

# Mathematical Studies Analysis

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# 1 Advanced topics in metric space theory

## 1.1 Baire category

**Definition.** Let  $X$  be a metric space.

1. We say that  $E \subset X$  is nowhere dense if  $(\overline{E})^\circ = \emptyset$ .
2. We say that  $E \subset X$  is meager in  $X$  if

$$E = \bigcup_{\alpha \in A} E_\alpha,$$

where  $A$  is a countable set and  $E_\alpha \subset X$  is nowhere dense for every  $\alpha \in A$ .

**Theorem.** Prove that the following are equivalent for  $E \subset X$ :

1.  $E$  is nowhere dense
2.  $\overline{E}$  is nowhere dense
3.  $(\overline{E})^c$  is open and dense in  $X$ .

*Proof.* (1)  $\implies$  (2). Suppose  $E$  is nowhere dense, then  $(\overline{E})^\circ = \emptyset$ . Note that the closure of  $\overline{E}$  is just  $\overline{E}$  itself. It follows that  $\overline{E}$  is also nowhere dense.

(2)  $\implies$  (3). Suppose  $\overline{E}$  is nowhere dense. Note that  $\overline{E}$  is closed, so  $(\overline{E})^c$  is open. Let  $x \in X$  be arbitrary. Since  $\overline{E}$  is nowhere dense,  $x \notin (\overline{E})^\circ$ . This implies that for arbitrary  $\varepsilon > 0$ , we have  $B(x, \varepsilon) \not\subset \overline{E}$ . This is equivalent to  $B(x, \varepsilon) \cap (\overline{E})^c \neq \emptyset$ . Hence,  $(\overline{E})^c$  is dense in  $X$ .

(3)  $\implies$  (1). Suppose  $(\overline{E})^c$  is dense in  $X$ . Let  $x \in X$  and  $\varepsilon > 0$  be arbitrary. It follows that  $B(x, \varepsilon) \cap (\overline{E})^c \neq \emptyset$ . This is equivalent to  $B(x, \varepsilon) \not\subset \overline{E}$ . Therefore,  $(\overline{E})^\circ = \emptyset$  and  $E$  is nowhere dense. □

**Theorem** (Baire category theorem). Let  $X$  be a complete metric space. Suppose that for each  $n \in \mathbb{N}$ ,  $U_n \subset X$  is open and dense in  $X$ . Prove that  $\bigcap_{n=0}^{\infty} U_n$  is dense in  $X$ . Hint: use the shrinking closed set property.

*Proof.* Consider any  $x \in X$  and arbitrary  $\varepsilon > 0$ , it suffices to show that  $U_n \cap B(x, \varepsilon) \neq \emptyset$  for each  $n \in \mathbb{N}$ . Now inductively choosing a sequence  $x_i \in X$  and  $\varepsilon_i > 0$  such that for each  $i \in \mathbb{N}$ ,  $B[x_i, \varepsilon_i] \subset U_i$ ,  $B[x_{i+1}, \varepsilon_{i+1}] \subset B[x_i, \varepsilon_i] \subset B(x, \varepsilon)$ , and  $\varepsilon_i < 2^{-i}\varepsilon$ .

Since  $U_0$  is dense in  $X$ ,  $B(x, \varepsilon) \cap U_0 \neq \emptyset$ . Note that both  $U_0$  and  $B(x, \varepsilon)$  are open, so we can choose  $x_0 \in B(x, \varepsilon) \cap U_0$  and  $\varepsilon_0 > 0$  so small that  $B[x_0, \varepsilon_0] \subset B(x, \varepsilon) \cap U_0$  and  $\varepsilon_0 < \varepsilon$ . Now suppose for  $0 \leq i \leq n$ , we have chosen  $x_i \in X$  and  $\varepsilon_i > 0$  such that  $B[x_i, \varepsilon_i] \subset U_i$  and  $\varepsilon_i < 2^{-i}\varepsilon$  for all  $0 \leq i \leq n$ ,  $B[x_{i+1}, \varepsilon_{i+1}] \subset B[x_i, \varepsilon_i]$  for all  $0 \leq i < n$ . Since  $U_{n+1}$  is dense in  $X$ ,  $B(x_n, \varepsilon_n) \cap U_{n+1} \neq \emptyset$ . Note also both  $U_{n+1}$  and  $B(x_n, \varepsilon_n)$  are open. Therefore, choose  $x_{n+1} \in B(x_n, \varepsilon_n) \cap U_{n+1}$  and  $\varepsilon_{n+1} > 0$  so small that  $B[x_{n+1}, \varepsilon_{n+1}] \subset B(x_n, \varepsilon_n) \cap U_{n+1}$  and  $\varepsilon_{n+1} < \frac{\varepsilon_n}{2}$ . It follows that  $B[x_{n+1}, \varepsilon_{n+1}] \subset U_{n+1}$  and  $B[x_{n+1}, \varepsilon_{n+1}] \subset B[x_n, \varepsilon_n] \subset B(x, \varepsilon)$ . Also,  $\varepsilon < \frac{\varepsilon_n}{2} < 2^{-n-1}\varepsilon$ . Now we have successfully constructing the desired sequence.

Since  $X$  is complete,  $\bigcap_{n=0}^{\infty} B[x_n, \varepsilon_n] = \{z\}$  for some  $z \in X$ . Note that for each  $n$ , we have  $z \in B[x_n, \varepsilon_n] \subset U_n$ . Also,  $z \in B[x_n, \varepsilon_n] \subset B(x, \varepsilon)$ . Therefore,  $z \in U_n \cap B(x, \varepsilon)$  for each  $n \in \mathbb{N}$  and  $\bigcap_{n=0}^{\infty} U_n$  is dense in  $X$ . □

**Remark.** An equivalent statement of the theorem is the following:

Let  $X$  be a complete metric space and  $\{C_n\}$  a countable collection of closed subsets of  $X$  such that  $X = \bigcup_{n \in \mathbb{N}} C_n$ . Then at least one of the  $C_n$  contains an open ball.

## 1.2 Open mapping theorem

### Linear surjections

**Theorem** (Open mapping theorem). Let  $X, Y$  be Banach spaces over a common field and assume that  $T \in \mathcal{L}(X; Y)$ . Prove that the following are equivalent.

1.  $T$  is surjective.
2. There exists  $\delta > 0$  such that  $B_Y(0, \delta) \subset \overline{T(B_X(0, 1))}$ .
3. For every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $B_Y(0, \delta) \subset T(B_X(0, \varepsilon))$ .
4.  $T$  is an open map: if  $U \subset X$  is open, then  $T(U) \subset Y$  is open.
5. There exists  $C \geq 0$  such that for each  $y \in Y$  there exists  $x \in X$  such that  $Tx = y$  and

$$\|x\|_X \leq C \|y\|_Y.$$

HINT: Prove that (1)  $\implies$  (2)  $\implies$  (3)  $\implies$  (4)  $\implies$  (5)  $\implies$  (1), keeping in mind the following suggestions.

1. For (1)  $\implies$  (2): Study the sets  $C_n = \overline{T(B_X(0, n))} \subset Y$  for  $n \geq 1$ .
2. For (2)  $\implies$  (3): Prove that  $\overline{T(B_X(0, 1))} \subset T(B_X(0, 3))$  by considering  $y \in \overline{T(B_X(0, 1))}$  and inductively constructing  $\{x_j\}_{j=0}^\infty \subset X$  such that  $\|x_j\|_X < 2^{-j}$  and  $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$  for all  $m \in \mathbb{N}$ .

*Proof.* (1)  $\implies$  (2). Following the hint, for  $n \geq 1$  let  $C_n = \overline{T(B_X(0, n))}$ . Then each of the  $C_n$  are closed. Since  $T$  is surjective,  $Y = \bigcup_{n=1}^\infty C_n$ . Suppose for contradiction that each  $C_n$  are nowhere dense. It then follows that  $C_n^c$  are dense in  $Y$ . By Baire Category Theorem,  $\bigcap_{n=1}^\infty C_n^c$  is dense in  $Y$ . However,  $\bigcap_{n=1}^\infty C_n^c = (\bigcup_{n=1}^\infty C_n)^c = \emptyset$ , a contradiction. Therefore, at least one  $C_n$  is not nowhere dense. That is, there exists some  $n \geq 1$ ,  $\overline{T(B_X(0, n))}$  contains an open ball. However, this is the same set as  $n\overline{T(B_X(0, 1))}$ . Therefore,  $\overline{T(B_X(0, 1))}$  contains an open ball  $B_Y(y_0, 4r)$  for some  $y_0 \in Y$  and  $r > 0$ .

Let  $y_1 = Tx_1$  for some  $x_1 \in B_X(0, 1)$  such that  $\|y_0 - y_1\| < 2r$ . It follows that  $B_Y(y_1, 2r) \subset B_Y(y_0, 4r) \subset \overline{T(B_X(0, 1))}$ . For any  $y \in Y$  such that  $\|y\| < r$ , we have

$$y = -\frac{1}{2}y_1 + \frac{1}{2}(2y + y_1) = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1).$$

However, notice that

$$\frac{1}{2}(2y + y_1) \subset \frac{1}{2}B_Y(y_1, 2r) \subset \frac{1}{2}\overline{T(B_X(0, 1))} = \overline{T(B_X(0, \frac{1}{2}))}.$$

It follows that

$$y = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1) \in -T\left(\frac{x_1}{2}\right) + \overline{T(B_X(0, \frac{1}{2}))}.$$

Note that  $-T(\frac{x_1}{2}) \in T(B_X(0, \frac{1}{2}))$ . Therefore,  $y \in \overline{T(B_X(0, 1))}$ . Since  $y$  is arbitrary with  $\|y\| < r$ , we have  $B_Y(0, r) \subset \overline{T(B_X(0, 1))}$ .

(2)  $\implies$  (3). Following the hint, we first show  $\overline{T(B_X(0, 1))} \subset T(B_X(0, 3))$ . By assumption, we have  $B_Y(0, R) \subset \overline{T(B_X(0, 1))}$  for some  $R > 0$ . It follows from homogeneity that for each  $m \in \mathbb{N}$ , we have

$$2^{-m}B_Y(0, R) = B_Y(0, 2^{-m}R) \subset 2^{-m}\overline{T(B_X(0, 1))} = \overline{T(B_X(0, 2^{-m}))}.$$

Let  $y \in \overline{T(B_X(0, 1))}$  and pick  $x_0 \in X$  with  $\|x_0\| < 1$  such that  $\|y - Tx_0\| < 2^{-1}R$ . Now suppose we have chosen  $x_j$  for  $0 \leq j \leq m$  such that  $\|x_j\| < 2^{-j}$  and  $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$  for all  $m \in \mathbb{N}$ . By the inclusion above, we can pick  $x_{m+1} \in X$  with  $\|x_{m+1}\| < 2^{-m-1}$  such that

$$\left\|y - \sum_{j=0}^m Tx_j - Tx_{m+1}\right\| = \left\|y - \sum_{j=0}^{m+1} Tx_j\right\| < 2^{-m-2}R.$$

Therefore,  $y - \sum_{j=0}^{m+1} Tx_j \in B_Y(0, 2^{-m-2})R$ . This completes the inductive construction, and we have found a sequence  $\{x_j\}$  such that  $\|x_j\| < 2^{-j}$  and  $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$  for each  $m \in \mathbb{N}$ . Note that

$$\sum_{j=0}^{\infty} \|x_j\| \leq \sum_{j=0}^{\infty} 2^{-j} = 2,$$

so  $\sum_{j=0}^{\infty} x_j$  converges absolutely. Since  $X$  is Banach,  $\sum_{j=0}^{\infty} x_j$  converges to some  $x \in X$  with  $\|x\| \leq 2$ . Also, since  $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$ , taking the limit where  $m$  approaches infinity we obtain

$$y = \sum_{j=0}^{\infty} Tx_j = T \left( \sum_{j=0}^{\infty} x_j \right) = Tx.$$

Therefore,  $y \in T(B_X(0, 3))$  and thus  $\overline{T(B_X(0, 1))} \subset T(B_X(0, 3))$ .

Now for every  $\varepsilon > 0$ , we have  $\frac{\varepsilon}{3} \overline{T(B_X(0, 1))} \subset \frac{\varepsilon}{3} T(B_X(0, 3)) = T(B_X(0, \varepsilon))$ . By assumption, there exists  $\delta > 0$  such that  $B_Y(0, \delta) \subset \overline{T(B_X(0, 1))}$ . Therefore,

$$B_Y \left( 0, \frac{\delta \varepsilon}{3} \right) = \frac{\varepsilon}{3} B_Y(0, \delta) \subset \frac{\varepsilon}{3} \overline{T(B_X(0, 1))} \subset T(B_X(0, \varepsilon)).$$

(3)  $\implies$  (4). Let  $U \subset X$  be open and  $y \in T(U)$ . There exists  $x \in U$  such that  $Tx = y$ . Since  $U$  is open, there exists  $\varepsilon > 0$  such that  $B_X(x, \varepsilon) \subset U$ . By assumption, there exists  $\delta > 0$  such that  $B_Y(0, \delta) \subset T(B_X(0, \varepsilon))$ . It follows that

$$B_Y(y, \delta) = y + B_Y(0, \delta) \subset Tx + T(B_X(0, \varepsilon)) = T(x + B_X(0, \varepsilon)) \subset T(U).$$

Therefore,  $T(U)$  is open and  $T$  is an open map.

(4)  $\implies$  (5). Since  $T$  is an open map,  $T(B_X(0, 1))$  is open. Also,  $T(0) = 0$  so there exists  $r > 0$  such that  $B_Y(0, r) \subset T(B_X(0, 1))$ . Now let  $y \in Y$ . Then,  $\frac{r}{2\|y\|}y \in B_Y(0, r)$  and there exists  $x \in B_X(0, 1)$  such that  $Tx = \frac{r}{2\|y\|}y$ . It follows that

$$T \left( \frac{2\|y\|}{r}x \right) = y,$$

and since  $x \in B_X(0, 1)$ ,

$$\left\| \frac{2\|y\|}{r}x \right\| = \frac{2\|y\|\|x\|}{r} < \frac{2}{r}\|y\|.$$

Letting  $C = \frac{2}{r}$  completes the proof.

(5)  $\implies$  (1). Since for each  $y \in Y$  there exists  $x \in X$  such that  $Tx = y$ ,  $T$  is surjective. □

### Linear homeomorphisms, norm equivalence, and closed graphs

**Theorem.** Let  $X$  and  $Y$  be Banach spaces and suppose that  $T \in \mathcal{L}(X, Y)$  is a bijection. Prove that  $T^{-1} \in \mathcal{L}(Y, X)$ , and in particular  $T$  is a linear (and thus bi-Lipschitz) homeomorphism.

*Proof.* Since  $T \in \mathcal{L}(X, Y)$  is a bijection,  $T$  is a surjection. It follows that  $T$  is an open map. In particular, for any  $U \subset X$  open,  $T(U) = (T^{-1})^{-1}(U)$  is open. Therefore,  $T^{-1}$  is continuous and thus  $T$  is a linear homeomorphism. □

**Theorem.** Let  $X$  be a vector space that is complete when equipped with both of the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ . Prove that if there exists a constant  $C_1 > 0$  such that  $\|x\|_2 \leq C_1 \|x\|_1$  for all  $x \in X$ , then there exists a constant  $C_0 > 0$  such that  $C_0 \|x\|_1 \leq \|x\|_2 \leq C_1 \|x\|_1$  for all  $x \in X$ .

*Proof.* Let  $T : X_1 \rightarrow X_2$ , where  $X_1$  and  $X_2$  are  $X$  equipped with norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ , respectively, be the identity map. Then for any  $x \in X$  with  $\|x\|_1 = 1$ , we have

$$\|Tx\|_2 = \|x\|_2 \leq C_1 \|x\|_1 = C_1.$$

Therefore,  $T \in \mathcal{L}(X_1, X_2)$ .  $T$  is also surjective. Therefore, there exists a constant  $C \geq 0$  such that each  $\|x\|_1 \leq C \|x\|_2$ . Hence, for each  $x \in X$

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C_1 \|x\|_1.$$

Letting  $C_0 = \frac{1}{C}$  completes the proof. □

**Theorem.** Let  $X$  and  $Y$  be Banach spaces and let  $T : X \rightarrow Y$  be linear (just the algebraic condition). Prove that the following are equivalent

1.  $T$  is continuous, i.e.  $T \in \mathcal{L}(X; Y)$ .
2. The graph of  $T$ ,  $\Gamma(T) = \{(x, Tx) : x \in X\} \subset X \times Y$ , is closed in  $X \times Y$ , where  $X \times Y$  is endowed with any of the usual  $p$ -norms.

*Proof.* (a)  $\implies$  (b). Let  $\{(x_n, Tx_n)\}$  be a convergent sequence in  $\Gamma(T)$ . Since  $X$  is Banach,  $x_n \rightarrow x$  for some  $x \in X$ . Since  $T \in \mathcal{L}(X; Y)$ , we have

$$\lim_{n \rightarrow \infty} Tx_n = T \left( \lim_{n \rightarrow \infty} x_n \right) = Tx.$$

Therefore,  $(x_n, Tx_n) \rightarrow (x, Tx) \in \Gamma(T)$ , and thus  $\Gamma(T)$  is closed.

(b)  $\implies$  (a). Let  $\pi_1 : \Gamma(T) \rightarrow X$  and  $\pi_2 : \Gamma(T) \rightarrow Y$  by  $\pi_1(x, Tx) = x$  and  $\pi_2(x, Tx) = Tx$ . Since  $\Gamma(T)$  is a closed in Banach space  $Y$ ,  $\Gamma(T)$  is Banach space. It is clear that both  $\pi_1$  and  $\pi_2$  are bounded linear maps. Moreover,  $\pi_1$  is a bijection. It follows that  $S = \pi_1^{-1}$  is a bounded linear map. Therefore,  $T = \pi_2 \circ S$  is a bounded linear map. □

### Linear injections with closed range

**Theorem.** Let  $X$  and  $Y$  be Banach spaces and  $T \in \mathcal{L}(X, Y)$ . Prove the following are equivalent.

1.  $T$  is injective and  $\text{range}(T)$  is closed.
2.  $T : X \rightarrow \text{range}(T)$  is a linear homeomorphism.
3. There exists  $C \geq 0$  such that  $\|x\|_X \leq C \|Tx\|_Y$  for all  $x \in X$ .

HINT: Prove that (1)  $\implies$  (2)  $\implies$  (3)  $\implies$  (1).

*Proof.* (1)  $\implies$  (2). If  $T$  is injective and  $\text{range}(T)$  is closed, then  $\Gamma(T) = \{(x, Tx) : x \in X\}$  is closed in  $X \times Y$ . Therefore,  $T : X \rightarrow \text{range}(T)$  is a bounded linear map. Since  $T$  is injective, this map is actually bijective from  $X$  to  $\text{range}(T)$ . Therefore,  $T$  is a linear homeomorphism.

(2)  $\implies$  (3). Since  $T$  is a bijective bounded linear map, from  $X$  to  $\text{range}(T)$ . There exists a constant  $C \geq 0$  such that for each  $y \in \text{range}(T)$  there exists a unique  $x \in X$  such that  $Tx = y$  and  $\|x\| \leq C \|y\| = C \|Tx\|$ . Since  $T$  is a bijection,  $\|x\| \leq C \|Tx\|$  for all  $x \in X$ .

(3)  $\implies$  (1). Let  $x \in X$  be such that  $Tx = 0$ . It follows that  $\|x\| \leq C \|Tx\| = 0$ . Therefore,  $x = 0$  and  $T$  is injective. To show that  $\text{range}(T)$  is closed, consider a convergent sequence  $\{y_n\} \subset \text{range}(T)$  with  $y_n = Tx_n$ . Since for any  $n, m \in \mathbb{N}$  we have

$$\|x_n - x_m\| \leq C \|T(x_n - x_m)\| = C \|y_n - y_m\|,$$

$\{x_n\}$  is Cauchy. Since  $X$  is Banach,  $x_n \rightarrow x$  for some  $x \in X$ . Therefore, for all  $n \in \mathbb{N}$  we have

$$\|y_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \|x_n - x\|,$$

and  $y_n \rightarrow Tx$ . Hence,  $\text{range}(T)$  is closed and the proof is complete.  $\square$

**Theorem.** Let  $X$  and  $Y$  be Banach spaces over a common field. Then, the following subsets of  $\mathcal{L}(X; Y)$  are open:

1.  $\{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\},$
2.  $\{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\},$
3.  $\mathcal{H}(X; Y) = \{T \in \mathcal{L}(X; Y) : T \text{ is a homeomorphism}\}.$

*Proof.* 1. Let  $T \in \mathcal{L}(X; Y)$  be surjective. By open mapping theorem, there is  $\delta > 0$  such that  $B_Y(0, \delta) \subset TB_X(0, 1)$ . By homogeneity we have  $B_Y(0, r) \subset TB_X(0, \alpha r)$  for all  $r > 0$  where  $\alpha = \delta^{-1}$ . Now let  $S \in \mathcal{L}(X; Y)$  be such that  $\|T - S\| < \beta < (2\alpha)^{-1}$ . Claim  $S$  is surjective.

Let  $y \in Y$ , inductively construct sequences  $\{x_n\}$  and  $\{y_n\}$ . First let  $y_0 = y$ . Then,  $\|y_0\| \in B(0, 2\|y_0\|)$ . Select  $x_0 \in X$  be such that  $Tx_0 = y_0$  and  $\|x_0\| \leq 2\alpha\|y_0\|$ . Suppose we have selected  $y_i, x_i$  for  $0 \leq i \leq n$ . Set  $y_{n+1} = y_n - Sx_n$  and select  $x_{n+1}$  be such that  $Tx_{n+1} = y_{n+1}$  and  $\|x_{n+1}\| \leq 2\alpha\|y_{n+1}\|$ . Then, we have

$$\|y_{n+1}\| = \|Tx_n - Sx_n\| \leq \|T - S\| \|x_n\| < 2\alpha\beta\|y_n\|$$

and

$$\|x_{n+1}\| = 2\alpha\|y_{n+1}\| \leq 2\alpha\|T - S\| \|x_n\| < 2\alpha\beta\|x_n\|.$$

Note that  $2\alpha\beta < 1$  and  $X$  is Banach, define

$$x = \sum_{n=0}^{\infty} x_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N x_n.$$

Also note that  $\lim_{n \rightarrow \infty} y_n = 0$ . It follows that

$$Sx = \sum_{n=0}^{\infty} Sx_n = \sum_{n=0}^{\infty} (y_n - y_{n+1}) = y_0 - \lim_{n \rightarrow \infty} y_{n+1} = y.$$

Therefore  $S$  is surjective and the set of surjective bounded linear maps are open.

