset \mathcal{E}_\delta$ such that $A_n \subset \bigcup_{m=0}^\infty E_{n,m}$. It follows that $X = \bigcup_{n=0}^\infty A_n = \bigcup_{n,m} E_{n,m}$. Therefore, $(\mathcal{E},\gamma)$ is fine.

We have the following observations:

1. μ^* is a metric outer measure. This implies that $\mathfrak{B}(X) \subset \mathfrak{M}$.
2. $G_\delta(X) \cup F_\sigma(X) \subset \mathfrak{B}(X) \subset \mathfrak{M}$ and $\mu^*(A) = 0$ implies $A \in \mathfrak{M}$.
3. By previous lemma and all sets in $\mathcal{E}$ are open, we know for each $A \subset X$ there is $E \in G_\delta(X)$ such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$. In particular, μ^* is cover regular.

Step 1: starting on (3).

For (b) $\implies$ (a), suppose (b) holds for $E \subset X$. Then $E = G \setminus (G \setminus E) \in \mathfrak{M}$ since $\mu^*(G \setminus E) = 0$.

For (c) $\implies$ (a), suppose (c) holds for $E \subset X$. Then $E = F \cup (E \setminus F) \in \mathfrak{M}$ since $\mu^*(E \setminus F) = 0$.

Next we show "(a) $\implies$ (c)" implies "(a) $\implies$ (b)". Suppose $E \in \mathfrak{M}$, then $E^c \in \mathfrak{M}$. By (a) $\implies$ (b) we know there exists $F \in F_\sigma$ such that $F \subset E^c$ and $\mu^*(E^c \setminus F) = 0$. Let $G = F^c \in G_\delta$ then $E \subset G$ and $G \subset E = E^c \subset F$.

Therefore, it remains to show (a) $\implies$ (c) to complete the proof for the theorem.

Step 2: reduction for (a) $\implies$ (c).

Claim it suffices to show it for E such that $\mu^*(E) < \infty$. Suppose we did this and $\mu^*(E) = \infty$. Using observation there exists $B_n \in \mathfrak{M}$ such that $A_n \subset B_n$ and $\mu^*(B_n) = \mu^*(A_n) < \infty$. Then $E_n = E \cap B_n \in \mathfrak{M}$ and $\mu^*(E_n) < \infty$. Then by special case there is $F_n \in F_\sigma(X)$ such that $F_n \subset E_n$ and $\mu^*(F_n \setminus E_n) = 0$. Let $F = \bigcup_{n=0}^\infty F_n \in F_\sigma$ then $F \subset \bigcup_{n=0}^\infty E_n = E$ and

$$\mu^*(E \setminus F) \leq \sum_{n=0}^\infty \mu^*(E_n \setminus F_n) = 0.$$

Step 3: further reduction.

Claim it suffices to show it for the case where $\mu^*(E) < \infty$ and $E \in G_\delta(X)$. Suppose we have proved this and consider $E \subset X$ such that $\mu^*(E) < \infty$. Observation 3 allows us to pick $G \in G_\delta(X)$ such that $E \subset G$ and $\mu^*(E) = \mu^*(G)$. Now pick $H \in G_\delta$ such that $G \setminus E \subset H$ and $\mu^*(H) = \mu^*(G \setminus E)$.

Now apply special case. This gives $F \in F_\sigma$ such that $F \subset G$ and $\mu^*(G \setminus F) = 0$. Let $K = F \setminus H = F \cap H^c \in F_\sigma$ and $K = F \cap H^c \subset G \cap (G \setminus E)^c \subset E$.

Note that $E, F, G, H, K \in \mathfrak{M}$, so

$$\begin{aligned}
\mu^*(E \setminus K) &= \mu^*(E) - \mu^*(K) \\
&= \mu^*(G) - \mu^*(F \setminus H) \\
&= \mu^*(G) - \mu^*(F) + \mu^*(F \cap H) \\
&\leq \mu^*(G) - \mu^*(F) + \mu^*(H) \\
&= \mu^*(G \setminus F) + \mu^*(H) \\
&= \mu^*(G \setminus E) \\
&= \mu^*(G) - \mu^*(E) \\
&= 0.
\end{aligned}$$

Therefore, K is the desired F_σ set.