2. Suppose  $T \in \mathcal{L}(X; Y)$  is injective with closed range. Then, closed range theorem gives  $C > 0$  such that  $\|x\| \leq C\|Tx\|$  for all  $x \in X$ . Now suppose  $S \in \mathcal{L}(X; Y)$  is such that  $\|T - S\| < (2C)^{-1}$ . Claim that  $S$  is also injective with closed range. Indeed,

$$\begin{aligned} \|x\| &\leq C\|Tx\| \leq C\|Sx\| + C\|(T - S)x\| \\ &\leq C\|Sx\| + \frac{1}{2}\|x\|. \end{aligned}$$

This shows that  $\|x\| \leq 2C\|Sx\|$  for all  $x \in X$ . By closed range theorem,  $S$  is injective with closed range. This implies that the set of injective bounded linear operator with closed range is open.

3. This directly follows from

$$\mathcal{H}(X; Y) = \{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\} \cap \{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\}.$$

$\square$

**Theorem.** Let  $X$  and  $Y$  be Banach spaces over a common field. Then the following holds.

1. The sets

$$\mathcal{L}_R(X; Y) = \{T \in \mathcal{L}(X; Y) : \text{there exists } S \in \mathcal{L}(Y; X) \text{ such that } ST = I_X\}$$

and

$$\mathcal{L}_L(X; Y) = \{T \in \mathcal{L}(X; Y) : \text{there exists } S \in \mathcal{L}(Y; X) \text{ such that } TS = I_Y\}$$

are open.

2. The following inclusion holds:

$$\mathcal{L}_L(X; Y) \subset \{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\}$$

and

$$\mathcal{L}_R(X; Y) \subset \{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\}.$$

3. The sets  $\mathcal{L}_L(X; Y) \setminus \mathcal{L}_R(X; Y)$  and  $\mathcal{L}_R(X; Y) \setminus \mathcal{L}_L(X; Y)$  are open.

*Proof.* 1. Let  $T_0 \in \mathcal{L}_R$  and  $S_0 \in \mathcal{L}(Y; X)$  be such that  $T_0 S_0 = I_Y$ . Note that  $I_X \in \mathcal{H}(X)$  and when  $\|P\| < 1$  for  $P \in \mathcal{L}(X)$ , we have  $I_X + P \in \mathcal{H}(X)$ . Suppose now  $T \in \mathcal{L}(X; Y)$  and  $\|T\| < \|S_0\|^{-1}$ . It follows that  $I_X + S_0 T \in \mathcal{H}(X)$ . For such  $T$ , we then have

$$T_0 + T = T_0(I_X + S_0 T).$$

Also,

$$(T_0 + T)(I_X + S_0 T)^{-1} S_0 = T_0(I_X + S_0 T)(I_X + S_0 T)^{-1} S_0 = T_0 S_0 = I_Y.$$

Therefore,  $T_0 + T \in \mathcal{L}_R$  for  $T \in B(T_0, \|S_0\|^{-1})$  and  $\mathcal{L}_R$  is open.

Now let  $T_0 \in \mathcal{L}_L$  and  $S_0 \in \mathcal{L}(Y; X)$  be such that  $S_0 T_0 = I_X$ . Again, for  $T \in \mathcal{L}(X; Y)$  with  $\|T\| < \|S_0\|^{-1}$ , we have

$$T_0 + T = (I_X + T S_0) T_0.$$

and

$$S_0(I_X + T S_0)^{-1}(T_0 + T) = I_X.$$

Therefore,  $\mathcal{L}_R$  is also open.

2. Let  $T \in \mathcal{L}_R$  and  $S \in \mathcal{L}(Y; X)$  be such that  $TS = I_Y$ . Then for any  $y \in Y$  let  $x = Sy$ . It follows that  $Tx = TSy = y$ . Also,  $\|x\| \leq \|S\| \|y\|$  so the 4th item in open mapping theorem guarantees that  $T$  is surjective. Hence,  $\mathcal{L}_L \subset \{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\}$ .

Now let  $T \in \mathcal{L}_L$  and  $S \in \mathcal{L}(Y; X)$  such that  $ST = I_X$ . Now for any  $x \in X$ , we have  $\|x\| = \|STx\| \leq \|S\| \|Tx\|$ . Then the closed range theorem guarantees that  $T$  is injective with closed range. Hence,  $\mathcal{L}_R \subset \{T \in \mathcal{L}_R(X; Y) : T \text{ is injective with closed range}\}$ .

3. \*\*\* TO-DO \*\*\*

□

### 1.3 Hahn-Banach theorem and duality

**Theorem** (Hahn-Banach theorem in  $\mathbb{R}$ ). Let  $X$  be a real vector space and suppose  $p : X \rightarrow \mathbb{R}$  is such that

$$p(tx + (1 - t)y) \leq tp(x) + (1 - t)p(y)$$

for all  $t \in [0, 1]$  and  $x, y \in X$ .

Suppose  $Y$  subspace of  $X$  and  $l : Y \rightarrow \mathbb{R}$  is a linear map such that  $l \leq p$  on  $Y$ . Then there exists linear map  $L : X \rightarrow \mathbb{R}$  such that  $L \leq p$  on  $X$  and  $L = l$  on  $Y$ .



*Proof.* Let

$$P = \{(Z, \lambda) : Y \subset Z \subset X, \lambda \text{ linear functional on } Z, \lambda \leq p \text{ on } Z \text{ and } l = \lambda \text{ on } Y\}$$

Define partial order  $(Z_1, \lambda_1) \preceq (Z_2, \lambda_2)$  if and only if  $Z_1 \subset Z_2$  and  $\lambda_1 = \lambda_2$  on  $Z_1$ . It is easy to verify that this is a partial order. Towards using Zorn's Lemma, let  $C \subset P$  be a chain and define

$$U = \bigcup_{(Z, \lambda) \in C} Z, \quad \Lambda = \bigcup_{(Z, \lambda) \in C} \lambda.$$

It is easy to verify that  $(U, \Lambda)$  is an upper bound for the chain. By Zorn's Lemma,  $P$  has a maximal element  $(M, L)$ . It remains to show that  $M = X$ .

Suppose for contradiction that  $M \neq X$ . Pick  $x_0 \in X \setminus M$ . For any  $x, y \in M$ , we have

$$\begin{aligned} \beta L(x) + \alpha L(y) &= L(\beta x + \alpha y) \\ &= \frac{1}{\alpha + \beta} L\left(\frac{\beta}{\alpha + \beta}x + \frac{\alpha}{\alpha + \beta}y\right) \\ &\leq (\alpha + \beta)p\left(\frac{\beta}{\alpha + \beta}x + \frac{\alpha}{\alpha + \beta}y\right) \\ &= (\alpha + \beta)p\left(\frac{\beta}{\alpha + \beta}(x - \alpha x_0) + \frac{\alpha}{\alpha + \beta}(y + \beta x_0)\right) \\ &\leq \beta p(x - \alpha x_0) + \alpha p(y + \beta x_0). \end{aligned}$$

This implies that

$$\sup_{\substack{x \in M \\ \alpha > 0}} \frac{1}{\alpha} [L(x) - p(x - \alpha x_0)] \leq \inf_{\substack{y \in M \\ \beta > 0}} \frac{1}{\beta} [p(y + \beta x_0) - L(y)].$$

Note that  $-p(-x_0) \leq \text{LHS}$  and  $\text{RHS} \leq p(x_0)$ , so  $\text{LHS}, \text{RHS} < \infty$ . Now pick  $v \in \mathbb{R}$  such that  $\text{LHS} \leq v \leq \text{RHS}$ . For  $x \in M$  and  $0 < t \in \mathbb{R}$  we have

$$L(x) - tv \leq p(x - tv_0), \quad L(x) + tv \leq p(x + tv_0).$$

Now define  $\widehat{L} : M \oplus \mathbb{R}x_0 \rightarrow \mathbb{R}$  by  $\widehat{L}(x + \alpha x_0) = L(x) + \alpha v$ . It follows that  $(M \oplus \mathbb{R}x_0, \widehat{L}) \in P$ . However,  $(M, L) \prec (M \oplus \mathbb{R}, \widehat{L})$ , a contradiction. Therefore,  $M = X$  and the proof is complete.  $\square$

**Theorem** (Hahn-Banach theorem in  $\mathbb{C}$ ). Let  $X$  be complex vector space and suppose  $p : X \rightarrow \mathbb{R}$  is such that

$$p(\alpha x + \beta y) \leq |\alpha|p(x) + |\beta|p(y)$$

for all  $\alpha, \beta \in \mathbb{C}$  such that  $|\alpha| + |\beta| = 1$  and  $x, y \in X$ .

Suppose  $Y$  subspace of  $X$  and  $l : Y \rightarrow \mathbb{C}$  is a linear map such that  $|l| \leq p$  on  $Y$ . Then there exists linear map  $L : X \rightarrow \mathbb{C}$  such that  $|L| \leq p$  on  $X$  and  $L = l$  on  $Y$ .

*Proof.* Define  $\lambda : Y \rightarrow \mathbb{R}$  by  $\lambda(x) = \text{Re}(l(x))$ . Note that

$$\lambda(ix) = \text{Re}(il(x)) = -\text{Im}(l(x)).$$

This implies that  $l(x) = \lambda(x) - i\lambda(ix)$ . Now treat  $X$  and  $Y$  as vector space over  $\mathbb{R}$  and apply Hahn-Banach theorem in  $\mathbb{R}$  to extend  $\lambda$  to  $\Lambda : X \rightarrow \mathbb{R}$  that agrees with  $\lambda$  on  $Y$ .

Define  $L : X \rightarrow \mathbb{C}$  by  $L(x) = \Lambda(x) - i\Lambda(ix)$ . It remains to show that  $|L| \leq p$ . For  $x \in X$ , write  $L(x) = |L(x)| e^{i\theta}$  for some  $\theta \in \mathbb{R}$ . It follows that

$$\begin{aligned} |L(x)| &= L(x)e^{-i\theta} \\ &= \Lambda(e^{-i\theta}x) - i\Lambda(ie^{-i\theta}x) \\ &= \Lambda(e^{-i\theta}x) \\ &\leq p(e^{-i\theta}x) \\ &\leq |e^{-i\theta}| p(x) \\ &= p(x), \end{aligned}$$

as desired. □

**Theorem** (Hahn-Banach theorem for bounded linear functionals). Let  $X$  be a normed vector space over  $\mathbb{F}$  and  $Y$  a subspace of  $X$ . If  $\lambda \in Y^*$  then there exists  $\Lambda \in X^*$  such that  $\Lambda = \lambda$  on  $Y$  and the operator norm  $\|\lambda\|_{Y^*} = \|\Lambda\|_{X^*}$ .

*Proof.* Consider  $p : X \rightarrow \mathbb{R}$  where  $p(x) = \|\lambda\|_{Y^*} \|x\|$ . Apply Hahn-Banach theorem. □

Next we show some useful implications of Hahn-Banach theorem.

**Theorem.** Let  $X$  be a normed vector space and fix  $x \in X$ . Then the following holds:

1. There exists  $\lambda \in X^*$  such that  $\|\lambda\| = \|x\|$  and

$$\lambda(x) = \|\lambda\| \|x\| = \|x\|^2.$$

2. We have

$$\|x\| = \max_{\substack{w \in X^* \\ \|w\|=1}} |w(x)|.$$

3.  $x = 0$  if and only if  $w(x) = 0$  for all  $w \in X^*$ .

*Proof.* 1. Let  $Y = \mathbb{F}x$  and define  $\lambda \in Y^*$  by  $\lambda(ax) = a\|x\|^2$ . Apply Hahn-Banach theorem.

2. Suppose  $x \neq 0$ . Define  $w = \frac{\lambda}{\|x\|}$  then it follows that  $|w(x)| = \|x\|$ .

3. Follows directly from (2). □

**Proposition.** Let  $X$  be normed vector space. Then the mapping  $\langle \cdot, \cdot \rangle : X^* \times X \rightarrow \mathbb{F}$  by  $(w, x) \mapsto w(x)$  is a bilinear map. That is,  $\langle \cdot, \cdot \rangle \in \mathcal{L}(X^*, X; \mathbb{F})$ . Moreover, if  $X \neq \{0\}$ , then  $\|\langle \cdot, \cdot \rangle\| = 1$ .

*Proof.* It is easy to see that  $\langle \cdot, \cdot \rangle$  is bilinear. For boundedness,

$$|\langle w, x \rangle| = |w(x)| \leq \|w\| \|x\|.$$

Hence,  $\|\langle \cdot, \cdot \rangle\| \leq 1$ . Meanwhile, pick some  $x \in X$  with  $\|x\| = 1$ . It follows that

$$1 = \|x\| = \max_{\substack{w \in X^* \\ \|w\|=1}} |w(x)| \leq \|\langle \cdot, \cdot \rangle\|.$$

Therefore,  $\|\langle \cdot, \cdot \rangle\| = 1$ . □

**Definition** (Norming set). Let  $X$  be normed vector space and  $E \subset X$ ,  $W \subset X^*$ . Say  $W$  is a **norming set** for  $E$  if

$$\|x\| = \sup_{\substack{w \in W \\ \|w\|=1}} |\langle w, x \rangle|$$

for all  $x \in E$ .

**Proposition.** Let  $X$  be normed vector space and  $S \subset X$  be a separable set. Let  $W$  be a norming set for  $S$ . Then, there exists  $\{w_n\}_{n=0}^\infty \subset W$  such that  $\|w_n\| = 1$ , and the sequence is norming for  $S$ . That is,

$$\|x\| = \sup_{n \in \mathbb{N}} |\langle w_n, x \rangle|.$$

*Proof.* Let  $\{v_n\}_{n=0}^\infty \subset S$  be dense. For any  $n, k \in \mathbb{N}$ , choose  $w_{n,k} \in W$  with  $\|w_{n,k}\| = 1$  such that

$$(1 - 2^{-k}) \|v_n\| \leq |w_{n,k}, v_n|.$$

Let  $x \in S$  and  $0 < \varepsilon < 1$  be arbitrary. Pick  $v_n \in S$  such that  $\|v_n - x\| < \varepsilon$  and pick  $j \in \mathbb{N}$  such that  $2^{-j} < \varepsilon$ . Then,

$$\begin{aligned} (1 - \varepsilon) \|x\| &\leq (1 - 2^{-j}) \|x\| \\ &\leq (1 - 2^{-j}) \|v_n\| + (1 - 2^{-j}) \|v_n - x\| \\ &\leq |\langle w_{n,j}, v_j \rangle| + \varepsilon \\ &\leq |\langle w_{n,j}, x \rangle| + |\langle w_{n,j}, x - v_n \rangle| + \varepsilon \\ &\leq |\langle w_{n,j}, x \rangle| + 2\varepsilon. \end{aligned}$$

This shows that  $\{w_{n,k}\}_{n,k=0}^\infty$  is a norming sequence. □

**Theorem.** Let  $X$  be normed vector space and define  $J : X \rightarrow X^{**}$  by  $\langle Jx, w \rangle = \langle w, x \rangle = w(x)$ . Then the following holds:

1.  $J \in \mathcal{L}(X, X^{**})$ .
2.  $J$  is an isometric embedding. In particular, it is injective.
3.  $\text{range}(J) \subset X^{**}$  is a norming set for  $X^*$ .
4.  $X$  is Banach if and only if  $\text{range}(J)$  is closed.

*Proof.* Note that we have

$$\begin{aligned} \|Jx\|_{X^{**}} &= \sup \{ |\langle Jx, w \rangle| : w \in X^* \text{ and } \|w\| \leq 1 \} \\ &= \sup \{ |\langle w, x \rangle| : w \in X^* \text{ and } \|w\| \leq 1 \} \\ &= \|x\|, \end{aligned}$$

where the last step is by a previous theorem that shows the existence of  $w \in X^*$  such that  $\|w\| = 1$  and  $|w(x)| = \|x\|$ . This implies (1) and (2). Now we know  $X$  is isometrically isomorphic to  $\text{range}(J) \subset X^{**}$ . Therefore,  $X$  is Banach if and only if  $\text{range}(J)$  is Banach. However,  $X^{**} = \mathcal{L}(X^*, \mathbb{F})$  is Banach, so  $\text{range}(J)$  is Banach if and only if  $\text{range}(J)$  is closed. This implies (4).

To show (3), note that we have

$$\begin{aligned} \|w\|_{X^*} &= \sup \{ |\langle w, x \rangle| : x \in X \text{ and } \|x\| \leq 1 \} \\ &= \sup \{ |\langle Jx, w \rangle| : x \in X \text{ and } \|x\| \leq 1 \} \\ &= \sup \{ |\langle v, w \rangle| : v \in \text{range}(J) \text{ and } \|v\|_{X^{**}} \leq 1 \}. \end{aligned}$$

This shows (3), completing the proof. □

## 2 Differential Calculus

### 2.1 Inverse and implicit function theorem

**Theorem** (Local injectivity theorem). Let  $X$  and  $Y$  be Banach spaces,  $z \in U \subset X$  with  $U$  open. Let  $f : U \rightarrow Y$  differentiable with  $Df$  continuous at  $z$ . Suppose  $Df(z) \in \mathcal{L}(X; Y)$  injective with closed range. Then for any  $0 < \varepsilon < 1$ , there exists  $r > 0$  such that

1.  $B[z, r] \subset U$ .
2.  $Df(x)$  injective with closed range for all  $x \in B[z, r]$ .
3. If  $x, y \in B(z, r)$ , then

$$(1 - \varepsilon) \|Df(z)(x - y)\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|Df(z)(x - y)\|.$$

4. The restriction  $f : B(z, r) \rightarrow f(B(z, r))$  is bi-Lipschitz homeomorphism.

*Proof.* Since  $Df(z)$  injective with closed range, there exists  $\theta > 0$  such that

$$\theta \|h\| \leq \|Df(z)h\|$$

for all  $h \in X$ . Since the set of bounded linear operator that is injective with closed range is open, there exists  $\delta > 0$  such that  $\|Df(z) - T\| < \delta$  implies  $T$  is injective with closed range.

Now let  $0 < \varepsilon < 1$ . Note that  $Df$  is continuous at  $z$ , so we can select  $r > 0$  so small that  $B[z, r] \subset U$ , and  $x \in B[z, r]$  implies

$$\|Df(x) - Df(z)\| < \min \{\delta, \theta\varepsilon\}.$$

In particular,  $Df(x)$  is injective with closed range for all  $x \in B[z, r]$ . By the mean value theorem, for any  $x, y \in B(x, r)$

$$\begin{aligned} \|f(x) - f(y) - Df(z)(x - y)\| &\leq \sup_{w \in B(z, r)} \|Df(w) - Df(z)\| \|x - y\| \\ &\leq \theta\varepsilon \|x - y\| \\ &\leq \varepsilon \|Df(z)(x - y)\|. \end{aligned}$$

It follows that

$$(1 - \varepsilon) \|Df(z)(x - y)\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|Df(z)(x - y)\|,$$

as desired.

This also implies that

$$(1 - \varepsilon)\theta \|x - y\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|Df(z)\| \|x - y\|,$$

so the restriction of  $f$  on  $B(z, r)$  is a bi-Lipschitz homeomorphism. □

**Theorem** (Local surjectivity theorem). Let  $X$  and  $Y$  be Banach spaces,  $z \in U \subset X$  with  $U$  open. Let  $f : U \rightarrow Y$  differentiable with  $Df$  continuous at  $z$ . Suppose  $Df(z) \in \mathcal{L}(X; Y)$  surjective. Then there exists  $r_0, \gamma > 0$  such that

1.  $B_X[z, r_0] \subset U$ .
2.  $Df(x)$  surjective for all  $x \in B_X[z, r_0]$ .
3.  $B_Y[f(z), \gamma r] \subset f(B_X[z, r])$  for all  $0 \leq r \leq r_0$ .

*Proof.* \*\*\* TO-DO \*\*\* □

**Definition** (diffeomorphism). Let  $X$  and  $Y$  be normed vector spaces and suppose that  $\emptyset \neq U \subset X$  is open. Let  $f : U \rightarrow Y$ . For  $k \geq 1$ , say  $f$  is a  $C^k$  diffeomorphism if

1.  $f : U \rightarrow f(U)$  homeomorphism with  $f(U) \subset Y$  open.
2.  $f \in C^k(U; Y)$ .
3.  $f^{-1} \in C^k(f(U); X)$ .

If  $f$  is a  $C^k$  diffeomorphism for all  $k \geq 1$ , say  $f$  is a smooth diffeomorphism.

**Theorem** (Inverse function theorem). Let  $X$  and  $Y$  be Banach spaces,  $U \subset X$  open and  $x_0 \in U$ . Suppose  $f : U \rightarrow Y$  differentiable,  $Df$  continuous at  $x_0$ ,  $Df(x_0)$  linear homeomorphism. Then there exists bounded and open  $V \subset U$  with  $x_0 \in V$  such that

1.  $f : V \rightarrow f(V)$  is bi-Lipschitz homeomorphism,  $Df(x)$  linear homeomorphism for all  $x \in V$ ,  $f(V) \subset Y$  bounded and open,  $f^{-1} : f(V) \rightarrow V$  differentiable with

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$$

for all  $y \in f(V)$  and  $Df^{-1}$  is continuous at  $f(x_0)$ . Also, there exists  $C_0, C_1 > 0$  such that

$$C_0 \leq \|Df(x)\| \leq C_1$$

for all  $x \in V$ , and

$$\frac{1}{C_1} \leq \|Df^{-1}(y)\| \leq \frac{1}{C_0}$$

for all  $y \in f(V)$ .

2. If  $f \in C^k(U; Y)$  for some  $1 \leq k \leq \infty$ , then  $f^{-1} \in C^k(f(V); X)$ . In particular,  $f$  is a local  $C^k$  diffeomorphism at  $x_0$ .
3. If  $f \in C^k(U; Y)$  for  $1 \leq k \in \mathbb{N}$ , then there exists open  $V_k \subset V$  such that  $x_0 \in V_k$ ,  $f \in C_b^k(V_k; Y)$  and  $f^{-1} \in C_b^k(f(V_k); X)$ .

*Proof.* \*\*\* TO-DO \*\*\* □

**Theorem** (Implicit function theorem). Let  $X$  and  $Y$  be Banach spaces,  $U \subset X \times Y$  be open with  $(x_0, y_0) \in U$ , and suppose  $f : U \rightarrow Z$  is differentiable in  $U$  with  $Df$  continuous at  $(x_0, y_0)$ . Further suppose  $z_0 = f(x_0, y_0)$  and  $D_2f(x_0, y_0) \in \mathcal{L}(Y; Z)$  is an isomorphism. Then there exists open sets  $x_0 \in V \subset X$ ,  $z_0 \in W \subset Z$ ,  $y_0 \in S \subset Y$ , and  $g \in C_b^{0,1}(V \times W; Y)$  such that the following holds:

1.  $g(x_0, z_0) = y_0$  and  $(x, g(x, z)) \in V \times S \subset U$  for all  $(x, z) \in V \times W$ . Also,  $g$  is differentiable on  $V \times W$  and  $Dg$  continuous at  $(x_0, z_0)$ .
2.  $f(x, g(x, z)) = z$  for all  $(x, z) \in V \times W$ . Moreover, if  $(x, y) \in V \times S$  such that  $f(x, y) = z$  for some  $z \in W$ , then  $y = g(x, z)$ .
3.  $D_2f(x, g(x, z))$  is an isomorphism for all  $(x, z) \in V \times W$ , and

$$\begin{aligned} D_1g(x, z) &= -[D_2f(x, g(x, z))]^{-1} D_1f(x, g(x, z)), \\ D_2g(x, z) &= [D_2f(x, g(x, z))]^{-1}. \end{aligned}$$

4. If  $f \in C^k$  then  $g \in C^k$  too for  $1 \leq k \leq \infty$ . If  $k$  finite and  $f \in C_b^k$  then the sets can be picked such that  $g \in C_b^k$ .

*Proof.* \*\*\* TO-DO \*\*\* □

### 3 Measure and integration

#### 3.1 Introduction to abstract measure theory

##### 3.1.1 Basic definitions

**Definition.** Let  $X$  be a set.

1. An **algebra** on  $X$  is  $\mathfrak{A} \subset \mathcal{P}(X)$  such that
  - (a)  $\emptyset \in \mathfrak{A}$ .
  - (b)  $E \in \mathfrak{A}$  implies  $E^c \in \mathfrak{A}$ .
  - (c)  $E, F \in \mathfrak{A}$  implies  $E \cup F \in \mathfrak{A}$ .
2. A  **$\sigma$ -algebra** is an algebra  $\mathfrak{M} \subset \mathcal{P}(X)$  such that if  $E_k \in \mathfrak{M}$  for all  $k \in \mathbb{N}$ , then  $\bigcup_{k=0}^{\infty} E_k \in \mathfrak{M}$ .
3. A pair  $(X, \mathfrak{M})$  with  $\mathfrak{M}$  a  $\sigma$ -algebra on  $X$  is called a **measurable space**.

**Theorem.** Let  $X$  be a set.

1. Suppose  $A \neq \emptyset$  is a set and  $\mathfrak{M}_\alpha$  is  $\sigma$ -algebra for each  $\alpha \in A$ , then  $\mathfrak{M} = \bigcap_{\alpha \in A} \mathfrak{M}_\alpha$  is a  $\sigma$ -algebra on  $X$ .
2. Suppose  $F \subset \mathcal{P}(X)$ , there is unique smallest  $\sigma$ -algebra  $\mathfrak{M}$  on  $X$  such that  $F \subset \mathfrak{M}$ . Write  $\mathfrak{M} = \sigma(F)$  and call this the  $\sigma$ -algebra generated by  $F$ .

**Theorem.** Let  $X$  and  $Y$  be sets and  $f : X \rightarrow Y$ .

1. Suppose  $\mathfrak{M}$  is a  $\sigma$ -algebra on  $X$  and set

$$\mathfrak{N} = \{E \subset Y : f^{-1}(E) \in \mathfrak{M}\}.$$

Then,  $\mathfrak{N}$  is a  $\sigma$ -algebra on  $Y$ . Call this the **push-forward** of  $\mathfrak{M}$  by  $f$ .

2. Suppose  $\mathfrak{N}$  is a  $\sigma$ -algebra on  $Y$  and set

$$\mathfrak{M} = \{f^{-1}(E) : E \in \mathfrak{N}\}.$$

Then,  $\mathfrak{M}$  is a  $\sigma$ -algebra on  $X$ . Call this the **pull-back** of  $\mathfrak{N}$  by  $f$ .

**Definition.** Let  $A \neq \emptyset$  be a set.

1. Let  $Y$  be a set and  $X_\alpha$  be sets with  $\sigma$ -algebra  $\mathfrak{M}_\alpha$  for all  $\alpha \in A$ . Suppose  $g_\alpha : X_\alpha \rightarrow Y$  for all  $\alpha \in A$ . Define

$$\sigma(\{E \subset Y : g_\alpha^{-1}(E) \in \mathfrak{M}_\alpha \text{ for all } \alpha \in A\})$$

to be the **push-forward** of  $\{g_\alpha\}_{\alpha \in A}$ .

2. Let  $X$  be a set and  $Y_\alpha$  be sets with  $\sigma$ -algebra  $\mathfrak{N}_\alpha$  for all  $\alpha \in A$ . Suppose  $f_\alpha : X \rightarrow Y_\alpha$  for all  $\alpha \in A$ . Define

$$\sigma(\{f_\alpha^{-1}(E) : E \in \mathfrak{N}_\alpha \text{ for some } \alpha \in A\})$$

to be the **pull-back** of  $\{f_\alpha\}_{\alpha \in A}$ .

**Definition.** Let  $A \neq \emptyset$  be a set and  $X_\alpha$  be sets with  $\sigma$ -algebra  $\mathfrak{M}_\alpha$  for all  $\alpha \in A$ . Then on the set  $X = \prod_{\alpha} X_\alpha$  we define the **product  $\sigma$ -algebra**  $\bigotimes_{\alpha} \mathfrak{M}_\alpha$  to be the pull-back of projection maps  $\pi_\alpha : X \rightarrow X_\alpha$ .

**Theorem.** Let  $A \neq \emptyset$  be a set and  $X_\alpha$  with  $\sigma$ -algebra  $\mathfrak{M}_\alpha$  for all  $\alpha \in A$ . Let  $X = \prod_{\alpha} X_\alpha$  and define

$$\mathcal{R} = \left\{ \prod_{\alpha} M_{\alpha} : M_{\alpha} \in \mathfrak{M}_{\alpha} \right\}.$$

Then,

1.  $\bigotimes_{\alpha} \mathfrak{M}_{\alpha} \subset \sigma(\mathcal{R})$ . If  $A$  countable then  $\sigma(\mathcal{R}) = \bigotimes_{\alpha} \mathfrak{M}_{\alpha}$ .

2. Suppose  $\mathfrak{M}_\alpha = \sigma(\mathcal{E}_\alpha)$  for all  $\alpha \in A$  and let

$$\mathcal{E} = \{\pi_\alpha^{-1}(E) : E \in \mathcal{E}_\alpha \text{ for some } \mathcal{E}_\alpha\}.$$

Then  $\bigotimes_\alpha \mathfrak{M}_\alpha = \sigma(\mathcal{E})$ . Moreover, if  $A$  is countable and  $X_\alpha \in \mathcal{E}_\alpha$  for all  $\alpha \in A$ , then  $\bigotimes_\alpha \mathfrak{M}_\alpha$  is generated by  $\mathcal{F} = \{\prod_\alpha E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}$

*Proof.* 1. For  $E \in \mathfrak{M}_\alpha$ , we have  $\pi_\alpha^{-1}(E) = \prod_\beta S_\beta$ , where

$$S_\beta = \begin{cases} E & (\beta = \alpha), \\ X_\beta & (\beta \neq \alpha). \end{cases}$$

Then,

$$\{\pi_\alpha^{-1}(M_\alpha) : M_\alpha \in \mathfrak{M}_\alpha\} \subset \left\{ \prod_\beta M_\beta : M_\beta \in \mathfrak{M}_\beta \right\} = \mathcal{R}.$$

This implies that  $\bigotimes_\alpha \mathfrak{M}_\alpha \subset \sigma(\mathcal{R})$ .

On the other hand, if  $A$  is countable, then

$$\prod_\alpha M_\alpha = \bigcap_\alpha \pi_\alpha^{-1}(M_\alpha) \in \bigotimes_\alpha \mathfrak{M}_\alpha$$

whenever  $M_\alpha \in \mathfrak{M}_\alpha$  for all  $\alpha \in A$ . This implies that  $\sigma(\mathcal{R}) \subset \bigotimes_\alpha \mathfrak{M}_\alpha$ .

2. It is clear that  $\sigma(\mathcal{E}) \subset \bigotimes_\alpha \mathfrak{M}_\alpha$ . On the other hand, for each  $\alpha \in A$ , let

$$\mathfrak{N}_\alpha = \{E \subset X_\alpha : \pi_\alpha^{-1}(E) \in \sigma(\mathcal{E})\}$$

be the push-forward of  $\sigma(\mathcal{E})$  to  $X_\alpha$  by  $\pi_\alpha$ . It is clear that  $\mathcal{E}_\alpha \subset \mathfrak{N}_\alpha$ . This implies  $\mathfrak{M}_\alpha = \sigma(\mathcal{E}) \subset \mathfrak{N}_\alpha$ . In particular,  $\pi_\alpha^{-1}(E) \in \sigma(\mathcal{E})$  for all  $E \in \mathfrak{M}_\alpha$ . This implies that  $\bigotimes_\alpha \mathfrak{M}_\alpha \subset \sigma(\mathcal{E})$ .

Now, assume  $A$  countable and  $X_\alpha \in \mathcal{E}_\alpha$  for all  $\alpha \in A$ . Then let  $E \in \mathfrak{M}_\alpha$  for some  $\alpha \in A$ . We have  $\pi_\alpha^{-1}(E) = \prod_\beta S_\beta$ , where

$$S_\beta = \begin{cases} E & (\beta = \alpha), \\ X_\beta & (\beta \neq \alpha). \end{cases}$$

Therefore,  $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F})$ .

On the other hand, since  $A$  is countable, we have

$$\prod_\alpha E_\alpha = \bigcap_\alpha \pi_\alpha^{-1}(E_\alpha) \in \sigma(\mathcal{E}).$$

This implies that  $\sigma(\mathcal{F}) \subset \sigma(\mathcal{E})$  and the proof is complete. □

**Corollary.** If  $\mathfrak{M}_i$  is  $\sigma$ -algebra for  $i = 1, 2, 3$ , then

$$\mathfrak{M}_1 \otimes (\mathfrak{M}_2 \otimes \mathfrak{M}_3) = (\mathfrak{M}_1 \otimes \mathfrak{M}_2) \otimes \mathfrak{M}_3 = \mathfrak{M}_1 \otimes \mathfrak{M}_2 \otimes \mathfrak{M}_3,$$

since they are all generated by

$$\{M_1 \times (M_2 \times M_3)\} = \{(M_1 \times M_2) \times M_3\} = \{M_1 \times M_2 \times M_3\}.$$

**Theorem.** Let  $X_1, \dots, X_n$  be metric spaces and  $X = \prod_{i=1}^n X_i$  be equipped with the usual metric. Then,  $\bigotimes_{i=1}^n \mathfrak{B}_{X_i} \subset \mathfrak{B}_X$ . However, if each  $X_i$  is separable, then  $\mathfrak{B}_X = \bigotimes_{i=1}^n \mathfrak{B}_{X_i}$ .

*Proof.* We know by the previous theorem that  $\bigotimes_{i=1}^n \mathfrak{B}_{X_i}$  is generated by  $\{\prod_i U_i : U_i \subset X_i \text{ open}\}$ . However,  $\prod_i U_i$  is open in  $X$ . Therefore,  $\bigotimes_{i=1}^n \mathfrak{B}_{X_i} \subset \mathfrak{B}_X$ .

Suppose now each  $X_i$  is separable and let  $D_i \subset X_i$  be countable and dense. Consider

$$\mathcal{E}_i = \{B(x_i, r) : X_i \in D_i, r = \infty \text{ or } r \in \mathbb{Q}^+\},$$

which is countable and  $\sigma(\mathcal{E}_i) = \mathfrak{B}_{X_i}$  since every open set in  $X_i$  is countable union of elements in  $\mathcal{E}_i$ . Similarly,  $\mathfrak{B}_X$  is generated by  $\{\prod_i E_i : E_i \in \mathcal{E}_i\}$ . But item 2 from the previous theorem implies that  $\bigotimes_{i=1}^n \mathfrak{B}_{X_i}$  is generated by the same set. Therefore,  $\bigotimes_{i=1}^n \mathfrak{B}_{X_i} = \mathfrak{B}_X$ . □

**Remark.** The above theorem is not true in general if  $X_i$  is not separable for some  $i$ .

**Definition.** Let  $X$  be a metric space. Define

$$F_\sigma(X) = \left\{ \bigcup_{k=0}^{\infty} C_k : C_k \subset X \text{ closed} \right\},$$

$$G_\delta(X) = \left\{ \bigcap_{k=0}^{\infty} U_k : U_k \subset X \text{ open} \right\}.$$

Note that  $F_\sigma(X) \subset \mathfrak{B}_X$  and  $G_\delta(X) \subset \mathfrak{B}_X$ .

**Theorem.** Let  $X$  be a metric space. Then the following holds:

1.  $F_\sigma$  and  $G_\delta$  are both closed under finite union and intersection.
2. If  $C \subset X$  is closed, then  $C \in G_\delta$ . If  $U \subset X$  is open, then  $U \in F_\sigma$ .
3. Suppose  $X$  is  $\sigma$ -compact, that is,  $X = \bigcup_{n=0}^{\infty} K_n$  for  $K_n \subset X$  compact, then each  $F \in F_\sigma$  is also  $\sigma$ -compact. In particular, all open sets are  $\sigma$ -compact.

**Theorem.** Let  $X$  and  $Y$  be metric spaces and  $f : X \rightarrow Y$  be continuous. Then the following holds:

1.  $E \in F_\sigma(Y)$  implies that  $f^{-1}(E) \in F_\sigma(X)$ , and  $E \in G_\delta(Y)$  implies that  $f^{-1}(E) \in G_\delta(X)$ .
2. If  $E \in \mathfrak{B}(Y)$ , then  $f^{-1}(E) \in \mathfrak{B}(X)$ .

**Theorem.** Let  $X$  and  $Y$  be metric spaces with  $X$   $\sigma$ -compact. Then,

1. If  $E \in F_\sigma(X)$  and  $f : E \rightarrow Y$  is continuous, then  $f(E) \in F_\sigma(Y)$  and  $\sigma$ -compact.
2. If  $f : X \rightarrow Y$  is a continuous injection, then  $E \in \mathfrak{B}(X)$  implies  $f(E) \in \mathfrak{B}(Y)$ .

**Corollary.** Let  $\emptyset \neq X \subset Y$  for  $Y$  a metric space. Then  $\mathfrak{B}(X) = \mathfrak{B}(Y)_X := \{X \cap E : E \in \mathfrak{B}(Y)\}$ .

*Proof.* We know  $V \subset X$  open if and only if  $V = X \cap U$  for some  $U$  open in  $Y$ . Therefore,

$$\{V \subset X : V \text{ open in } X\} \subset \mathfrak{B}(Y)_X.$$

This implies that  $\mathfrak{B}(X) \subset \mathfrak{B}(Y)_X$ .

On the other hand, the inclusion map  $I : X \rightarrow Y$  is a continuous injection, so if  $E \in \mathfrak{B}(Y)$ , then  $I^{-1}(E) \in \mathfrak{B}(X)$ . However,  $I^{-1}(E) = E \cap X$ . Therefore,  $\mathfrak{B}(Y)_X \subset \mathfrak{B}(X)$ . □



### 3.1.2 Measures

**Definition** (Measure). Let  $X$  be a set with  $\mathfrak{M}$  a  $\sigma$ -algebra on  $X$ . A **measure** is a map  $\mu : \mathfrak{M} \rightarrow [0, \infty]$  such that

1.  $\mu(\emptyset) = 0$ .
2. If  $\{E_k\}_{k=0}^\infty \subset \mathfrak{M}$  pairwise disjoint, then  $\mu(\bigcup_{k=0}^\infty E_k) = \sum_{k=0}^\infty \mu(E_k)$ .

Such a triple  $(X, \mathfrak{M}, \mu)$  is a **measure space**.

**Definition.** We say  $(X, \mathfrak{M}, \mu)$  is **finite** if  $\mu(X) < \infty$ . We say  $(X, \mathfrak{M}, \mu)$  is  **$\sigma$ -finite** if  $X = \bigcup_{n=0}^\infty X_n$  for  $X_n \in \mathfrak{M}$  and  $\mu(X_n) < \infty$ .

**Theorem.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space. Then the following holds:

1. If  $E$  and  $F$  is measurable and  $E \subset F$ , then  $\mu(E) \leq \mu(F)$ .
2. If  $E_k \in \mathfrak{M}$  for all  $k \in \mathbb{N}$ , then  $\mu(\bigcup_{k=0}^\infty E_k) \leq \sum_{k=0}^\infty \mu(E_k)$ .

### 3.1.3 Outer measures and Carathéodory construction

**Definition** (Outer measure). Let  $X$  be a set. An **outer measure** is a map  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  such that

1.  $\mu^*(\emptyset) = 0$ .
2.  $E \subset F$  implies  $\mu^*(E) \leq \mu^*(F)$ .
3. If  $E_k \subset X$  for all  $k \in \mathbb{N}$ , then  $\mu^*(\bigcup_{k=0}^\infty E_k) \leq \sum_{k=0}^\infty \mu^*(E_k)$ .

**Proposition.** Let  $\mu_\alpha^* : \mathcal{P}(X) \rightarrow [0, \infty]$  be an outer measure for all  $\alpha \in A \neq \emptyset$ . Then  $\lambda : \mathcal{P}(X) \rightarrow [0, \infty]$  defined by  $\lambda(E) = \sup_{\alpha \in A} \mu_\alpha^*(E)$  is an outer measure.

*Proof.* 1.  $\mu_\alpha^*(\emptyset) = 0$  for all  $\alpha \in A$  implies that  $\lambda(\emptyset) = 0$ .

2. Suppose  $E \subset F$ , then  $\mu_\alpha^*(E) \leq \mu_\alpha^*(F) \leq \lambda(F)$  for all  $\alpha \in A$ . Take the sup and we obtain  $\lambda(E) \leq \lambda(F)$ .

3. Let  $E_k \subset X$  for each  $k \in \mathbb{N}$ . Then,

$$\mu_\alpha^*\left(\bigcup_{k=0}^\infty E_k\right) \leq \sum_{k=0}^\infty \mu_\alpha^*(E_k) \leq \sum_{k=0}^\infty \lambda(E_k)$$

This implies that  $\lambda(\bigcup_{k=0}^\infty E_k) \leq \sum_{k=0}^\infty \lambda(E_k)$ .

□

**Definition.** Let  $X$  be a set with outer measure  $\mu^*$ . Say a set  $E \subset X$  is measurable with respect to  $\mu^*$  if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for all  $A \subset X$ .

**Theorem** (Carathéodory construction). Let  $X$  be a set with outer measure  $\mu^*$ , the following holds.

1. The collection  $\mathfrak{M} = \{E \subset X : E \text{ measurable}\}$  is a  $\sigma$ -algebra.
2. If  $E \subset X$  is such that  $\mu^*(E) = 0$ , then  $E \in \mathfrak{M}$ .
3. The restriction  $\mu = \mu^*|_{\mathfrak{M}}$  is a measure, and  $(X, \mathfrak{M}, \mu)$  is a complete measure space.

**Definition** (Cover regular). Let  $\mu^*$  be an outer measure on  $X$ . Say  $\mu^*$  is cover-regular if for any  $A \subset X$ , there exists  $E \in \mathfrak{M}$  such that  $A \subset E$  and  $\mu^*(A) = \mu(E)$ .

**Proposition.** Let  $\mu^*$  be an outer measure on  $X$ . Then  $\mu^*$  is outer-regular if and only if for any  $A \subset X$ ,  $\mu^*(A) = \inf \{\mu(E) : A \subset E \in \mathfrak{M}\}$ . In either case, the inf is a min.

**Proposition.** Let  $X$  be a set with cover-regular outer measure  $\mu^*$ . Suppose for  $n \in \mathbb{N}$ , we have  $A_n \subset A_{n+1}$ . Then,

$$\mu^* \left( \bigcup_{n=0}^{\infty} A_n \right) = \lim_{n \rightarrow \infty} \mu^*(A_n).$$

*Proof.* First note that  $\mu^*(A_n) \leq \mu^*(A_{n+1}) \leq \mu^*(A)$ , where  $A = \bigcup_{n=0}^{\infty} A_n$ . Therefore,

$$\lim_{n \rightarrow \infty} \mu^*(A_n) \leq \mu^*(A).$$

On the other hand, by cover regularity, there exists  $A_n \subset E_n \in \mathfrak{M}$  such that  $\mu^*(A_n) = \mu(E_n)$ . In particular,  $\lim_{n \rightarrow \infty} \mu^*(A_n) = \lim_{n \rightarrow \infty} \mu(E_n)$ . Then,

$$A = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} A_k \subset \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} E_k \in \mathfrak{M},$$

and

$$\mu^*(A) \leq \mu \left( \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} E_k \right) = \lim_{n \rightarrow \infty} \mu \left( \bigcap_{k=n}^{\infty} E_k \right) \leq \lim_{n \rightarrow \infty} \mu(E_n) = \lim_{n \rightarrow \infty} \mu(A_n),$$

where we have used monotone continuity of **measure**. Therefore,  $\lim_{n \rightarrow \infty} \mu^*(A_n) = \mu^* \left( \bigcup_{n=0}^{\infty} A_n \right)$ .  $\square$

### 3.1.4 Constructing outer measures

**Definition.** Let  $X$  be a set. A gauge on  $X$  is a pair  $(\mathcal{E}, \gamma)$  where  $\mathcal{E} \subset \mathcal{P}(X)$  is such that  $\emptyset \in \mathcal{E}$  and  $\gamma : \mathcal{E} \rightarrow [0, \infty]$  is such that  $\gamma(\emptyset) = 0$ .

**Theorem.** Let  $X$  be a set and  $(\mathcal{E}, \gamma)$  be a gauge on  $X$ . Define  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  via

$$\mu^*(E) = \inf \left\{ \sum_{n=0}^{\infty} \gamma(E_n) : E \subset \bigcup_{n=0}^{\infty} E_n \text{ and } \{E_n\}_{n=0}^{\infty} \subset \mathcal{E} \right\}.$$

Then  $\mu^*$  is an outer measure on  $X$  and hence generates  $(X, \mathfrak{M}, \mu)$ , a complete measure space thorough Carathéodory construction.

*Proof.* \*\*\* TO-DO \*\*\*  $\square$

**Theorem.** Let  $(X, d)$  be a metric space with gauge  $(\mathcal{E}, \gamma)$  and outer measures  $\mu_\delta^* : \mathcal{P}(X) \rightarrow [0, \infty]$  produced by  $(\mathcal{E}_\delta, \gamma_\delta)$  for  $\delta > 0$ . Define  $\mu_d^* : \mathcal{P}(X) \rightarrow [0, \infty]$  by

$$\mu_d^*(A) = \sup_{\delta > 0} \mu_\delta^*(A).$$

Then  $\mu_d^*$  is a metric outer measure. Moreover,  $\mu_d^*(A) = \lim_{\delta \rightarrow 0} \mu_\delta^*(A)$  for  $A \subset X$ .