Step 4: finishing (a) $\implies$ (c).

Suppose $E \in G_\delta(X)$ and $\mu^*(E) < \infty$. Write $E = \bigcup_{n=0}^\infty V_n$ where $V_n \subset X$ open. For $m, n \in \mathbb{N}$, let

$$C_{n,m} = \left\{x \in V_n : \operatorname{dist}(x, V_n^c) \geq 2^{-m}\right\} \subset V_n.$$

Note that $C_{n,m}$ is closed, $C_{n,m} \subset C_{n,m+1}$, $V_n = \bigcup_m C_{n,m}$. Since $E, C_{n,m}, V_n \in \mathfrak{M}$, we have

$$\mu^*(E) = \mu^*(E \cap V_n) = \lim_{m\to\infty} \mu^*(E \cap C_{n,m}).$$

Thus, there exists $M(n,k)$ such that $\mu^*(E \setminus C_{n,M(n,k)}) < 2^{-n-k}$. Now let $D_k = \bigcup_{n=0}^\infty C_{n,M(n,k)}$ closed. Also, $D_k \subset \bigcup_{n=0}^\infty V_n = E$ and

$$\mu^*(E) - \mu^*(D_k) = \mu^*(E \setminus D_k) \leq \sum_{n=0}^\infty \mu^*(E \setminus C_{n,M(n,k)}) \leq 2^{-k+1}.$$

Let $F = \bigcup_{k=0}^\infty D_k \subset E$ and note that $F \in F_\sigma$. Then

$$\mu^*(E \setminus F) = \mu^*(E) - \mu^*(F) \leq \mu^*(E) - \mu^*(D_k) < 2^{-k+1}$$

for all $k \in \mathbb{N}$. Therefore, $\mu^*(E \setminus F) = 0$.

□

Lemma. Suppose (X, d) metric space with metric outer measure μ^*. Suppose $X = \bigcup_{n=0}^\infty V_n$ for $V_n \subset X$ open and $\mu^*(V_n) < \infty$. Suppose $E \subset G \in G_\delta(X)$ such that $\mu^*(G \setminus E) = 0$. Then for each $\varepsilon > 0$, there exists open $U \subset X$ such that $E \subset U$ and $\mu^*(U \setminus E) < \varepsilon$.

Proof. Let $E_n = E \cap V_n$ and $G = G \cap V_n$. Write $G = \bigcap_{j=0}^\infty W_j$ where W_j open. Now set

$$Z_{n,m} = V_n \cap \bigcap_{j=0}^m W_j,$$

which are open for all $n, m \in \mathbb{N}$. Now notice that $G_n \subset Z_{n,m+1} \subset Z_{n,m} \subset V_n$. Note that $\mu^*(V_n) < \infty$, so $\mu^*(G_n) = \lim_{m\to\infty} \mu^*(Z_{n,m})$. Therefore, for all $\varepsilon > 0$, there exists $M(n)$ such that

$$\mu^*(Z_{n,M(n)} \setminus G_n) < \varepsilon 2^{-n-2}.$$

Then set $U = \bigcup_{n=0}^{\infty} Z_{n,M(n)} \supset \bigcup_{n=0}^{\infty} G_n = G \supset E$ open, then we have

$$\begin{aligned}
\mu^*(U \setminus E) &= \mu^*(U \setminus G) + \mu^*(G \setminus E) \\
&= \mu^* \left(\bigcup_{n=0}^{\infty} Z_{n,M(n)} \cap \bigcap_{n=0}^{\infty} G_n^c \right) \\
&\leq \sum_{n=0}^{\infty} \mu^*(Z_{n,M(n)} \setminus G_n) \\
&< \varepsilon,
\end{aligned}$$

as desired.

Definition (Outer-regular)**.** Let X be a metric space, $\mathfrak{M}$ a σ-algebra with $\mathfrak{B}(X) \subset \mathfrak{M}$ and suppose $\mu : \mathfrak{M} \to [0, \infty]$ is a measure. Say μ is outer-regular if

$$\mu(E) = \inf \{\mu(U) : E \subset U \text{ open}\}.$$

□