*Proof.* \*\*\* TO-DO \*\*\*  $\square$

**Definition.** We call  $\mu_d^*$  the metric outer measure generated by  $(\mathcal{E}, \gamma)$ .

**Lemma.** Let  $X$  be a set with gauge  $(\mathcal{E}, \gamma)$  that covers  $X$ . Let  $A \subset X$ , then the following holds:

1. Let  $\mu^*$  be the outer measure generated by  $(\mathcal{E}, \gamma)$ . Then there exists collection  $\{E_{m,n}\}_{m,n=0}^{\infty} \subset \mathcal{E}$  such that  $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$  such that  $A \subset E$  and  $\mu^*(A) = \mu^*(E)$ .
2. Suppose  $(X, d)$  is metric space and the gauge is fine. Let  $\mu_d^*$  be the metric outer measure. Then there exists collection  $\{E_{m,n}\}_{m,n=0}^{\infty} \subset \mathcal{E}$  such that  $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$  such that  $A \subset E$  and  $\mu^*(A) = \mu^*(E)$ .

*Proof.* The proof for (1) is very similar to the proof for (2), so we only show (2) as follows. Since the gauge is fine,  $(\mathcal{E}_\delta, \gamma_\delta)$  covers  $X$  for all  $\delta > 0$ . Then, for any  $m \in \mathbb{N}$ , there exists  $\{E_{m,n}\}_n \subset \mathcal{E}_{2^{-m}}$  such that  $A \subset \bigcup_{n=0}^{\infty} E_{m,n}$  and  $\sum_{n=0}^{\infty} \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}$ . Now let  $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$ . Note that  $A \subset E$  and for any  $m \in \mathbb{N}$ , we have

$$\mu_{2^{-m}}^*(E) \leq \mu_{2^{-m}}^* \left( \bigcup_{n=0}^{\infty} E_{m,n} \right) \leq \sum_{n=0}^{\infty} \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}.$$

Taking the limit as  $m \rightarrow \infty$ , we have

$$\mu_d^*(E) \leq \mu_d^*(A) \leq \mu_d^*(E),$$

as desired. □

**Theorem.** Let  $(X, d)$  be metric space with  $(\mathcal{E}, \gamma)$  such that all sets in  $\mathcal{E}$  are open. Assume that  $\mu^*$  is a metric outer measure on  $X$  such that either

1.  $\mu^*$  is generated by  $(\mathcal{E}, \gamma)$ , or
2.  $\mu^* = \mu_d^*$  is generated by  $(\mathcal{E}_\delta, \gamma_\delta)$ .

Further suppose that  $X = \bigcup_{n=0}^{\infty} A_n$  where  $A_n \subset X$  is such that  $\mu^*(A_n) < \infty$ . Then the following holds:

1. The gauge covers  $X$  in case 1 and is fine in case 2.
2. In both cases,  $\mu^*$  is cover-regular. More precisely, for each  $A \subset X$ , there is  $G \in G_\delta(X) \subset \mathfrak{B}(X) \subset \mathfrak{M}$  such that  $A \subset G$  and  $\mu^*(A) = \mu^*(G)$ .
3. In both cases, the following are equivalent for  $E \subset X$ :
  - (a)  $E \in \mathfrak{M}$ , i.e.  $E$  is measurable.
  - (b) there exists  $G \in G_\delta(X)$  such that  $E \subset G$  and  $\mu^*(G \setminus E) = 0$ .
  - (c) there exists  $F \in F_\sigma(X)$  such that  $F \subset E$  and  $\mu^*(E \setminus F) = 0$ .

*Proof. Step 0: proof for (1) and (2).*

We know  $X = \bigcup_{n=0}^{\infty} A_n$  for some  $\mu^*(A_n) < \infty$ . For case (1), we can pick  $\{E_{n,m}\} \subset \mathcal{E}$  such that  $A_n \subset \bigcup_{m=0}^{\infty} E_{n,m}$ . Then  $X = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n,m} E_{n,m}$ . Therefore,  $\mathcal{E}$  covers  $X$ . For case (2), note that  $\mu_d^*(A_n) < \infty$  and  $\mu_d^*(A_n) \geq \mu_\delta^*(A_n)$  for each  $\delta > 0$  and  $n \in \mathbb{N}$ . Then for each  $\delta > 0$ , there exists  $\{E_{n,m}\} \subset \mathcal{E}_\delta$  such that  $A_n \subset \bigcup_{m=0}^{\infty} E_{n,m}$ . It follows that  $X = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n,m} E_{n,m}$ . Therefore,  $(\mathcal{E}, \gamma)$  is fine.

We have the following observations:

1.  $\mu^*$  is a metric outer measure. This implies that  $\mathfrak{B}(X) \subset \mathfrak{M}$ .
2.  $G_\delta(X) \cup F_\sigma(X) \subset \mathfrak{B}(X) \subset \mathfrak{M}$  and  $\mu^*(A) = 0$  implies  $A \in \mathfrak{M}$ .
3. By previous lemma and all sets in  $\mathcal{E}$  are open, we know for each  $A \subset X$  there is  $E \in G_\delta(X)$  such that  $A \subset E$  and  $\mu^*(A) = \mu^*(E)$ . In particular,  $\mu^*$  is cover regular.

**Step 1: starting on (3).**

For (b)  $\implies$  (a), suppose (b) holds for  $E \subset X$ . Then  $E = G \setminus (G \setminus E) \in \mathfrak{M}$  since  $\mu^*(G \setminus E) = 0$ .

For (c)  $\implies$  (a), suppose (c) holds for  $E \subset X$ . Then  $E = F \cup (E \setminus F) \in \mathfrak{M}$  since  $\mu^*(E \setminus F) = 0$ .

Next we show “(a)  $\implies$  (c)” implies “(a)  $\implies$  (b)”. Suppose  $E \in \mathfrak{M}$ , then  $E^c \in \mathfrak{M}$ . By (a)  $\implies$  (b) we know there exists  $F \in F_\sigma$  such that  $F \subset E^c$  and  $\mu^*(E^c \setminus F) = 0$ . Let  $G = F^c \in G_\delta$  then  $E \subset G$  and  $G \setminus E = E^c \setminus F$ .

Therefore, it remains to show (a)  $\implies$  (c) to complete the proof for the theorem.

**Step 2: reduction for (a)  $\implies$  (c).**

Claim it suffices to show it for  $E$  such that  $\mu^*(E) < \infty$ . Suppose we did this and  $\mu^*(E) = \infty$ . Using observation there exists  $B_n \in \mathfrak{M}$  such that  $A_n \subset B_n$  and  $\mu^*(B_n) = \mu^*(A_n) < \infty$ . Then  $E_n = E \cap B_n \in \mathfrak{M}$  and  $\mu^*(E_n) < \infty$ . Then by special case there is  $F_n \in F_\sigma(X)$  such that  $F_n \subset E_n$  and  $\mu^*(F_n \setminus E_n) = 0$ . Let  $F = \bigcup_{n=0}^\infty F_n \in F_\sigma$  then  $F \subset \bigcup_{n=0}^\infty E_n = E$  and

$$\mu^*(E \setminus F) \leq \sum_{n=0}^\infty \mu^*(E_n \setminus F_n) = 0.$$

**Step 3: further reduction.**

Claim it suffices to show it for the case where  $\mu^*(E) < \infty$  and  $E \in G_\delta(X)$ . Suppose we have proved this and consider  $E \subset X$  such that  $\mu^*(E) < \infty$ . Observation 3 allows us to pick  $G \in G_\delta(X)$  such that  $E \subset G$  and  $\mu^*(E) = \mu^*(G)$ . Now pick  $H \in G_\delta$  such that  $G \setminus E \subset H$  and  $\mu^*(H) = \mu^*(G \setminus E)$ .

Now apply special case. This gives  $F \in F_\sigma$  such that  $F \subset G$  and  $\mu^*(G \setminus F) = 0$ . Let  $K = F \setminus H = F \cap H^c \in F_\sigma$  and  $K = F \cap H^c \subset G \cap (G \setminus E)^c \subset E$ .

Note that  $E, F, G, H, K \in \mathfrak{M}$ , so

$$\begin{aligned} \mu^*(E \setminus K) &= \mu^*(E) - \mu^*(K) \\ &= \mu^*(G) - \mu^*(F \setminus H) \\ &= \mu^*(G) - \mu^*(F) + \mu^*(F \cap H) \\ &\leq \mu^*(G) - \mu^*(F) + \mu^*(H) \\ &= \mu^*(G \setminus F) + \mu^*(H) \\ &= \mu^*(G \setminus E) \\ &= \mu^*(G) - \mu^*(E) \\ &= 0. \end{aligned}$$

Therefore,  $K$  is the desired  $F_\sigma$  set.

**Step 4: finishing (a)  $\implies$  (c).**

Suppose  $E \in G_\delta(X)$  and  $\mu^*(E) < \infty$ . Write  $E = \bigcup_{n=0}^\infty V_n$  where  $V_n \subset X$  open. For  $m, n \in \mathbb{N}$ , let

$$C_{n,m} = \{x \in V_n : \text{dist}(x, V_n^c) \geq 2^{-m}\} \subset V_n.$$

Note that  $C_{n,m}$  is closed,  $C_{n,m} \subset C_{n,m+1}$ ,  $V_n = \bigcup_m C_{n,m}$ . Since  $E, C_{n,m}, V_n \in \mathfrak{M}$ , we have

$$\mu^*(E) = \mu^*(E \cap V_n) = \lim_{m \rightarrow \infty} \mu^*(E \cap C_{n,m}).$$

Thus, there exists  $M(n, k)$  such that  $\mu^*(E \setminus C_{n, M(n, k)}) < 2^{-n-k}$ . Now let  $D_k = \bigcup_{n=0}^\infty C_{n, M(n, k)}$  closed. Also,  $D_k \subset \bigcup_{n=0}^\infty V_n = E$  and

$$\mu^*(E) - \mu^*(D_k) = \mu^*(E \setminus D_k) \leq \sum_{n=0}^\infty \mu^*(E \setminus C_{n, M(n, k)}) \leq 2^{-k+1}.$$

Let  $F = \bigcup_{k=0}^\infty D_k \subset E$  and note that  $F \in F_\sigma$ . Then

$$\mu^*(E \setminus F) = \mu^*(E) - \mu^*(F) \leq \mu^*(E) - \mu^*(D_k) < 2^{-k+1}$$

for all  $k \in \mathbb{N}$ . Therefore,  $\mu^*(E \setminus F) = 0$ .

□

**Lemma.** Suppose  $(X, d)$  metric space with metric outer measure  $\mu^*$ . Suppose  $X = \bigcup_{n=0}^\infty V_n$  for  $V_n \subset X$  open and  $\mu^*(V_n) < \infty$ . Suppose  $E \subset G \in G_\delta(X)$  such that  $\mu^*(G \setminus E) = 0$ . Then for each  $\varepsilon > 0$ , there exists open  $U \subset X$  such that  $E \subset U$  and  $\mu^*(U \setminus E) < \varepsilon$ .

*Proof.* Let  $E_n = E \cap V_n$  and  $G = G \cap V_n$ . Write  $G = \bigcap_{j=0}^{\infty} W_j$  where  $W_j$  open. Now set

$$Z_{n,m} = V_n \cap \bigcap_{j=0}^m W_j,$$

which are open for all  $n, m \in \mathbb{N}$ . Now notice that  $G_n \subset Z_{n,m+1} \subset Z_{n,m} \subset V_n$ . Note that  $\mu^*(V_n) < \infty$ , so  $\mu^*(G_n) = \lim_{m \rightarrow \infty} \mu^*(Z_{n,m})$ . Therefore, for all  $\varepsilon > 0$ , there exists  $M(n)$  such that

$$\mu^*(Z_{n,M(n)} \setminus G_n) < \varepsilon 2^{-n-2}.$$

Then set  $U = \bigcup_{n=0}^{\infty} Z_{n,M(n)} \supset \bigcup_{n=0}^{\infty} G_n = G \supset E$  open, then we have

$$\begin{aligned} \mu^*(U \setminus E) &= \mu^*(U \setminus G) + \mu^*(G \setminus E) \\ &= \mu^*\left(\bigcup_{n=0}^{\infty} Z_{n,M(n)} \cap \bigcap_{n=0}^{\infty} G_n^c\right) \\ &\leq \sum_{n=0}^{\infty} \mu^*(Z_{n,M(n)} \setminus G_n) \\ &< \varepsilon, \end{aligned}$$

as desired. □

**Definition** (Outer-regular). Let  $X$  be a metric space,  $\mathfrak{M}$  a  $\sigma$ -algebra with  $\mathfrak{B}(X) \subset \mathfrak{M}$  and suppose  $\mu : \mathfrak{M} \rightarrow [0, \infty]$  is a measure. Say  $\mu$  is outer-regular if

$$\mu(E) = \inf \{ \mu(U) : E \subset U \text{ open} \}.$$

### 3.2 Lebesgue and Hausdorff measure

\*\*\* TO-DO \*\*\*

### 3.3 Measurable and $\mu$ -measurable functions

**Definition** (Measurable functions). Let  $(X, \mathfrak{M})$  and  $(Y, \mathfrak{N})$  be measurable sets. A map  $f : X \rightarrow Y$  is called  $(\mathfrak{M}, \mathfrak{N})$  measurable if  $f^{-1}(E) \in \mathfrak{M}$  for all  $E \in \mathfrak{N}$ .

\*\*\* TO-DO \*\*\*

**Definition** (Simple functions). Let  $(X, \mathfrak{M})$  and  $(Y, \mathfrak{N})$  be measurable sets. A map  $f : X \rightarrow Y$  is called simple if it is measurable and  $f(X)$  is finite. Write the set of all simple functions from  $X$  to  $Y$  as  $S(X, Y)$ .

**Theorem** (Characterization of  $\overline{\mathbb{R}}$  measurability). Let  $(X, \mathfrak{M})$  be measure space and  $f : X \rightarrow \overline{\mathbb{R}}$ . The following are equivalent:

1.  $f$  is measurable.
2. There exists  $\{\varphi_k\}_{k=0}^{\infty} \subset S(X; \overline{\mathbb{R}})$  such that  $\varphi_k \rightarrow f$  pointwise as  $k \rightarrow \infty$ .

Moreover, if  $f$  is measurable, the sequence can be built such that

- On the set  $\{f \geq 0\}$ , we have  $0 \leq \varphi_k \leq \varphi_{k+1} \leq f$ .
- On the set  $\{f < 0\}$ , we have  $f \leq \varphi_{k+1} \leq \varphi_k \leq 0$ .
- If  $f$  is actually from  $X$  to  $\mathbb{R}$  and is bounded, then  $\varphi_k \rightarrow f$  uniformly.

*Proof.* (2)  $\implies$  (1). Pointwise limit of measurable functions are measurable.

(1)  $\implies$  (2). Suppose  $f : X \rightarrow [0, \infty]$  is measurable. For  $k \in \mathbb{N}$ , define  $\varphi_k : [0, \infty)$  by

$$\varphi_k(x) = \begin{cases} (j-1)2^{-k} & \text{if } (j-1)2^{-k} \leq f(x) < j2^{-k} \text{ for } 1 \leq j \leq k2^k, \\ k & \text{if } f(x) > k. \end{cases}$$

Because  $f$  is measurable,  $\varphi_k$  is simple for each  $k \in \mathbb{N}$ .

Note that  $0 \leq \varphi_k \leq \varphi_{k+1} \leq f$ . Also, if  $f(x) < \infty$ , then  $0 \leq f(x) - \varphi_k(x) \leq 2^{-k}$ . If  $f(x) = \infty$ , then  $\varphi_k(x) = k$ . This shows that  $\varphi_k \rightarrow f$ . Moreover, if  $f$  is bounded then  $\varphi_k \rightarrow f$  uniformly.

In the general case, apply the special case to  $f$  on  $\{f \geq 0\}$  and  $-f$  on  $\{f < 0\}$ .

□

**Definition** (Separably-valued). Let  $X$  be a set and  $Y$  a metric space. A map  $f : X \rightarrow Y$  is **separably-valued** if  $f(X) \subset Y$  is separable.

**Theorem.** Let  $(X, \mathfrak{M})$  be measure space and  $Y$  be metric space,  $f : X \rightarrow Y$ . The following are equivalent for  $f : X \rightarrow Y$ :

1.  $f$  is  $(\mathfrak{M}, \mathfrak{B}(Y))$  measurable and separably valued.
2. There exists  $\{\varphi_k\}_{k=0}^{\infty} \in S(X; Y)$  such that  $\varphi_k \rightarrow f$  pointwise.

*Proof.* (2)  $\implies$  (1). The pointwise limit of measurable function is measurable. On the other hand,  $f(X) = \bigcup_{k=0}^{\infty} \varphi_k(X)$ , which is separable since  $\varphi_k(X)$  finite for any  $k \in \mathbb{N}$ .

(1)  $\implies$  (2). Assume initially that  $Y$  is totally bounded. Then for each  $n \in \mathbb{N}$  there exists  $y_0^n, \dots, y_{K(n)}^n \in Y$  such that  $Y = \bigcup_{k=0}^{K(n)} B(y_k^n, 2^{-n})$ . Let  $V_0^n = B(y_0^n, 2^{-n})$  and for  $k \geq 1$  define  $V_k^n = B(y_k^n, 2^{-n}) \setminus \bigcup_{j=0}^{k-1} B(y_j^n, 2^{-n})$ . Then,  $Y = \bigcup_{k=0}^{M(n)} V_k^n$  where  $V_k^n = \emptyset$  for  $M(n) < k \leq K(n)$ .

Define  $\varphi_n : Y \rightarrow \{y_0^n, \dots, y_{M(n)}^n\}$  via  $\varphi_n(y) = y_k^n$  if  $y \in V_k^n$ . Clearly  $\varphi_n$  is simple and  $d(\varphi_n(y), y) < 2^{-n}$  for all  $n \in \mathbb{N}$  and  $y \in Y$ . Therefore,  $\varphi_n(y) \rightarrow (y)$  pointwise. Then  $f_n = \varphi_n \circ f$  are simple functions from  $X$  to  $Y$ . Also, since  $\varphi_n \rightarrow \text{id}$  pointwise,  $f_n \rightarrow f$  pointwise.

Now consider the general case in which  $f(X)$  is a separable subset of  $Y$ . Then there exists a homeomorphism  $h : f(X) \rightarrow Z$  for  $Z$  a totally bounded metric space, for example take  $Z$  a subset of Hilbert cube  $H^\infty$  since all separable metric space is homeomorphism to a subset of the Hilbert cube. Thus  $h \circ f : X \rightarrow Z$  is measurable with  $Z$  totally bounded, so the special case provides a sequence  $\{\varphi_n\}_{n=0}^{\infty} \subset S(X; Z)$  such that  $\varphi_n \rightarrow h \circ f$  pointwise. Then,  $h^{-1} \circ \varphi_n \in S(X; Y)$  is such that  $h^{-1} \circ \varphi_n \rightarrow h^{-1} \circ h \circ f = f$  pointwise, using continuity of  $h$  and  $h^{-1}$ .

□

**Definition** (Almost everywhere). Let  $(X, \mathfrak{M}, \mu)$  be a measure space and let  $P(x)$  be a proposition for every  $x \in X$ . Say  $P$  is true **almost everywhere** (a.e.) if there exists a set  $N \in \mathfrak{M}$  such that  $\mu(N) = 0$  and  $P(x)$  is true for all  $x \in N^c$ .

**Theorem.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space. Let  $Y$  be a metric space,  $f : X \rightarrow Y$ . The following are equivalent:

1. There exists  $\{\psi_n\}_{n=0}^{\infty} \subset S(X; Y)$  such that  $\psi_n \rightarrow f$  pointwise a.e. in  $X$ .
2. There exists a measurable and separably valued  $F : X \rightarrow Y$  such that  $f = F$  a.e.
3. There exists a null set  $N \in \mathfrak{M}$  and a measurable  $F : X \rightarrow Y$  such that  $f = F$  on  $N^c$  and  $f(N^c)$  is separable in  $Y$ .

*Proof.* (1)  $\implies$  (2). There exists  $N \in \mathfrak{M}$  null such that  $\psi_n \rightarrow f$  pointwise in  $N^c$ . Thus,  $f : N^c \rightarrow Y$  is measurable and separably valued by the previous theorem. Note the constant map  $N \ni x \mapsto y \in Y$  for  $y \in Y$  fixed is measurable. Thus we can define  $F : X \rightarrow Y$  by

$$F(x) = \begin{cases} f(x) & (x \in N^c), \\ y & (x \in N). \end{cases}$$

Then  $F$  is measurable. It is also separably valued since  $F(X) = f(N^c) \cup \{y\}$ .

(2)  $\implies$  (3). Trivial.

(3)  $\implies$  (1). Note that  $F : N^c \rightarrow Y$  is measurable and  $F(N^c) = f(N^c)$  is separable. By previous theorem, there exists  $\{\varphi_n\}_{n=0}^\infty \in S(N^c; Y)$  such that  $\varphi_n \rightarrow F = f$  pointwise on  $N^c$ . Now let  $\psi_n \in S(X; Y)$  be  $\varphi_n$  in  $N^c$  and  $y \in Y$  fixed in  $N$ . Then  $\psi_n \rightarrow f$  pointwise in  $N^c$ .

□

**Definition.** Let  $(X, \mathfrak{M})$  be measurable,  $Y$  be either a normed vector space or  $\overline{\mathbb{R}}$ . Let  $\psi \in S(X; Y)$ .

1. A **representation** of  $\psi$  is a finite and well-defined sum  $\psi = \sum_{k=1}^K v_k \chi_{E_k}$  for  $v_k \in Y$  and  $E_k \in \mathfrak{M}$ .
2. A **canonical representation** is  $\psi = \sum_{v \in \psi(X)} v \chi_{\psi^{-1}(\{v\})}$
3. Now suppose  $\mu$  is a measure. We say a representation  $\psi = \sum_{k=1}^K v_k \chi_{E_k}$  is **finite** if  $\mu(E_k) < \infty$  for all  $k$  such that  $v_k \neq 0$ . We say  $\psi$  is a **finite simple function** if it has a finite representation.

We write  $S_{\text{fin}}(X; Y) = \{f \in S(X; Y) : f \text{ is finite}\}$ . Note that it is clear  $\psi$  is finite if and only if the canonical representation is finite if and only if  $\mu(\text{supp}(\psi)) < \infty$  where  $\text{supp}(\psi) = \{x \in X : \psi(x) \neq 0\}$  is the support of  $\psi$ .

**Definition.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and  $Y$  be a metric space.

1. We say  $f : X \rightarrow Y$  is **almost measurable** if  $f = F$  a.e. with  $F : X \rightarrow Y$  is measurable.
2. We say  $f : X \rightarrow Y$  is **almost separably valued** if there exists a null set  $N \in \mathfrak{M}$  such that  $f(N^c)$  is separable.
3. We say  $f : X \rightarrow Y$  is  **$\mu$ -measurable** if it is almost measurable and almost separably valued. Equivalently,  $f$  is the a.e. limit of simple functions.
4. Suppose  $Y$  is a normed vector space or  $\overline{\mathbb{R}}$ . We say  $f : X \rightarrow Y$  is **strongly  $\mu$ -measurable** if there exists  $\{\psi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; Y)$  such that  $\psi_n \rightarrow f$  a.e. as  $n \rightarrow \infty$ .

**Example.** Let  $X = \{1, 2, 3\}$  and  $\mathfrak{M} = \{\emptyset, \{1, 2\}, \{3\}, \{1, 2, 3\}\}$ . Let  $f, g : X \rightarrow \mathbb{R}$  via  $f(x) = x$  and  $g(x) = 3$ . Then  $f$  is not measure since  $f^{-1}(\{1\}) = \{1\} \notin \mathfrak{M}$  but  $g$  is measurable.

Now equip  $(X, \mathfrak{M})$  with the measure  $\delta_3$ . Then,  $f = g$  a.e. This shows that equality almost everywhere does not preserve measurability. The problem is that  $(X, \mathfrak{M}, \delta_3)$  is not **complete**.

This brings us to the next theorem.

**Theorem.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space. Then the following are equivalent:

1.  $(X, \mathfrak{M}, \mu)$  is complete.
2. If  $(Y, \mathfrak{N})$  is a measure space,  $f, g : X \rightarrow Y$ ,  $f$  is measurable and  $f = g$  a.e., then  $g$  is measurable.
3. If  $Y$  is a metric space with  $\text{card } Y = 2$ ,  $f, g : X \rightarrow Y$ ,  $f$  measurable,  $f = g$  a.e., then  $g$  is measurable.

*Proof.* (1)  $\implies$  (2). Suppose  $f, g : X \rightarrow Y$ ,  $f$  is measurable,  $f = g$  a.e. Pick null set  $N \in \mathfrak{M}$  such that  $f = g$  on  $N^c$ . Take  $E \in \mathfrak{N}$ , then

$$\begin{aligned} g^{-1}(E) &= (g^{-1}(E) \cap N) \cup (g^{-1}(E) \cap N^c) \\ &= (g^{-1}(E) \cap N) \cup (f^{-1}(E) \cap N^c). \end{aligned}$$

Note that  $f^{-1}(E) \cap N^c$  is measurable, and  $g^{-1}(E) \cap N \subset N$  null, so it is also measurable. Therefore,  $g^{-1}(E)$  is measurable and  $g$  is measurable.

(2)  $\implies$  (3). Clear.

(3)  $\implies$  (1). Prove the contrapositive. Suppose  $(X, \mathfrak{M}, \mu)$  is not complete and  $Y = \{y, z\}$  a metric space. Find  $\emptyset \neq A \subsetneq B$  such that  $\mu(B) = 0$  and  $A \notin \mathfrak{M}$ . Define  $f, g : X \rightarrow Y$  by

$$g(x) = \begin{cases} y & (x \notin A), \\ z & (x \in A). \end{cases}$$

and  $f(x) = y$  be constant. Then  $f = g$  a.e.,  $f$  is measurable, and  $g$  is not measurable. □

**Corollary.** Let  $(X, \mathfrak{M}, \mu)$  be a complete measurable space,  $Y$  a separable metric space, and  $f : X \rightarrow Y$ . Then,  $f$  is  $\mu$ -measurable if and only if  $f$  is measurable.

**Proposition.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and  $Y$  be a metric space. The following holds:

1. Let  $f, g : X \rightarrow Y$ . If  $f$  is  $\mu$ -measurable and  $f = g$  a.e., then  $g$  is  $\mu$ -measurable.
2. Suppose  $Y$  is a normed vector space or  $\mathbb{R}$ . If  $f, g : X \rightarrow Y$ ,  $f$  is strongly  $\mu$ -measurable,  $f = g$  a.e., then  $g$  is strong  $\mu$ -measurable.

*Proof.* 1. Let  $\{\varphi_n\}_{n=0}^\infty \subset S(X; Y)$  be such that  $\varphi_n \rightarrow g$  pointwise a.e. Pick null set  $N \in \mathfrak{M}$  such that  $f = g$  on  $N^c$ . Pick null set  $Z \in \mathfrak{M}$  such that  $f = \lim_{n \rightarrow \infty} \varphi_n$ . This implies that  $g = \lim_{n \rightarrow \infty} \varphi_n$  on  $(N \cup Z)^c$ .

2. Same proof as the first item but let  $\{\varphi_n\}_{n=0}^\infty \in S_{\text{fin}}(X; Y)$ . □

**Theorem.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and  $Y$  be a normed vector space with  $V \neq \{0\}$ . Then the following are equivalent:

1.  $(X, \mathfrak{M}, \mu)$  is  $\sigma$ -finite.
2. If  $f : X \rightarrow Y$  is  $\mu$ -measurable, then  $f$  is strongly  $\mu$ -measurable.
3. Let  $f : X \rightarrow Y$ , then  $f$  is  $\mu$ -measurable if and only if  $f$  is strongly  $\mu$ -measurable.
4. If  $y \in Y \setminus \{0\}$ , then  $f : X \rightarrow Y$  via  $f(x) = y$  strongly  $\mu$ -measurable.

*Proof.* (1)  $\implies$  (2). Suppose  $(X, \mathfrak{M}, \mu)$  is  $\sigma$ -finite. We can find  $\{X_n\}_{n=0}^\infty \subset \mathfrak{M}$  such that  $X_n \subset X_{n+1}$ ,  $\mu(X_n) < \infty$  and  $\bigcup_{n=0}^\infty X_n = X$ . Let  $f : X \rightarrow Y$  be  $\mu$ -measurable. Pick  $\{\psi_n\}_{n=0}^\infty \subset S(X; Y)$  such that  $\psi_n \rightarrow f$  pointwise a.e. Define  $\varphi_n = \chi_{X_n} \psi_n$ . This shows that  $f$  is strongly  $\mu$ -measurable.

(2)  $\iff$  (3). Trivial since strongly  $\mu$ -measurability implies  $\mu$ -measurability.

(2)  $\implies$  (4). Constant function are  $\mu$ -measurable.

(4)  $\implies$  (1). Let  $y \in Y \setminus \{0\}$  and define  $f : X \rightarrow Y$  via  $f(x) = y$ . This is strongly  $\mu$ -measurable by assumption. Then there exists  $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; Y)$  such that  $\varphi_n \rightarrow f$  pointwise on  $N^c$  where  $N$  is null.

Pick  $\varepsilon > 0$  such that  $\{0\} \cap B(y, \varepsilon) = \emptyset$ . Set  $X_n = \varphi_n^{-1}(B(y, \varepsilon))$ . Then we have  $\mu(X_n) < \infty$ . For any  $x \in N^c$  and  $n$  sufficiently large,  $\varphi_n(x) \in B(y, \varepsilon)$ . Therefore,  $N^c \subset \bigcup_{n=0}^\infty X_n$  and the proof we are complete. □

Finally, we present a useful characterization of  $\mu$ -measurability of Banach-valued maps.

**Theorem (Pettis).** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and  $V$  be a Banach space over  $\mathbb{F}$ . Suppose  $W \subset V^*$  is a norming subspace. Let  $f : X \rightarrow V$ . Then the following are equivalent:



1.  $f$  is  $\mu$ -measurable.
2.  $f$  is almost separably valued, and  $w \circ f : X \rightarrow \mathbb{F}$  is  $\mu$ -measurable for each  $w \in V^*$ .
3.  $f$  is almost separably valued, and  $w \circ f : X \rightarrow \mathbb{F}$  is  $\mu$ -measurable for each  $w \in W$ .

In any case, there exists  $\{\varphi_n\}_{n=0}^\infty \subset S(X; V)$  such that  $\|\varphi_n\| \leq 2\|f\|$  on  $X$  such that  $\varphi_n \rightarrow f$  pointwise a.e. as  $n \rightarrow \infty$ . Moreover, the same equivalence holds with  $\mu$ -measurability replaced by strongly  $\mu$ -measurability and  $\{\varphi_n\}_{n=0}^\infty$  replaced by  $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; V)$ .

*Proof.* (1)  $\implies$  (2). Suppose  $f$  is  $\mu$ -measurable, which means it is almost separably valued. Each  $w \in V^*$  is also continuous so  $w \circ f$  is  $\mu$ -measurable.

(2)  $\implies$  (3). Trivial since  $W \subset V^*$ .

(3)  $\implies$  (1). Suppose  $f$  is almost separably valued. Then there exists null set  $N_* \subset X$  such that  $f(X \setminus N_*) \subset V$  separable. Define the subspace

$$M = \text{span}(f(X \setminus N_*)) \subset V,$$

which is separable by construction. Pick a dense set  $\{v_n\}_{n=0}^\infty \subset M$  such that  $v_0 = 0$ . Then by a previous theorem, we know there exists a norming sequence  $\{w_n\}_{n=0}^\infty \subset W$  for  $M$ .

Now, given any  $v \in V$  and  $n \in \mathbb{N}$ , define the function  $\Phi_{n,v} : X \rightarrow [0, \infty)$  by

$$\Phi_{n,v}(x) = |\langle w_n, f(x) - v \rangle| = |w_n(f(x) - v)|.$$

Note that  $X \ni x \mapsto \langle w_n, v \rangle \in \mathbb{F}$  is  $\mu$ -measurable and the map  $X \ni x \mapsto \langle w_n, f(x) \rangle \in \mathbb{F}$  is also  $\mu$ -measurable by assumption. It follows that  $\Phi_{n,v}$  is  $\mu$ -measurable. Therefore, there exists null set  $N_{n,v} \subset X$  and a measurable map  $\Psi_{n,v} : X \rightarrow [0, \infty)$  such that  $\Psi_{n,v} = \Phi_{n,v}$  on  $X \setminus N_{n,v}$ . For each  $v \in V$  define null set

$$N(v) = N_* \cup \bigcup_{n=0}^\infty N_{n,v} \subset X,$$

with  $\Psi_{n,v} = \Phi_{n,v}$  on  $X \setminus N(v)$  for all  $n \in \mathbb{N}$ .

For  $v \in M$  define the map  $\Phi_v : X \rightarrow [0, \infty]$  by  $\Phi_v(x) = \|f(x) - v\|$  and note that  $\{w_n\}_{n=0}^\infty$  is norming sequence for  $M$ . This implies that

$$\Phi_v(x) = \sup_{n \in \mathbb{N}} |\langle w_n, f(x) - v \rangle|$$

for all  $x \in X \setminus N_*$ . We also have that

$$\Phi_v(x) = \sup_{n \in \mathbb{N}} \Phi_{n,v}(x) = \sup_{n \in \mathbb{N}} \Psi_{n,v}(x)$$

for all  $x \in X \setminus N(v)$ , so  $\Phi_v$  is measurable when restricted to  $X \setminus N(v)$ . We can then define the set

$$N = \bigcup_{m=0}^\infty N(v_m) \subset X,$$

which is null. By construction, each  $\Phi_{v_m}$  is measurable when restricted to  $N^c$ . In particular,  $\Phi_0 = \Phi_{v_0} = \|f\|$  is measurable when restricted to  $N^c$ .

For  $u \in M$  and  $n \in \mathbb{N}$ , define

$$k(n, u) = \min \left\{ 0 \leq k \leq n : \|u - v_k\| = \min_{0 \leq j \leq n} \|u - v_j\| \right\}.$$

By construction,

$$\|v_{k(n,u)}\| \leq \|u - v_{k(n,u)}\| + \|u\| \leq \|u - v_0\| + \|u\| = 2\|u\|.$$

We then define  $S_n : M \rightarrow \{v_0, \dots, v_n\}$  via  $S_n(u) = v_{k(n,u)}$ . Note that  $\|S_n(u)\| \leq 2\|u\|$ . Also,  $\{v_m\}_{m=0}^\infty$  dense in  $M$  implies  $S_n(u) \rightarrow u$  as  $n \rightarrow \infty$ .

Finally, for  $n \in \mathbb{N}$ , define  $\psi_n : N^c \rightarrow \{v_0, \dots, v_n\} \subset V$  via  $\psi_n = S_n \circ f$ . For  $0 \leq k \leq n$ , we compute

$$\begin{aligned} & \{x \in N^c : \psi_n(x) = v_k\} \\ &= \left\{x \in N^c : \|f(x) - v_k\| = \min_j \|f(x) - v_j\|\right\} \cap \bigcap_{j=0}^{k-1} \{x \in N^c : \|f(x) - v_k\| < \|f(x) - v_j\|\} \end{aligned}$$

This set is measurable since  $\Phi_{v_m}$  measurable on  $N^c$  for each  $m \in \mathbb{N}$ . It follows that  $\psi_n$  is measurable on  $N^c$ . Let  $\varphi_n \in S(X; V)$  by

$$\varphi_n(x) = \begin{cases} \psi_n(x) & (x \in N^c), \\ 0 & (x \in N). \end{cases}$$

Then,  $\|\varphi_n\| \leq 2\|f\|$  and  $\varphi_n(x) = \psi_n(x) \rightarrow f(x)$  as  $n \rightarrow \infty$  for  $x \in N^c$ . Therefore,  $\varphi_n \rightarrow f$  a.e. and thus  $f$  is  $\mu$ -measurable. □

### 3.4 Lebesgue-Bochner Integral

**Lemma.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and  $Y \in \{V, [0, \infty]\}$ . Let  $\psi : X \rightarrow Y$  be simple such that

$$\psi = \sum_{i=1}^I \alpha_i \chi_{E_i} = \sum_{j=1}^J \beta_j \chi_{F_j}.$$

Additionally, if  $Y = V$  suppose both representation are finite. Then,

$$\sum_{i=1}^I \alpha_i \mu(E_i) = \sum_{j=1}^J \beta_j \mu(F_j).$$

Based on this lemma, we can define

$$\int_X \psi \, d\mu = \sum_{i=1}^I \alpha_i \mu(E_i).$$

This induces maps  $\int_X \cdot \, d\mu : S(X; [0, \infty]) \rightarrow [0, \infty]$  and  $\int_X \cdot \, d\mu : S_{\text{fin}}(X; V) \rightarrow V$ .

**Proposition.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and  $Y \in \{V, [0, \infty]\}$ . Then the following holds:

1. If  $Y = V$ , then

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu$$

for all  $\alpha, \beta \in \mathbb{F}$  and  $f, g \in S_{\text{fin}}(X; V)$ . If  $Y = [0, \infty]$ , the same equality holds for any  $\alpha, \beta > 0$  and  $f, g \in S(X; V)$ .

2. If  $Y = V$ , then  $\|f\| \in S_{\text{fin}}(X; [0, \infty))$  and

$$\left\| \int_X f \, d\mu \right\| \leq \int_X \|f\| \, d\mu.$$

3. If  $E \in \mathfrak{M}$ , then

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu.$$

4. If  $N \in \mathfrak{M}$  is a null set, then

$$\int_N f \, d\mu = 0.$$

5. If  $A, B \in \mathfrak{M}$  is such that  $A \cap B = \emptyset$ , then

$$\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu.$$

6. Suppose  $\{X_n\}_{n=0}^\infty \subset \mathfrak{M}$  is such that  $X_n \subset X_{n+1}$  and  $\mu(X_n) < \infty$ . Then

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_{X_n} f \, d\mu.$$

*Proof.* Write  $f = \sum_k f_k \chi_{E_k}$  be the canonical representation. We then have

$$\int_{X_n} f \, d\mu = \sum_k f_k \mu(X_n \cap E_k).$$

For each  $k$ , we have  $X_n \cap E_k \subset X_{n+1} \cap E_k$  and  $\bigcup_{n=0}^\infty (X_n \cap E_k) = E_k$ . It follows that

$$\lim_{n \rightarrow \infty} \mu(X_n \cap E_k) = \mu(E_k).$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{X_n} f \, d\mu = \sum_k f_k \mu(E_k) = \int_X f \, d\mu.$$

□

7. If  $Y = \mathbb{R}$  or  $Y = [0, \infty]$  and  $f \leq g$  a.e., then

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

### 3.4.1 Integration of $\overline{\mathbb{R}}$ -valued functions

Note that if  $(X, \mathfrak{M}, \mu)$  is a measure space and  $\varphi \in S(X; [0, \infty])$ , then

$$\int_X \varphi \, d\mu = \sup \left\{ \int_X \psi \, d\mu : \psi \in S(X; [0, \infty]) \text{ and } \psi \leq \varphi \text{ a.e.} \right\}.$$

**Definition.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space. Let  $f : X \rightarrow [0, \infty]$  be  $\mu$ -measurable. We define

$$\int_X f \, d\mu = \sup \left\{ \int_X \psi \, d\mu : \psi \in S(X; [0, \infty]) \text{ and } \psi \leq f \text{ a.e.} \right\} \in [0, \infty].$$

We say  $f$  is **integrable** if  $\int_X f \, d\mu < \infty$ .

**Remark.** There are two remarks with regard to the definition above.

1. In principle we do not need  $f$  to be  $\mu$ -measurable here. We build this into the definition because the resulting integral is more-or-less useless without this assumption.
2.  $[0, \infty]$  is a separable metric space, so for  $f : X \rightarrow [0, \infty]$ ,  $f$  is measurable implies  $f$  is  $\mu$ -measurable, and  $f$  almost measurable implies  $f$  is  $\mu$ -measurable.

**Theorem.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space,  $f, g : X \rightarrow [0, \infty]$  be  $\mu$ -measurable functions. The following holds:

1. For  $\alpha \in [0, \infty)$ , we have

$$\int_X \alpha f \, d\mu = \alpha \int_X f \, d\mu.$$

2. If  $f \leq g$  a.e., then

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

3. If  $f = g$  a.e., then

$$\int_X f \, d\mu = \int_X g \, d\mu.$$

4. For  $E \in \mathfrak{M}$ , we have

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu.$$

5. If  $N \in \mathfrak{M}$  is null, then

$$\int_N f \, d\mu = 0.$$

*Proof.* Follow directly from corresponding results in  $S(X; [0, \infty])$  and the definition of  $\int_X f \, d\mu$ .  $\square$

**Theorem** (Monotone convergence theorem, basic version). Let  $(X, \mathfrak{M}, \mu)$  be a measure space and suppose for each  $n \in \mathbb{N}$ , we have  $f_n : X \rightarrow [0, \infty]$  **measurable**. Further suppose that  $f_n \leq f_{n+1}$  on  $X$  and  $f : X \rightarrow [0, \infty]$  is given by  $f = \lim_{n \rightarrow \infty} f_n$ . Then  $f$  is measurable and

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu = \sup_{n \in \mathbb{N}} \int_X f_n \, d\mu.$$

*Proof.* We already know  $f$  is measurable. Also,  $f_n \leq f_{n+1} \leq f$  on  $X$ , so

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f \, d\mu.$$

It follows that

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu.$$

To show the opposite inequality, let  $\varphi \in S(X; [0, \infty])$  such that  $\varphi \leq f$  a.e. and  $\alpha \in (0, 1)$ . Let  $N \in \mathfrak{M}$  be a null set and  $\varphi \leq f$  on  $N^c$ . Also, for each  $n \in \mathbb{N}$ , let  $E_n = \{x \in X : f_n(x) \geq \alpha\varphi(x)\}$ . Note the following:

1. Since  $f_n \leq f_{n+1}$ , we have  $E_n \subset E_{n+1}$ .
2. Since  $f_n \rightarrow f$  pointwise, we have  $X = N \cup \bigcup_{n=0}^{\infty} E_n$ .
3. We have

$$\alpha \int_{N \cup E_n} \varphi \, d\mu = \int_{E_n} \alpha\varphi \, d\mu \leq \int_{E_n} f_n \, d\mu \leq \int_X f_n \, d\mu$$

4. We have

$$\int_X \varphi \, d\mu = \lim_{n \rightarrow \infty} \int_{N \cup E_n} \varphi \, d\mu.$$

Therefore,

$$\alpha \int_X \varphi \, d\mu = \lim_{n \rightarrow \infty} \alpha \int_{N \cup E_n} \varphi \, d\mu \leq \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

Since the above inequality holds for all  $\alpha \in (0, 1)$ , we know  $\int_X \varphi \, d\mu \leq \lim_{n \rightarrow \infty} \int_X f_n \, d\mu$ . This is then true for all simple function  $\varphi$  such that  $\varphi \leq f$  a.e. Taking the sup gives

$$\int_X f \, d\mu \leq \lim_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

The proof is then complete.  $\square$

**Theorem.** Let  $(X, \mathfrak{M}, \mu)$  be measure space,  $f, g : X \rightarrow [0, \infty]$  be  $\mu$ -measurable. Then

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

*Proof.* Recall that  $\mu$ -measurable functions are almost measurable. Choose measurable functions  $F, G : X \rightarrow [0, \infty]$  such that  $f = F$  and  $g = G$  a.e. We may then choose  $\{\varphi_n\}_{n=0}^\infty, \{\psi_n\}_{n=0}^\infty \subset S(X; [0, \infty])$  such that  $\lim_{n \rightarrow \infty} \varphi_n = F$  and  $\lim_{n \rightarrow \infty} \psi_n = G$ ,  $0 \leq \varphi_n \leq \varphi_{n+1} \leq F$  and  $0 \leq \psi_n \leq \psi_{n+1} \leq G$ . Then

$$0 \leq \varphi_n + \psi_n \leq \varphi_{n+1} + \psi_{n+1} \leq F + G = \lim_{n \rightarrow \infty} (\varphi_n + \psi_n).$$

It follows then from monotone convergence theorem that

$$\begin{aligned} \int_X (F + G) d\mu &= \lim_{n \rightarrow \infty} \int_X (\varphi_n + \psi_n) d\mu \\ &= \lim_{n \rightarrow \infty} \int_X \varphi_n d\mu + \lim_{n \rightarrow \infty} \int_X \psi_n d\mu \\ &= \int_X F d\mu + \int_X G d\mu. \end{aligned}$$

Since  $f = F$  and  $g = G$  a.e., we have

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

□

**Recall:** given  $f : X \rightarrow \overline{\mathbb{R}}$ , we write  $f^\pm : X \rightarrow [0, \infty]$  via

$$f^+ = \max\{0, f\}, \quad f^- = \max\{0, -f\}.$$

Then we have  $f = f^+ - f^-$  and  $|f| = f^+ + f^-$ . Also, if  $f$  is measurable or  $\mu$ -measurable, then  $f^\pm$  is also measurable or  $\mu$ -measurable since they are composition of a continuous function (namely  $x \mapsto \max\{0, x\}$ ) with a measurable or  $\mu$ -measurable function.

**Definition.** Let  $(X, \mathfrak{M}, \mu)$  be measure space and  $f : X \rightarrow \overline{\mathbb{R}}$  be  $\mu$ -measurable. If either  $f^+$  or  $f^-$  is **integrable**, we say  $f$  is **extended integrable** and set

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu \in \overline{\mathbb{R}}.$$

We say  $f$  is **integrable** if  $f^\pm$  are both integrable.

**Proposition** (absolute integrability). Let  $(X, \mathfrak{M}, \mu)$  be a measure space,  $f : X \rightarrow \overline{\mathbb{R}}$  be  $\mu$ -measurable. Then  $f$  is integrable if and only if  $|f|$  is integrable.

*Proof.* We know  $f$  is integrable if and only if  $f^\pm$  are both integrable, but  $|f| = f^+ + f^-$ . Therefore,  $f$  integrable implies  $|f|$  is integrable. Conversely, if  $|f|$  is integrable, then  $0 \leq f^\pm \leq |f|$ , so  $f^\pm$  are both integrable. □

**Theorem.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space,  $f, g : X \rightarrow \overline{\mathbb{R}}$  are extended integrable. The following holds:

1. For all  $E \in \mathfrak{M}$ , we have  $\int_E f d\mu = \int_X f \chi_E d\mu$ .
2. For all  $\alpha \in \mathbb{R}$ , we have  $\alpha \int_X f d\mu = \int_X \alpha f d\mu$ .
3.  $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$ , provided that all operations are well-defined.
4.  $\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$  for all  $A, B \in \mathfrak{M}$  such that  $A \cap B = \emptyset$ .
5. If  $f \leq g$  a.e. then  $\int_X f d\mu \leq \int_X g d\mu$ .

6.  $|\int_X f \, d\mu| \leq \int_X |f| \, d\mu$ .
7. If  $|f| \leq g$  a.e. and  $g$  integrable, then  $f$  is integrable.

**Theorem** (Chebyshev's inequality). If  $f$  is measurable, then

$$\mu(\{x \in X : |f(x)| \geq \alpha\}) \leq \frac{1}{\alpha} \int_X |f| \, d\mu$$

for all  $\alpha \in (0, \infty)$ .

*Proof.*

$$\text{LHS} = \int_{\{|f| \geq \alpha\}} 1 \, d\mu = \int_{\{|f| \geq \alpha\}} \frac{|f|}{\alpha} \, d\mu = \frac{1}{\alpha} \int_X |f| \, d\mu = \text{RHS}.$$

□

**Corollary.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and  $f : X \rightarrow \overline{\mathbb{R}}$ .

1. If  $f$  is integrable, then there exists a null set  $N \in \mathfrak{M}$  and a  $\sigma$ -finite set  $E \in \mathfrak{M}$  such that  $\{|f| = \infty\} \subset N$  and  $\text{supp}(f) \subset E$ .
2. If  $f$  is extended integrable, then there exists a null set  $N \in \mathfrak{M}$  such that either  $\{f = \infty\} \subset N$  or  $\{f = -\infty\} \subset N$ .

*Proof.* 1. Suppose initially that  $f$  is measurable and integrable, then Chebyshev's inequality implies that

$$\mu(\{|f| = \infty\}) \leq \mu(\{|f| > 2^k\}) \leq 2^{-k} \int_X |f| \, d\mu$$

for all  $k \in \mathbb{N}$ . It follows that  $\mu(\{|f| = \infty\})$  is null.

On the other hand,  $\text{supp}(f) = \bigcup_{k=0}^{\infty} \{|f| > 2^{-k}\}$ , but

$$\mu(\{|f| > 2^{-k}\}) \leq 2^k \int_X |f| \, d\mu < \infty.$$

It follows that  $\text{supp}(f)$  is  $\sigma$ -finite.

In general, if  $f$  is integrable and  $\mu$ -measurable, pick  $F = f$  a.e. for  $F$  measurable and integrable and apply the argument above.

2. Next, if  $f$  is extended integrable but not integrable, then either  $f^+$  is integrable or  $f^-$  is integrable. If  $f^+$  is integrable, then  $\{f = +\infty\}$  is contained in some null set. If  $f^-$  is integrable,  $\{f = -\infty\}$  is contained in a null set.

□

To prove the more general form of monotone convergence theorem, we first need a useful lemma.

**Lemma.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and suppose that  $f : X \rightarrow \overline{\mathbb{R}}$  is  $\mu$ -measurable and  $g : X \rightarrow \mathbb{R}$  is integrable. Further suppose  $g \leq f$  a.e. Then,  $f$  and  $f - g$  are extended integrable, and

$$\int_X (f - g) \, d\mu = \int_X f \, d\mu - \int_X g \, d\mu.$$

*Proof.* Since  $g \leq f$  a.e., we have  $f^- \leq g^-$  a.e. Since  $g$  is integrable,  $f^-$  is integrable and thus  $f$  is extended-integrable. We also have  $f - g$  well defined on all of  $X$  and  $f - g \geq 0$  a.e. Therefore,  $f - g$  is extended-integrable.

If  $f$  is integrable, then we immediately have the desired equality. Suppose not  $f$  is not integrable but only extended-integrable. This implies  $f^+$  is not integrable. We must then have  $f - g$  not integrable, otherwise  $f = (f - g) + g$  is integrable. Therefore,  $\int_X (f - g) \, d\mu = \int_X f \, d\mu = \infty$ , and the desired equality holds. □

**Theorem** (Monotone convergence theorem, general form). Let  $(X, \mathfrak{M}, \mu)$  be a measure space and suppose  $f_k : X \rightarrow \overline{\mathbb{R}}$  is  $\mu$ -measurable for all  $k \in \mathbb{N}$ . Suppose that  $f : X \rightarrow \overline{\mathbb{R}}$  is such that  $f_k \rightarrow f$  a.e. Then,  $f$  is  $\mu$ -measurable and the following holds:

1. Suppose that  $\{f_k\}_{k=0}^\infty$  is almost everywhere nondecreasing, that is,  $f_k \leq f_{k+1}$  a.e. Suppose also that there exists an integrable function  $g : X \rightarrow \overline{\mathbb{R}}$  such that  $g \leq f_k$  a.e. for all  $k \in \mathbb{N}$ . Then,  $f$  and  $f_k$  are extended integrable for all  $k \in \mathbb{N}$ , and

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu.$$

2. Suppose that  $\{f_k\}_{k=0}^\infty$  is almost everywhere nonincreasing, that is,  $f_k \geq f_{k+1}$  a.e. Suppose also that there exists an integrable function  $g : X \rightarrow \overline{\mathbb{R}}$  such that  $g \geq f_k$  a.e. for all  $k \in \mathbb{N}$ . Then,  $f$  and  $f_k$  are extended integrable for all  $k \in \mathbb{N}$ , and

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu.$$

*Proof.* Since  $g$  is integrable, there exists a null set  $\tilde{N} \in \mathfrak{M}$  such that  $\{|g| = \infty\} \subset \tilde{N}$ . Now  $g$  is  $\mathbb{R}$ -valued in  $N^c$ . We can also select a null set  $N \supset \tilde{N}$  such that the following holds:

- $g$  is measurable on  $N^c$ .
- $f_k \rightarrow f$  as  $k \rightarrow \infty$  on  $N^c$ .
- For each  $k \in \mathbb{N}$ ,  $f_k$  is measurable on  $N^c$ ,  $f_k \leq f_{k+1} \leq f$  on  $N^c$ , and  $g \leq f_k \leq f$  on  $N^c$ .

By Lemma 10.3.22, we know  $f, f - g$  are extended integrable on  $N^c$  and  $f_k, f_k - g$  are extended integrable on  $N^c$  for each  $k \in \mathbb{N}$ . Additionally, we have

$$\int_{N^c} (f - g) d\mu = \int_{N^c} f d\mu - \int_{N^c} g d\mu,$$

and for each  $k \in \mathbb{N}$

$$\int_{N^c} (f_k - g) d\mu = \int_{N^c} f_k d\mu - \int_{N^c} g d\mu.$$

Note now  $f_k - g$  is measurable function on  $N^c$  taking values in  $[0, \infty]$ . Also,  $f_k - g \leq f_{k+1} - g$  on  $N^c$  and  $f_k - g \rightarrow f - g$  pointwise as  $k \rightarrow \infty$  on  $N^c$ . By the basic version of monotone convergence theorem, we have

$$\lim_{k \rightarrow \infty} \int_{N^c} (f_k - g) d\mu = \int_{N^c} (f - g) d\mu.$$

Therefore,

$$\lim_{k \rightarrow \infty} \int_{N^c} f_k d\mu - \int_{N^c} g d\mu = \int_{N^c} f d\mu - \int_{N^c} g d\mu.$$

However, note that  $\int_{N^c} g d\mu \in \mathbb{R}$  and it then follows that

$$\lim_{k \rightarrow \infty} \int_{N^c} f_k d\mu = \int_{N^c} f d\mu.$$

Since both  $f_k$  and  $f$  are extended integrable and  $N$  is null, we have

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu,$$

as desired. □

**Corollary.** 1. Let  $(X, \mathfrak{M}, \mu)$  be a measure space,  $f_k : X \rightarrow (-\infty, \infty]$  be  $\mu$ -measurable for all  $k \in \mathbb{N}$  and  $f_k \geq 0$  a.e. Then,

$$\int_X \sum_{k=0}^\infty f_k d\mu = \sum_{k=0}^\infty \int_X f_k d\mu.$$

2. Suppose  $(X, \mathfrak{M}, \mu)$  is a measure space,  $X = \bigcup_{k=0}^{\infty} E_k$  such that  $\{E_k\}_{k=0}^{\infty} \subset \mathfrak{M}$  and  $\mu(E_k \cap E_j) = 0$  for all  $k \neq j$ . Given  $f : X \rightarrow [0, \infty]$   $\mu$ -measurable, we then have

$$\int_X f \, d\mu = \sum_{k=0}^{\infty} \int_{E_k} f \, d\mu.$$

*Proof.* 1. Note that  $\text{supp}(f_k^-)$  is in a null set, so each  $f_k$  is extended integrable. The same holds for  $\sum_{k=0}^{\infty} f_k : X \rightarrow [-\infty, \infty]$ . On the other hand, the partial sums  $\sum_{k=0}^m f_k \leq \sum_{k=0}^{m+1} f_k$  a.e. Apply monotone convergence theorem gives the desired equality.

2. Use the first claim on  $f_k = f\chi_{E_k}$ .

□

**Theorem** (Fatou's lemma). Let  $(X, \mathfrak{M}, \mu)$  be a measure space, and suppose that  $f_k : X \rightarrow \overline{\mathbb{R}}$  are  $\mu$ -measurable for all  $k \in \mathbb{N}$ . Suppose that  $g : X \rightarrow \overline{\mathbb{R}}$  is extended integrable,  $\int_X g \, d\mu > -\infty$ , and  $g \leq f_k$  a.e. for all  $k \in \mathbb{N}$ . Then the following holds:

1. For each  $k \in \mathbb{N}$ ,  $f_k$  is extended integrable.
2. The function  $\liminf_{k \rightarrow \infty} f_k$  is extended integrable.
3. We have

$$\int_X g \, d\mu \leq \int_X \liminf_{k \rightarrow \infty} f_k \, d\mu \leq \liminf_{k \rightarrow \infty} \int_X f_k \, d\mu.$$

*Proof.* Note that  $\int_X g \, d\mu > -\infty$  implies  $g^-$  is integrable. Write

$$f = \liminf_{k \rightarrow \infty} f_k,$$

which is a  $\mu$ -measurable function. Then,  $g \leq f_k$  a.e. implies  $g \leq f$  a.e. as well. It follows that  $-f_k \leq -g$  and  $-f \leq -g$ . Therefore,  $f_k^- \leq g^-$  and  $f^- \leq g^-$ . This shows that  $f_k$  and  $f$  are extended-integrable. Next, note that

$$\int_X g \, d\mu \leq \int_X \inf_{j \geq k} f_j \, d\mu \leq \int_X f_k \, d\mu.$$

By monotone convergence theorem, we know the middle term converges when  $k$  approaches infinity. Taking the  $\liminf$ , we have

$$\int_X g \, d\mu \leq \liminf_{k \rightarrow \infty} \int_X \inf_{j \geq k} f_j \, d\mu = \lim_{k \rightarrow \infty} \int_X \inf_{j \geq k} f_j \, d\mu = \int_X \liminf_{k \rightarrow \infty} f_k \, d\mu \leq \liminf_{k \rightarrow \infty} \int_X f_k \, d\mu.$$

□

**Theorem** (Dominated convergence theorem). Let  $(X, \mathfrak{M}, \mu)$  be a measure space and suppose  $f_k, g_k : X \rightarrow \overline{\mathbb{R}}$   $\mu$ -measurable for each  $k \in \mathbb{N}$ . Suppose that  $f, g : X \rightarrow \overline{\mathbb{R}}$  are such that  $f_k \rightarrow f$  a.e. and  $g_k \rightarrow g$  a.e. Suppose further that  $g_k$  is integrable and  $|f_k| \leq g_k$  a.e. for each  $k \in \mathbb{N}$ . Suppose also  $g$  is integrable and that

$$\lim_{k \rightarrow \infty} \int_X g_k \, d\mu = \int_X g \, d\mu.$$

Then,  $f_k$  is integrable for each  $k \in \mathbb{N}$ ,  $f$  is integrable, and

$$\lim_{k \rightarrow \infty} \int_X f_k \, d\mu = \int_X f \, d\mu.$$

Moreover,  $f_k - f$  is well-defined for all  $k \in \mathbb{N}$  outside a null set  $N \subset X$ , and

$$\lim_{k \rightarrow \infty} \int_{N^c} |f_k - f| \, d\mu = 0$$



*Proof.* We know  $|f_k| \leq g_k$  a.e.,  $g_k \rightarrow g$  a.e., and  $f_k \rightarrow f$  a.e. Then,  $|f| \leq g$  a.e., so  $f_k$  and  $f$  are integrable. In turn, we can use a previous corollary to pick  $N \in \mathfrak{M}$  null such that  $f_k, f, g_k, g$  are all  $\mathbb{R}$ -valued and all assumed inequalities hold on  $N^c$ . Then,  $|f - f_k| \leq g + g_k$  on  $N^c$ , and so

$$0 \leq g + g_k - |f - f_k|.$$

Apply Fatou's lemma, we then have

$$\begin{aligned} \int_{N^c} 2g \, d\mu &= \int_{N^c} \liminf_{k \rightarrow \infty} (g + g_k - |f - f_k|) \, d\mu \\ &\leq \liminf_{k \rightarrow \infty} \int_{N^c} (g + g_k - |f - f_k|) \, d\mu \\ &= \liminf_{k \rightarrow \infty} \int_{N^c} (g + g_k - |f - f_k|) \, d\mu + \liminf_{k \rightarrow \infty} \int_{N^c} -(g + g_k) \, d\mu + \int_{N^c} 2g \, d\mu \\ &\leq \liminf_{k \rightarrow \infty} \int_{N^c} -|f - f_k| \, d\mu + \int_{N^c} 2g \, d\mu. \end{aligned}$$

It follows that

$$0 \leq \limsup_{k \rightarrow \infty} \int_{N^c} |f - f_k| \, d\mu = -\liminf_{k \rightarrow \infty} \int_{N^c} -|f - f_k| \, d\mu \leq 0.$$

Therefore,

$$\lim_{k \rightarrow \infty} \int_{N^c} |f - f_k| \, d\mu = 0.$$

Note that  $f_k$  and  $f$  are integrable, so

$$\left| \int_X f \, d\mu - \int_X f_k \, d\mu \right| = \left| \int_{N^c} f \, d\mu - \int_{N^c} f_k \, d\mu \right| \leq \int_{N^c} |f - f_k| \, d\mu.$$

This then implies

$$\lim_{k \rightarrow \infty} \int_X f_k \, d\mu = \int_X f \, d\mu,$$

and the proof is complete.  $\square$

**Remark.** Usually, dominated convergence theorem is applied with  $g_k = g$ , in which case the assumption  $\int_X g_k \, d\mu \rightarrow \int_X g \, d\mu$  becomes trivial.

### 3.4.2 Bochner integration

**Lemma.** Suppose  $(X, \mathfrak{M}, \mu)$  is a measure space and  $V$  a normed vector space, and  $\varphi : X \rightarrow V$  simple. Note then  $\|\varphi\| : X \rightarrow [0, \infty)$  is a simple function now. Then,  $\varphi$  is a **finite** simple function if and only if  $\|\varphi\|$  is integrable.

*Proof.* ( $\implies$ ) Suppose  $\varphi$  is finite, then  $\|\varphi\|$  is finite. Then,  $\|\varphi\|$  is integrable.

( $\impliedby$ ) Suppose  $\|\varphi\|$  is integrable. We know  $\varphi$  is simple, so  $\varphi(X) \setminus \{0\}$  is a finite set in  $V$ . Then, there exists  $0 < m \in \mathbb{R}$  such that  $\|v\| \geq m$  for all  $v \in \varphi(X) \setminus \{0\}$ . Then,

$$\mu(\text{supp}(\varphi)) = \mu(\{x \in X : \|\varphi(x)\| > 0\}) = \mu(\{\|\varphi\| \geq m\}).$$

By Chebyshev's inequality, we have

$$\mu(\text{supp}(\varphi)) \leq \frac{1}{m} \int_X \|\varphi\| \, d\mu < \infty.$$

This completes the proof.  $\square$

**Lemma.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space,  $V$  be a Banach space,  $f : X \rightarrow V$   $\mu$ -strongly measurable. Suppose that for  $j \in \{0, 1\}$ , we have  $\{\varphi_k^j\}_{k=0}^\infty \subset S_{\text{fin}}(X; V)$  such that

$$\lim_{k \rightarrow \infty} \int_X \|f - \varphi_k^j\| d\mu = 0.$$

Then,  $\{\int_X \varphi_k^j\}_{k=0}^\infty$  is convergent in  $V$  for both  $j \in \{0, 1\}$  and

$$\lim_{k \rightarrow \infty} \int_X \varphi_k^0 d\mu = \lim_{k \rightarrow \infty} \int_X \varphi_k^1 d\mu.$$

*Proof.* For  $k, m \in \mathbb{N}$ , we have

$$\begin{aligned} \left\| \int_X \varphi_m^j d\mu - \int_X \varphi_k^j d\mu \right\| &= \left\| \int_X (\varphi_m^j - \varphi_k^j) d\mu \right\| \\ &\leq \int_X \|\varphi_m^j - \varphi_k^j\| d\mu \\ &\leq \int_X \|f - \varphi_m^j\| d\mu + \int_X \|f - \varphi_k^j\| d\mu. \end{aligned}$$

This shows that  $\{\int_X \varphi_k^j\}_{k=0}^\infty$  is Cauchy and hence convergent.

On the other hand,

$$\begin{aligned} \left\| \int_X \varphi_k^0 d\mu - \int_X \varphi_k^1 d\mu \right\| &\leq \int_X \|\varphi_k^0 - \varphi_k^1\| d\mu \\ &\leq \int_X \|f - \varphi_k^0\| d\mu + \int_X \|f - \varphi_k^1\| d\mu \\ &\rightarrow 0, \end{aligned}$$

completing the proof.  $\square$

This leads to the following definition for Bochner integration.

**Definition.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and  $V$  a Banach space. A map  $f : X \rightarrow V$  is (Bochner) integrable if it is strongly  $\mu$ -measurable and there exists a sequence  $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; V)$  such that  $\varphi_n \rightarrow f$  a.e. and

$$\lim_{n \rightarrow \infty} \int_X \|f - \varphi_n\| d\mu = 0,$$

in which case we define

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X \varphi_n d\mu \in V.$$

Note that this is well-defined by the previous lemmas.

**Theorem** (absolute integrability). Let  $(X, \mathfrak{M}, \mu)$  be a measure space,  $V$  a Banach space,  $f : X \rightarrow V$ . Then,  $f$  is integrable if and only if  $f$  is  $\mu$ -measurable and  $\|f\| : X \rightarrow [0, \infty]$  is integrable. In either case,

$$\left\| \int_X f d\mu \right\| \leq \int_X \|f\| d\mu.$$

*Proof.* ( $\implies$ ) Suppose  $f$  is integrable. This implies that  $f$  is strongly  $\mu$ -measurable and in particular  $\mu$ -measurable. Also,  $\|f\| : X \rightarrow [0, \infty)$  is  $\mu$ -measurable. Suppose  $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; V)$  is such that  $\varphi_n \rightarrow f$  a.e. and  $\int_X \|f - \varphi_n\| d\mu \rightarrow 0$ . Then,

$$\int_X \|f\| d\mu \leq \int_X \|f - \varphi_n\| d\mu + \int_X \|\varphi_n\| d\mu < \infty$$

for  $n$  sufficiently large. This implies that  $\|f\|$  is integrable.

( $\Leftarrow$ ) Suppose  $f$  is  $\mu$ -measurable and  $\int_X \|f\| d\mu < \infty$ . Then, Pettis theorem gives a sequence  $\{\varphi_n\}_{n=0}^\infty \in S(X; V)$  such that  $\varphi_n \rightarrow f$  a.e. and  $\|\varphi_n\| \leq 2\|f\|$ . Then,

$$\int_X \|\varphi_n\| d\mu \leq 2 \int_X \|f\| d\mu < \infty.$$

Therefore,  $\{\varphi_n\}_{n=0}^\infty$  is actually a sequence of finite simple functions. This implies that  $f$  is actually strongly  $\mu$ -measurable. On the other hand,  $\|f - \varphi_n\| \leq 3\|f\|$ , so dominated convergence theorem implies

$$\int_X \|f - \varphi_n\| d\mu \rightarrow 0$$

as  $n \rightarrow \infty$ . By definition,  $f$  is now integrable. Moreover,

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X \varphi_n d\mu.$$

It follows then from the dominated convergence theorem that

$$\left\| \int_X f d\mu \right\| = \lim_{n \rightarrow \infty} \left\| \int_X \varphi_n d\mu \right\| \leq \lim_{n \rightarrow \infty} \int_X \|\varphi_n\| d\mu = \int_X \|f\| d\mu.$$

□

**Theorem** (dominated convergence theorem for Bochner). Let  $(X, \mathfrak{M}, \mu)$  be a measure space,  $V$  a Banach space, and suppose  $f_n : X \rightarrow V$ ,  $g_n : X \rightarrow \mathbb{R}$  are  $\mu$ -measurable  $n \in \mathbb{N}$ . Further suppose  $f : X \rightarrow V$  and  $g : X \rightarrow \mathbb{R}$  are such that  $f_n \rightarrow f$  a.e. and  $g_n \rightarrow g$  a.e. Also, suppose  $g_n, g$  are integrable. Finally suppose  $\|f_n\| \leq g_n$  a.e. and

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X g d\mu.$$

Then,  $f_n, f$  are integrable and

$$\lim_{n \rightarrow \infty} \int_X \|f_n - f\| d\mu = 0,$$

so we also have

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

*Proof.* Since  $\|f_n\| \leq g_n$  and  $\|f\| \leq g$ , we have  $f_n$  and  $f$  integrable. Note that  $\|f - f_n\| \leq g + g_n$  and  $g + g_n \rightarrow 2g$  as  $n \rightarrow \infty$ . Dominated convergence theorem then implies

$$\lim_{n \rightarrow \infty} \int_X \|f - f_n\| d\mu = 0,$$

completing the proof. □

**Proposition.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space and  $V$  a Banach space over  $\mathbb{F}$ . Let  $f : X \rightarrow V$  integrable. The following holds:

1. If  $W$  is a Banach space over  $F$  and  $T \in \mathcal{L}(V, W)$ , then  $T \circ f : X \rightarrow W$  is integrable and

$$\int_X T \circ f d\mu = T \int_X f d\mu.$$

2. Suppose  $g : X \rightarrow V$  is integrable, then  $\int_X f d\mu = \int_X g d\mu$  if and only if  $\int_X w \circ f d\mu = \int_X w \circ g d\mu$  for every  $w \in V^*$ .

*Proof.* 1. Let  $\{\varphi_n\}_{n=0}^\infty \subset S_{\text{fin}}(X; V)$  such that  $\varphi_n \rightarrow f$  a.e. and  $\int_X \|f - \varphi_n\| d\mu \rightarrow 0$ . Then we have  $T \circ \varphi_n \rightarrow T \circ f$  a.e. and

$$\int_X \|T \circ f - T \circ \varphi_n\| d\mu \leq \|T\| \int_X \|f - \varphi_n\| d\mu \rightarrow 0.$$

Therefore,  $T \circ f$  is integrable and

$$\int_X T \circ f d\mu = \lim_{n \rightarrow \infty} \int_X T \circ \varphi_n d\mu = \lim_{n \rightarrow \infty} T \int_X \varphi_n d\mu = T \int_X f d\mu.$$

2. Let  $w \in V^*$ , then  $\int_X f d\mu = \int_X g d\mu$  clearly implies  $\int_X w \circ f d\mu = \int_X w \circ g d\mu$ . On the other hand, if  $\int_X w \circ f d\mu = \int_X w \circ g d\mu$  for all  $w \in V^*$ , then

$$w \left[ \int_X f d\mu - \int_X g d\mu \right] = 0$$

for all  $w \in V^*$ . By Hahn-Banach theorem, this implies  $\int_X f d\mu = \int_X g d\mu$ .

□

### 3.5 Constructing product measures

**Definition** (Pre-measure). Let  $X$  be a set and  $\mathfrak{A}$  be an algebra on  $X$ . A map  $\gamma : \mathfrak{A} \rightarrow [0, \infty]$  is a **pre-measure** if the following is satisfied:

1.  $\gamma(\emptyset) = 0$ .
2. If  $\{A_i\}_{i=0}^\infty \subset \mathfrak{A}$  is disjoint and  $\bigcup_{i=0}^\infty A_i \in \mathfrak{A}$ , then  $\gamma(\bigcup_{i=0}^\infty A_i) = \sum_{i=0}^\infty \gamma(A_i)$ .

**Theorem** (Pre-measure extension theorem). Let  $X$  be a set,  $\mathfrak{A}$  is an algebra on  $X$ , and  $\gamma$  a pre-measure. Let  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  be the outer measure constructed from  $(X, \gamma)$ . Denote  $\mathfrak{M}$  as the measurable space and  $\mu : \mathfrak{M} \rightarrow [0, \infty]$  the corresponding measure. Then the following holds:

1.  $\mathfrak{A} \subset \mathfrak{M}$  and  $\mu = \gamma$  on  $\mathfrak{A}$ .
2. Suppose  $\mathfrak{N}$  is a  $\sigma$ -algebra on  $X$  such that  $\mathfrak{A} \subset \mathfrak{N} \subset \mathfrak{M}$ , and  $\nu : \mathfrak{N} \rightarrow [0, \infty]$  is a measure such that  $\nu = \gamma$  on  $\mathfrak{A}$ . Then  $\nu \leq \mu$  on  $\mathfrak{N}$  and  $\nu(E) = \mu(E)$  whenever  $E$  is  $\sigma$ -finite w.r.t.  $\mu$ .

In particular, if  $X$  is “ $\gamma$   $\sigma$ -finite”, then  $\mu = \nu$  on  $\mathfrak{N}$ .

*Proof.* First show  $\mu = \gamma$  on  $\mathfrak{A}$ . It suffices to show that  $\mu^* = \gamma$  on  $\mathfrak{A}$ .

For any  $E \in \mathfrak{A}$ , we know  $E$  is covered by  $E$ , so  $\mu^* \leq \gamma$ . On the other hand, let  $E \subset \mathfrak{A}$  and  $\{A_k\}_{k=0}^\infty \subset \mathfrak{A}$  be a cover of  $E$ . Define  $B_0 = E \cap A_0 \in \mathfrak{A}$  and  $B_k = E \cap (A_k \setminus \bigcup_{i=0}^{k-1} A_i) \in \mathfrak{A}$ . Then  $\{B_k\}_{k=0}^\infty$  is pairwise disjoint and  $\bigcup_{k=0}^\infty B_k = E$ . It follows that

$$\gamma(E) = \gamma\left(\bigcup_{k=0}^\infty B_k\right) = \sum_{k=0}^\infty \gamma(B_k) \leq \sum_{k=0}^\infty \gamma(A_k).$$

Therefore,  $\mu^* = \gamma$  on  $\mathfrak{A}$ .

Next we show  $\mathfrak{A} \subset \mathfrak{M}$ . Let  $E \in \mathfrak{A}$  be arbitrary and we want to show  $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$  for all  $A \subset X$ . Fix arbitrary  $A \subset X$  and  $\varepsilon > 0$ . Pick  $\{A_k\}_{k=0}^\infty \subset \mathfrak{A}$  covering  $A$  such that

$$\sum_{k=0}^\infty \gamma(A_k) < \mu^*(A) + \varepsilon.$$

It follows that

$$\begin{aligned}
 \mu^*(A \cap E) + \mu^*(A \cap E^c) &\leq \mu^*\left(\bigcup_{k=0}^{\infty} A_k \cap E\right) + \mu^*\left(\bigcup_{k=0}^{\infty} A_k \cap E^c\right) \\
 &\leq \sum_{k=0}^{\infty} \mu^*(A_k \cap E) + \mu^*(A_k \cap E^c) \\
 &= \sum_{k=0}^{\infty} \gamma(A_k \cap E) + \gamma(A_k \cap E^c) \\
 &= \sum_{k=0}^{\infty} \gamma(A_k).
 \end{aligned}$$

This implies that  $E$  is measurable, completing the proof for the first item.

For the second item, we first show that  $\nu \leq \mu$ . Let  $E \in \mathfrak{N} \subset \mathfrak{M}$  and  $\{A_k\}_{k=0}^{\infty} \subset \mathfrak{A}$  that covers  $E$ . It follows that

$$\nu(E) \leq \nu\left(\bigcup_{k=0}^{\infty} A_k\right) = \lim_{n \rightarrow \infty} \nu\left(\bigcup_{i=0}^n A_i\right).$$

Note that  $\bigcup_{i=0}^n A_i \in \mathfrak{A}$ , so  $\nu(\bigcup_{i=0}^n A_i) = \mu(\bigcup_{i=0}^n A_i)$ . This implies that

$$\nu(E) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=0}^n A_i\right) = \mu\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} \gamma(A_k).$$

Therefore,  $\nu \leq \mu$ .

Next we show  $\nu(E) = \mu(E)$  for  $\mu(E) < \infty$ . Let  $\varepsilon > 0$  and select  $\{A_k\}_{k=0}^{\infty} \subset \mathfrak{A}$  covering  $E$  such that

$$\sum_{k=0}^{\infty} \gamma(A_k) < \mu^*(E) + \varepsilon = \mu(E) + \varepsilon.$$

Then,

$$\mu\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} \gamma(A_k) < \mu(E) + \varepsilon.$$

It follows that  $\mu(\bigcup_{k=0}^{\infty} A_k \setminus E) < \varepsilon$  and thus

$$\mu(E) \leq \mu\left(\bigcup_{k=0}^{\infty} A_k\right) = \nu\left(\bigcup_{k=0}^{\infty} A_k\right) = \nu(E) + \nu\left(\bigcup_{k=0}^{\infty} A_k \setminus E\right) \leq \nu(E) + \varepsilon,$$

where for  $\mu(\bigcup_{k=0}^{\infty} A_k) = \nu(\bigcup_{k=0}^{\infty} A_k)$  we used the same limit argument as the previous part.

For the case where  $E$  is  $\sigma$ -finite, it follows from a similar argument.

□

**Theorem** (Product measures). Let  $2 \leq n \in \mathbb{N}$  and suppose  $(X_i, \mathfrak{M}_i, \mu_i)$  is measure space for  $1 \leq i \leq n$ . Let  $X = \prod_i X_i$  and

$$\mathcal{E} = \left\{ E = \prod_i E_i : E_i \in \mathfrak{M}_i \text{ for } 1 \leq i \leq n \right\}.$$

The following holds:

1.  $\mathfrak{A} = \left\{ \bigcup_{k=0}^K A^k : \{A^k\}_k \subset \mathcal{E} \text{ and disjoint} \right\}$  is an algebra.
2. Suppose  $\{E^k\}_{k=0}^{\infty} \subset \mathcal{E}$  and  $\{F^k\}_{k=0}^{\infty} \subset \mathcal{E}$  are both pairwise disjoint sequences of sets and  $\bigcup_{k=0}^{\infty} E^k = \bigcup_{k=0}^{\infty} F^k$ , then

$$\sum_{k=0}^{\infty} \prod_{i=1}^n \mu_i(E_i^k) = \sum_{k=0}^{\infty} \prod_{i=1}^n \mu_i(F_i^k).$$

3. The map  $\gamma : \mathfrak{A} \rightarrow [0, \infty]$  defined by

$$\gamma \left( \bigcup_{k=0}^K \prod_{i=1}^n E_i^k \right) = \sum_{k=0}^K \prod_{i=1}^n \mu_i(E_i^k)$$

is a well-defined pre-measure.

4. If  $(X_i, \mathfrak{M}_i, \mu_i)$  is  $\sigma$ -finite, then  $X$  is  $\gamma$   $\sigma$ -finite.

*Proof.* 1. Since  $\emptyset \in \mathfrak{M}_i$  for all  $1 \leq i \leq n$ , we know  $\emptyset \in \mathcal{E}$ . Next let  $E, F \in \mathcal{E}$  be such that  $E = \prod_{i=1}^n E_i$  and  $F = \prod_{i=1}^n F_i$ . Then,

$$E \cap F = \prod_{i=1}^n (E_i \cap F_i) \in \mathcal{E}.$$

Similarly,

$$E^c = \bigcup_{i=1}^n \left( E_i^c \times \prod_{j \neq i} E_j \right) \in \mathcal{E}.$$

This shows that  $\mathfrak{A}$  is an algebra.

2. Suppose  $\bigcup_{k=0}^{\infty} E^k = \bigcup_{k=0}^{\infty} F^k$ , then we have

$$\sum_{k=0}^{\infty} \prod_{i=1}^n \chi_{E_i^k}(x_i) = \sum_{k=0}^{\infty} \prod_{i=1}^n \chi_{F_i^k}(x_i)$$

for all  $x = (x_1, \dots, x_n) \in X$ . Now fix  $(x_2, \dots, x_n)$ , we then have

$$\sum_{k=0}^{\infty} \chi_{E_1^k}(x_1) \alpha_1^k = \sum_{k=0}^{\infty} \chi_{F_1^k}(x_1) \beta_1^k,$$

where  $\alpha_1^k = \prod_{i=2}^n \chi_{E_i^k}(x_i)$  and  $\beta_1^k = \prod_{i=2}^n \chi_{F_i^k}(x_i)$ . Using the monotone convergence theorem and integrate both sides, we have

$$\sum_{k=0}^{\infty} \mu_1(E_1) \alpha_1^k = \sum_{k=0}^{\infty} \mu_1(F_1) \beta_1^k.$$

Iterate this argument gives the desired equality.

3. Suppose  $\{A_i\}_{i=0}^{\infty} \subset \mathfrak{A}$  disjoint such that  $\bigcup_{i=0}^{\infty} A_i \in \mathfrak{A}$ . By construction, there exists sequence  $\{F^j\}_{j=0}^J \subset \mathfrak{A}$  with  $J < \infty$  such that  $\bigcup_{i=0}^{\infty} A_i = \bigcup_{j=0}^J F_j$ . Also,  $A_i \in \mathfrak{A}$  for each  $i \in \mathbb{N}$ , so  $\bigcup_{i=0}^{\infty} A_i = \bigcup_{k=0}^{\infty} E^k$  where  $\{E^k\}_{k=0}^{\infty} \subset \mathcal{E}$  disjoint. It follows that

$$\gamma \left( \bigcup_{i=0}^{\infty} A_i \right) = \gamma \left( \bigcup_{j=0}^J F^j \right) = \sum_{j=0}^J \prod_{i=1}^n \mu_i(F_i^j) = \sum_{k=0}^{\infty} \prod_{i=1}^n \mu_i(E_i^k),$$

where the last equality is by item 2. However,

$$\gamma \left( \bigcup_{i=0}^{\infty} A_i \right) = \sum_{k=0}^{\infty} \prod_{i=1}^n \mu_i(E_i^k) = \sum_{i=0}^{\infty} \gamma(A_i).$$

This shows that  $\gamma$  is a pre-measure.

4. For each  $1 \leq i \leq n$ , there exists  $\{S_i^k\}_{k=0}^{\infty} \subset \mathfrak{M}_i$  such that  $S_i^k \subset S_i^{k+1}$ ,  $\bigcup_{k=0}^{\infty} S_i^k = X_i$ , and  $\mu_i(S_i^k) < \infty$ . Consider  $\{A^k\}_{k=0}^{\infty}$  where  $A^k = \prod_{i=1}^n S_i^k$ . Note that

$$X = \bigcup_{k=0}^{\infty} A^k \quad \text{and} \quad \gamma(A^k) = \prod_{i=1}^n \mu_i(S_i^k) < \infty.$$

This completes the proof. □

**Corollary.** Suppose that  $\{(X_i, \mathfrak{M}_i, \mu_i)\}_{i=1}^n$  be a sequence of  $\sigma$ -finite measure space. Let  $X = \prod_{i=1}^n X_i$  be endowed with the product  $\sigma$ -algebra  $\bigotimes_{i=1}^n \mathfrak{M}_i$ . Let  $\mathfrak{A}$  and  $\gamma : \mathfrak{A} \rightarrow [0, \infty]$  be the algebra and pre-measure from the previous theorem. Then, there exists a unique measure  $\nu : \bigotimes_{i=1}^n \mathfrak{M}_i \rightarrow [0, \infty]$  such that  $\nu = \gamma$  on  $\mathfrak{A}$ . Moreover,  $\nu$  is  $\sigma$ -finite.

*Proof.* Use the previous theorem and extend the pre-measure.  $\square$

### 3.6 Area formula and change of variable formula

#### 3.6.1 Area formula

We first need to develop a few facts in linear algebra.

**Proposition.** Let  $V_1, \dots, V_n, W$  be vector space over  $\mathbb{F}$  and  $T \in L(V_1, \dots, V_n; W)$ . Suppose  $x_i^j \in V_i$  for  $j = 0, 1$  and  $1 \leq i \leq n$ . Then,

$$\begin{aligned} T(x_1^0 + x_1^1, \dots, x_n^0 + x_n^1) &= \sum_{\beta \in B(n)} T(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}) \\ &= \sum_{m=0}^n \sum_{\beta \in B_m(n)} T(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}), \end{aligned}$$

where

$$\begin{aligned} B(n) &= \{\beta : \{1, \dots, n\} \rightarrow \{0, 1\}\}, \\ B_m(n) &= \left\{ \beta \in B(n) : \sum \beta(k) = m \right\}. \end{aligned}$$

*Proof.* Induction on  $n \geq 1$ .  $\square$

**Definition.** 1. For  $1 \leq k \leq n$  we set

$$\mathcal{A}(n, k) = \left\{ (\alpha_1, \dots, \alpha_k) \in \{1, \dots, n\}^k : \alpha_1 < \alpha_2 < \dots < \alpha_k \right\}.$$

We also set  $\mathcal{A}(n, 0) = \{0\}$ .

2. For  $1 \leq k \leq n$ , let  $M \in \mathbb{F}^{n \times k}$ ,  $N \in \mathbb{F}^{k \times n}$ ,  $P \in \mathbb{F}^{n \times n}$ . For  $\alpha \in \mathcal{A}(n, k)$ , we set  $M_\alpha, N_\alpha, P_\alpha \in \mathbb{F}^{k \times k}$  via

$$(M_\alpha)_{i,j} = M_{\alpha_i,j}, \quad (N_\alpha)_{i,j} = N_{i,\alpha_j}, \quad (P_\alpha)_{i,j} = P_{\alpha_i,\alpha_j}.$$

**Theorem.** Let  $M \in \mathbb{F}^{n \times n}$  and  $Z \in \mathbb{F}$ . Then,

$$\det(zI + M) = z^n + \sum_{k=0}^{n-1} z^k \sum_{\alpha \in \mathcal{A}(n, n-k)} \det(M_\alpha).$$

*Proof.* Fix  $z \in \mathbb{F}$ . Let  $x_i^0 = ze_i \in \mathbb{F}^n$  and  $x_i^1 = M_i \in \mathbb{F}^n$  be the  $i$ -th column of  $M$ . Recall that  $\det \in L^n(\mathbb{F}^n; \mathbb{F})$ . Therefore,

$$\begin{aligned} \det(zI + M) &= \det(x_1^0 + x_1^1, \dots, x_n^0 + x_n^1) \\ &= \sum_{k=0}^n \sum_{\beta \in B_k(n)} \det(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}) \\ &= z^n + \sum_{k=1}^n \sum_{\beta \in B_k(n)} \det(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}). \end{aligned}$$

Now given  $1 \leq k \leq n$  and  $\beta \in B_k(n)$ , we set  $\alpha \in \mathcal{A}(n, k)$  to be an increasing enumeration of  $\{1 \leq i \leq n : \beta(i) = 1\}$ . This gives a bijection from  $\mathcal{A}(n, k)$  to  $B_k(n)$ . On the other hand, if  $\beta \in B_k(n)$ , then

$$\det(x_1^{\beta(1)}, \dots, x_n^{\beta(n)}) = z^{n-k} \det(M_\alpha),$$

for the  $\alpha \in \mathcal{A}(n, k)$  that corresponds to the  $\beta \in B_k(n)$ . This completes the proof.  $\square$

**Theorem.** Let  $1 \leq n \leq m$ ,  $A \in \mathbb{F}^{m \times n}$ ,  $B \in \mathbb{F}^{n \times m}$ . The following holds:

1. (Sylvester's formula)  $\det(I_m + AB) = \det(I_n + BA)$ .
2. (Cauchy-Binet formula)  $\det(BA) = \sum_{\alpha \in \mathcal{A}(m,n)} \det A_\alpha \det B^\alpha$ .

In particular, if  $A^* \in \mathbb{F}^{n \times m}$  given by  $A_{ij}^* = \overline{A_{ji}}$ , then  $\det(A^*A) = \sum_{\alpha \in \mathcal{A}(m,n)} |\det A_\alpha|^2$ .

*Proof.* 1. We have

$$\begin{bmatrix} I_m & A \\ 0_{n \times m} & I_n \end{bmatrix} \begin{bmatrix} I_m & -A \\ B & I_n \end{bmatrix} = \begin{bmatrix} I_m + AB & 0_{m \times n} \\ B & I_n \end{bmatrix}$$

and

$$\begin{bmatrix} I_m & -A \\ B & I_n \end{bmatrix} \begin{bmatrix} I_m & A \\ 0_{n \times m} & I_n \end{bmatrix} = \begin{bmatrix} I_m & 0_{m \times n} \\ B & I_n + BA \end{bmatrix}.$$

It follows that  $\det(I_m + AB) = \det(I_n + BA)$ .

2. Fix  $z \in \mathbb{F} \setminus \{0\}$ . Then,

$$\begin{aligned} z^{-m} \det(zI_m + AB) &= \det(I_m + z^{-1}AB) \\ &= \det(I_n + B(z^{-1}A)) \\ &= z^{-n} \det(zI_n + BA). \end{aligned}$$

It follows that  $z^n \det(I_m + AB) = z^m \det(I_n + BA)$ . By our previous propositions, we have

$$z^{n+m} + \sum_{k=0}^{m-1} z^{k+n} \sum_{\alpha \in \mathcal{A}(m, m-k)} \det(AB)_\alpha^\alpha = z^{n+m} + \sum_{k=0}^{n-1} z^{k+m} \sum_{\alpha \in \mathcal{A}(n, n-k)} \det(BA)_\alpha^\alpha.$$

Consider the coefficients of degree  $m$ , we obtain

$$\sum_{\alpha \in \mathcal{A}(n, n)} \det(BA)_\alpha^\alpha = \sum_{\alpha \in \mathcal{A}(m, n)} \det(AB)_\alpha^\alpha.$$

Note that LHS =  $\det BA$  and  $(AB)_\alpha^\alpha = A_\alpha B^\alpha$ . This completes the proof.  $\square$

**Definition** (Jacobian map). Let  $\emptyset \neq U \subset \mathbb{R}^n$  be an open set and  $f \in C^1(U; \mathbb{R}^m)$  with  $n \leq m$ . Define the **Jacobian map**  $J_f \in C^0(U; [0, \infty))$  by

$$J_f = \|Df\| = \sqrt{\det(Df)^T Df}.$$

**Lemma.** Let  $\emptyset \neq U \subset \mathbb{R}^n$  be open,  $f \in C^1(U; \mathbb{R}^m)$  for some  $n \leq m$ . Suppose  $z \in U$  is such that  $Df(z)$  is injective. Then for  $0 < \varepsilon < 1$ , there exists  $B(z, r) \subset U$  such that

1.  $f|_{B(z, r)}$  is a Lipschitz injection.
2. If  $E \subset B(z, r)$  is Lebesgue measurable, then  $f(E) \in \mathfrak{H}^n(\mathbb{R}^m)$  and

$$(1 - \varepsilon)^{n+1} \int_E J_f d\lambda \leq \mathcal{H}^n(f(E)) \leq (1 + \varepsilon)^{n+1} \int_E J_f d\lambda.$$

*Proof.* Define the following  $M = Df(z)$ ,  $L \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$  such that  $LM = I_n$ , and  $g = f \circ L$ , so  $f = g \circ M$ .

Let  $0 < \varepsilon < 1$  and pick  $r > 0$  such that

$$(1 - \varepsilon) \|M(x - y)\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|M(y - x)\| \quad (\text{A})$$

for all  $x, y \in B(z, r)$  and

$$(1 + \varepsilon)^{-1} J_f(z) \leq J_f(x) \leq (1 - \varepsilon)^{-1} J_f(z) \quad (\text{B})$$



for all  $x \in B(z, r)$ . Note that

$$\mathcal{H}^n(ME) = J_f(z)\lambda(E).$$

Since  $ML$  is the identity on range  $M$ , equation (A) gives  $[g] \leq 1 + \varepsilon$  and  $[M \circ f^{-1}] \leq (1 - \varepsilon)^{-1}$ . It follows that

$$\mathcal{H}^n(f(E)) = \mathcal{H}^n(g(ME)) \leq (1 + \varepsilon)^n \mathcal{H}^n(ME) = (1 + \varepsilon)^n J_f(z)\lambda(E).$$

Also,

$$J_f(z)\lambda(E) = \mathcal{H}^n(ME) = \mathcal{H}^n(M \circ f^{-1}(f(E))) \leq (1 - \varepsilon)^{-n} \mathcal{H}^n(f(E)).$$

Now, equation (B) gives

$$J_f(z)\lambda(E) = \int_E J_f(z) d\lambda \leq (1 + \varepsilon) \int_E J_f d\lambda$$

and

$$J_f(z)\lambda(E) = \int_E J_f(z) d\lambda \geq (1 - \varepsilon) \int_E J_f d\lambda.$$

This completes the proof. □

**Definition.** Let  $X$  be a set equipped with counting measure  $\mathcal{H}^0 : \mathcal{P}(X) \rightarrow [0, \infty]$ . Let  $Y$  be a set and  $f : X \rightarrow Y$ . For any  $E \subset X$ , define  $\mathcal{N}_f(\cdot, E) : Y \rightarrow [0, \infty]$  by

$$\mathcal{N}_f(y, E) = \mathcal{H}^0(E \cap f^{-1}(\{y\})) = \mathcal{H}^0(\{x \in E : f(x) = y\}).$$

**Theorem.** Let  $F \in F_\sigma(\mathbb{R}^n)$  and  $f : F \rightarrow \mathbb{R}^m$  be locally Lipschitz with  $n \leq m$ . If  $E \subset F$  is Lebesgue measurable, then  $\mathcal{N}_f(\cdot, E) : \mathbb{R}^m \rightarrow [0, \infty]$  is  $\mathfrak{H}^n(\mathbb{R}^m)$  measurable.

*Proof.* Homework. □

**Lemma.** Let  $\emptyset \neq U \subset \mathbb{R}^n$  be open,  $f \in C^1(U; \mathbb{R}^m)$  for  $n \leq m$ . Suppose  $Df(x)$  is injective for all  $x \in U$ . Then for all  $E \subset U$  Lebesgue measurable, and

$$\int_E J_f d\lambda = \int_{\mathbb{R}^m} \mathcal{N}_f(\cdot, E) d\mathcal{H}^n.$$

*Proof.* Let  $E \subset U$  be Lebesgue measurable and  $0 < \varepsilon < 1$ . Using the previous lemma, we can pick  $\{B(x_k, r_k)\}_{k=0}^\infty$  such that  $B(x_k, r_k) \subset U$ ,  $f : B(x_k, r_k) \rightarrow \mathbb{R}^m$  is Lipschitz injection,  $E \subset \bigcup_{k=0}^\infty B(x_k, r_k)$ , and

$$(1 - \varepsilon)^{n+1} \int_F J_f d\lambda \leq \mathcal{H}^n(f(F)) \leq (1 + \varepsilon)^{n+1} \int_F J_f d\lambda$$

for all  $F \subset B(x_k, r_k)$ .

Let  $E_0 = E \cap B(x_0, r_0)$  and for  $k > 0$  let  $E_k = E \cap B(x_k, r_k) \setminus \bigcup_{j=0}^{k-1} B(x_j, r_j)$ . Then  $E = \bigsqcup_{k=0}^\infty E_k$ . Applying the inequality, we obtain

$$(1 - \varepsilon)^{n+1} \int_{E_k} J_f d\lambda \leq \mathcal{H}^n(f(E_k)) \leq (1 + \varepsilon)^{n+1} \int_{E_k} J_f d\lambda.$$

However, since  $f$  is injective when restricted to  $E_k$ , we have

$$\mathcal{H}^n(f(E_k)) = \int_{f(E_k)} 1 d\mathcal{H}^n = \int_{\mathbb{R}^m} \mathcal{N}_f(\cdot, E_k) d\mathcal{H}^n.$$

Summing the inequalities, we can then obtain from monotone convergence theorem that

$$(1 - \varepsilon)^{n+1} \int_E J_f d\lambda \leq \int_{\mathbb{R}^m} \sum_{k=0}^\infty \mathcal{N}_f(\cdot, E_k) d\mathcal{H}^n = \int_{\mathbb{R}^m} \mathcal{N}_f(\cdot, E) d\mathcal{H}^n \leq (1 + \varepsilon)^{n+1} \int_E J_f d\lambda.$$

Since this holds for all  $\varepsilon > 0$ , we have

$$\int_E J_f d\lambda = \int_{\mathbb{R}^m} \mathcal{N}_f(\cdot, E) d\mathcal{H}^n.$$

□

**Theorem** (Sard's theorem, special case). Let  $\emptyset \neq U \subset \mathbb{R}^n$  be open,  $f \in C^1(U; \mathbb{R}^m)$  for  $n \leq m$ . Then the set

$$Z = \{x \in U : J_f(x) = 0\}$$

is Lebesgue measurable and  $f(Z) \in \mathfrak{H}^n(\mathbb{R}^m)$  and  $\mathcal{H}^n(f(Z)) = 0$ .

*Proof.* Note that  $Z$  is relatively closed, so it is Lebesgue measurable. It then suffices to show that the outer measure  $\mathcal{H}^n(f(Z)) = 0$ .

Write  $U = \bigcup_{k=0}^{\infty} Q_k$  where  $\{Q_k\}_{k=0}^{\infty}$  is a sequence of almost disjoint cubes. It suffices to show  $\mathcal{H}^n(f(Z_k)) = 0$ , where  $Z_k = Z \cap Q_k$ . Let  $0 < \varepsilon < 1$  and let  $f_\varepsilon \in C^1(U; \mathbb{R}^{m+n})$  by  $f_\varepsilon(x) = (f(x), \varepsilon x)$ . Then  $f_\varepsilon$  is injective, and

$$Df_\varepsilon(x) = \begin{bmatrix} Df(x) \\ \varepsilon I_n \end{bmatrix} \in \mathbb{R}^{(m+n) \times n},$$

which is also injective for each  $x \in U$ . Also,

$$(Df_\varepsilon)^T Df_\varepsilon = \begin{bmatrix} Df^T & \varepsilon I \end{bmatrix} \begin{bmatrix} Df \\ \varepsilon I \end{bmatrix} = (Df)^T Df + \varepsilon^2 I.$$

It follows that

$$\begin{aligned} J_{f_\varepsilon}^2 &= \det((Df_\varepsilon)^T Df_\varepsilon) \\ &= \det(\varepsilon^2 I + (Df)^T Df) \\ &= \varepsilon^{2n} + \sum_{j=0}^{n-1} \varepsilon^{2j} \sum_{\alpha \in \mathcal{A}(n, n-j)} \det((Df)^T Df)_\alpha^\alpha \\ &= \det(Df)^T Df + \varepsilon^{2n} + \sum_{j=1}^{n-1} \varepsilon^{2j} \sum_{\alpha \in \mathcal{A}(n, n-j)} \det((Df)^T Df)_\alpha^\alpha \\ &\leq J_f^2 + \varepsilon^2 \left( 1 + \sum_{j=1}^{n-1} \sum_{\alpha \in \mathcal{A}(n, n-j)} \det((Df)^T Df)_\alpha^\alpha \right). \end{aligned}$$

Therefore, for  $x \in Q_k$ , we have  $J_{f_\varepsilon}^2(x) \leq J_f^2(x) + \varepsilon^2 C_k$  for a constant  $C_k > 0$  depending only on  $f$  and  $k \in \mathbb{N}$ . If  $x \in Z_k$ , then  $x \in Q_k \cap Z$ , so  $J_{f_\varepsilon}(x) \leq \varepsilon \sqrt{C_k}$ . Note that  $f_\varepsilon$  is injective and  $Df_\varepsilon(x)$  are injective for all  $x \in Z_k$ , the previous lemma gives

$$\mathcal{H}^n(f_\varepsilon(Z_k)) = \int_{Z_k} J_{f_\varepsilon} d\lambda \leq \varepsilon \sqrt{C_k} \lambda(Q_k),$$

but  $f(Z_k) = \pi_m(f_\varepsilon(Z_k))$  where  $\pi_m$  is the projection map. Therefore,

$$\mathcal{H}^n(f(Z_k)) \leq \mathcal{H}^n(f_\varepsilon(Z_k)) \leq \varepsilon \sqrt{C_k} \lambda(Q_k).$$

This then implies that  $\mathcal{H}^n(f(Z_k)) = 0$ . □

**Theorem** ( $C^1$  area formula). Let  $\emptyset \neq U \subset \mathbb{R}^n$  be open,  $f \in C^1(U; \mathbb{R}^m)$  for  $n \leq m$ . If  $E \subset U$  is Lebesgue measurable, then

$$\int_E J_f d\lambda = \int_{\mathbb{R}^m} \mathcal{N}_f(\cdot, E) d\mathcal{H}^n = \int_{f(E)} \mathcal{N}_f(\cdot, E) d\mathcal{H}^n.$$

In particular, if  $f$  is injective, then

$$\mathcal{H}^n(f(E)) = \int_E J_f d\lambda.$$

*Proof.* Let  $Z = \{J_f = 0\}$ , which is closed in  $U$ . Therefore,  $V = U \setminus Z$  is open. Note that  $J_f(x) \neq 0$  implies  $Df(x)$  injective. Then, previous lemma implies

$$\int_{V \cap E} J_f d\lambda = \int_{\mathbb{R}^m} \mathcal{N}_f(\cdot, E \cap V) d\mathcal{H}^n.$$

On the other hand,

$$\int_{E \cap Z} J_f d\lambda = 0 = \int_{f(E \cap Z)} \mathcal{N}_f(\cdot, E \cap Z) d\mathcal{H}^n = \int_{\mathbb{R}^m} \mathcal{N}_f(\cdot, E \cap Z) d\mathcal{H}^n.$$

Adding the equality together gives

$$\int_E J_f d\lambda = \int_{\mathbb{R}^m} \mathcal{N}_f(\cdot, E) d\mathcal{H}^n.$$

□

### 3.6.2 Change of variable

**Theorem** (change of variable, non-injective form). Let  $\emptyset \neq U \subset \mathbb{R}^n$  be open and  $f \in C^1(U; \mathbb{R}^m)$  with  $n \leq m$ . Let  $E \subset U$  be measurable. Then the following holds:

1. Suppose  $g : E \rightarrow [0, \infty]$  is  $\lambda$ -measurable. Then the map

$$\mathbb{R}^m \ni y \mapsto \int_{E \cap f^{-1}(\{y\})} g d\mathcal{H}^0 \in [0, \infty] \quad (*)$$

is  $\mathcal{H}^n$ -measurable, and

$$\int_E g J_f d\lambda = \int_{\mathbb{R}^m} \int_{E \cap f^{-1}(\{y\})} g d\mathcal{H}^0 d\mathcal{H}^n.$$

In particular,  $g J_f$  is  $\lambda$ -integrable if and only if the map  $(*)$  is  $\mathcal{H}^n$ -integrable.

2. Let  $Y \in \{V, \overline{\mathbb{R}}\}$  with  $V$  a Banach space. Suppose  $g : E \rightarrow Y$  is  $\lambda$ -measurable and  $g J_f$  is  $\lambda$ -integrable. Then for  $\mathcal{H}^n$ -a.e.  $y \in \mathbb{R}^m$ , the restriction  $g : E \cap f^{-1}(\{y\}) \rightarrow Y$  is  $\mathcal{H}^0$ -integrable. Moreover, the now  $Y$  valued map  $(*)$  is  $\mathcal{H}^n$ -integrable and

$$\int_E g J_f d\lambda = \int_{\mathbb{R}^m} \int_{E \cap f^{-1}(\{y\})} g d\mathcal{H}^0 d\mathcal{H}^n.$$

**Example.** As an example, say  $V \subset \mathbb{R}^n$  and  $f : V \rightarrow f(V) \subset \mathbb{R}^n$  is a diffeomorphism. Then

$$J_f = \sqrt{\det(Df)^T Df} = |\det Df|$$

and

$$\int_{E \cap f^{-1}(\{y\})} g d\mathcal{H}^0 = g \circ f^{-1}(y).$$

The theorem then gives

$$\int_E g |\det Df| d\lambda = \int_{f(E)} g \circ f^{-1} d\lambda.$$

This is the usual change of variable formula we encountered before in calculus.

*Proof sketch.* 1. We first prove the theorem assuming  $g : E \rightarrow [0, \infty]$  is Lebesgue measurable. Let  $\{\varphi_k\}_{k=0}^\infty$  be a sequence of simple functions such that  $\varphi_k \rightarrow g$  pointwise as  $k \rightarrow \infty$ . WLOG also assume  $\varphi_k \leq \varphi_{k+1}$ . Let

$$\varphi_k = \sum_{j=0}^{J_k} \varphi_{k,j} \chi_{E_{k,j}}$$

be the canonical representation of  $\varphi_k$ .

For  $y \in \mathbb{R}^m$ , we compute

$$\begin{aligned} \int_{E \cap f^{-1}(\{y\})} \varphi_k d\mathcal{H}^0 &= \sum_j \varphi_{k,j} \mathcal{H}^0(E_{k,j} \cap f^{-1}(\{y\})) \\ &= \sum_j \varphi_{k,j} \mathcal{N}_f(y, E_{k,j}) \end{aligned}$$

Therefore, the map

$$y \mapsto I_k := \int_{E \cap f^{-1}(\{y\})} \varphi_k d\mathcal{H}^0$$

is  $\mathfrak{H}^n(\mathbb{R}^m)$  measurable. Note that  $\varphi_k \leq \varphi_{k+1}$ , so  $I_k \leq I_{k+1}$ . Monotone convergence theorem then implies that the map

$$y \mapsto I := \int_{E \cap f^{-1}(\{y\})} g d\mathcal{H}^0$$

is  $\mathfrak{H}^n(\mathbb{R}^m)$  measurable and  $I = \lim_{k \rightarrow \infty} I_k$

On the other hand,

$$\begin{aligned} \int_E \varphi_k J_f d\lambda &= \sum_j \varphi_{k,j} \int_{E_{k,j}} J_f d\lambda \\ &= \sum_j \varphi_{k,j} \int_{\mathbb{R}^m} \mathcal{N}_f(\cdot, E_{k,j}) d\mathcal{H}^n \\ &= \int_{\mathbb{R}^m} \int_{E \cap f^{-1}(\{y\})} \varphi_k d\mathcal{H}^0 d\mathcal{H}^n(y). \end{aligned}$$

Using monotone convergence theorem again, we obtain

$$\int_E g J_f d\lambda = \int_{\mathbb{R}^m} \int_{E \cap f^{-1}(\{y\})} g d\mathcal{H}^0 d\mathcal{H}^n(y).$$

Therefore, item 1 is proved in the special case. The general case follows by considering null sets and using the more general convergence theorems.

2. To promote from  $Y = [0, \infty]$  to  $Y = \overline{\mathbb{R}}$  by splitting  $g = g^+ - g^-$  and applying item 1 to  $g^\pm$ . Then promote to  $Y = \mathbb{C}$  by splitting  $g = \operatorname{Re} g + i \operatorname{Im} g$ . Finally, promote to  $V$  a Banach space over  $\mathbb{F}$  as follows: let  $w \in V^*$  and consider  $w \circ g : E \rightarrow \mathbb{F}$ . Then show

$$\int w \circ g J_f d\lambda = \iint w \circ g d\mathcal{H}^0 d\mathcal{H}^n$$

for all  $w \in V^*$ . This will then give the desired result. □

**Theorem** (change of variable, local injective form). Let  $\emptyset \neq U \subset \mathbb{R}^n$  be open,  $f \in C^1(U; \mathbb{R}^m)$  for  $n \leq m$ . Suppose  $E \subset U$  is Lebesgue measurable such that  $E^\circ \neq \emptyset$  and  $\lambda(\partial E \cap U) = \lambda(Z \cap E) = 0$  where  $Z = \{J_f = 0\}$ . Further suppose the restriction  $f : E^\circ \rightarrow f(E^\circ)$  is injective. Finally let  $g : f(E) \rightarrow Y$ , where  $Y \in \{V, \overline{\mathbb{R}}\}$  with  $V$  a Banach space. Then the following holds:

1.  $f(E) \in \mathfrak{H}^n(\mathbb{R}^m)$ .
2.  $g$  is  $\mathcal{H}^n$ -measurable if and only if  $g \circ f$  is  $\lambda$ -measurable.
3.  $g$  is  $\mathcal{H}^n$ -integrable on  $f(E)$  if and only if  $g \circ f J_f$  is  $\lambda$ -integrable on  $E$ . In either case,

$$\int_E g \circ f J_f d\lambda = \int_{f(E)} g d\mathcal{H}^n.$$

*Proof sketch.* Apply the previous theorem to see that

$$\int_{E^\circ} g \circ f J_f d\lambda = \int_{f(E^\circ)} g d\mathcal{H}^n.$$

However,

$$\int_{\partial E \cap E} g \circ f J_f d\lambda = 0 = \int_{f(\partial E \cap E)} g d\mathcal{H}^n.$$

□

**Example.** Let  $E = [0, \infty) \times [0, 2\pi] \subset \mathbb{R}^2$ . Let  $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  via

$$\rho(r, \theta) = (r \cos \theta, r \sin \theta).$$

Then  $\rho$  is a smooth function. Note

$$E^\circ = (0, \infty) \times (0, 2\pi)$$

and the restriction  $\rho : E^\circ \rightarrow \rho(E^\circ)$  is injective. Additionally,

$$\partial E = (\{0\} \times [0, 2\pi]) \cup ([0, \infty) \times \{0\}) \cup ([0, \infty) \times \{2\pi\})$$

is null. We also know that  $|\det D\rho(r, \theta)| = r$ , so  $Z = \{J_\rho = 0\}$  is null.

Note that  $\rho(E^\circ) = \mathbb{R}^2 \setminus N$  where  $N$  is null. Then,  $f : \mathbb{R}^2 \rightarrow V$  is integrable if and only if  $f \circ \rho : E^\circ \rightarrow V$  is integrable, and in either case,

$$\begin{aligned} \int_{\mathbb{R}^2} f d\lambda &= \int_{E^\circ} f \circ \rho J_\rho d\lambda \\ &= \int_0^\infty \int_0^{2\pi} f(r \cos \theta, r \sin \theta) r d\theta dr \\ &= \int_0^{2\pi} \int_0^\infty f(r \cos \theta, r \sin \theta) r dr d\theta. \end{aligned}$$

**Example.** Let  $E = [0, \infty) \times [0, 2\pi] \times [0, \pi] \subset \mathbb{R}^3$  and  $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  via

$$\zeta(r, \theta, \varphi) = (r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi).$$

This is 3D spherical coordinates. Similar to the previous example,  $\zeta : E^\circ \rightarrow \zeta(E^\circ)$  is injective and  $\mathbb{R}^3 \setminus \zeta(E^\circ)$  is null. In addition,

$$J_\zeta(r, \theta, \varphi) = |\det D\zeta(r, \theta, \varphi)| = r^2 \sin \varphi.$$

Then for  $f : \mathbb{R}^3 \rightarrow V$  we have  $f$  is integrable if and only if  $f \circ \zeta : E^\circ \rightarrow V$  is integrable, and in either case,

$$\begin{aligned} \int_{\mathbb{R}^3} f d\lambda &= \int_{E^\circ} f \circ \zeta J_\zeta d\lambda \\ &= \int_0^\infty \int_0^{2\pi} \int_0^\pi f \circ \zeta(r, \theta, \varphi) r^2 \sin \varphi d\varphi d\theta dr. \end{aligned}$$

**Example (Hyperspherical coordinates).** Let  $n \geq 3$  and  $E = [0, \infty) \times [0, 2\pi] \times [0, \pi]^{n-2} \in \mathbb{R}^n$ . Let  $\Psi : E \rightarrow \mathbb{R}^n$  via

$$\begin{aligned} \Psi_n &= r \cos(\varphi_{n-2}) \\ \Psi_{n-1} &= r \sin(\varphi_{n-2}) \cos(\varphi_{n-3}) \\ &\dots \\ \Psi_3 &= r \sin(\varphi_{n-2}) \sin(\varphi_{n-3}) \cdots \sin(\varphi_2) \cos(\varphi_1) \\ \Psi_2 &= r \sin(\varphi_{n-2}) \sin(\varphi_{n-3}) \cdots \sin(\varphi_2) \sin(\varphi_1) \sin(\theta) \\ \Psi_1 &= r \sin(\varphi_{n-2}) \sin(\varphi_{n-3}) \cdots \sin(\varphi_2) \sin(\varphi_1) \cos(\theta) \end{aligned}$$

An induction argument shows that

$$\det D\Psi = r^{n-1} \sin(\varphi_1) \sin^2(\varphi_2) \dots \sin^{n-2}(\varphi_{n-2}) \geq 0.$$

Then for  $f : \mathbb{R}^n \rightarrow V$  we have

$$\int_{\mathbb{R}^n} f \, d\lambda = \int_E f \circ \Psi J_\Psi$$

### 3.7 Spaces of integrable functions

In this subsection,  $(X, \mathfrak{M}, \mu)$  will denote a measure space and  $V$  will denote a Banach space.

**Definition.** 1. For  $\mu$ -measurable  $f$ , for  $1 \leq p < \infty$  define  $\|f\|_p$  via

$$\|f\|_p = \left[ \int_X \|f\|^p \, d\mu \right]^{\frac{1}{p}},$$

and for  $p = \infty$  define  $\|f\|_\infty$  via

$$\|f\|_\infty = \inf \{s \in [0, \infty] : f \leq s \text{ a.e.}\}.$$

2. Define

$$\text{Leb}_\mu^p(X; V) = \{f : X \rightarrow V, f \text{ is } \mu\text{-measurable and } \|f\|_p < \infty\}$$

to be the set of  $p$ -integrable functions.

**Example.** 1. If  $X \in \{\mathbb{N}, \mathbb{Z}\}$  with  $\mu$  being the counting measure, then  $\ell^p(X; V) = \text{Leb}_\mu^p(X; V)$ .

2. For all  $p \in [1, \infty]$ , we have  $S_{\text{fin}}(X; V) \subset \text{Leb}_\mu^p(X; V)$ .

**Theorem** (Hölder's inequalities). Suppose  $p, q, r \in [1, \infty]$  satisfy  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ , then the following holds:

1. If  $f \in \text{Leb}_\mu^p(X; \mathbb{F})$ ,  $g \in \text{Leb}_\mu^q(X; \mathbb{F})$ , then  $fg \in \text{Leb}_\mu^r(X; \mathbb{F})$  and

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

In particular, if  $r = 1$ , then

$$\int_X |fg| \, d\mu \leq \|f\|_p \|g\|_q.$$

2. If  $V_1, V_2, W$  are Banach spaces and  $T \in \mathcal{L}(V_1, V_2; W)$  then

$$\|T(f, g)\| \leq \|T\|_{\mathcal{L}} \|f\|_p \|g\|_q.$$

**Theorem** (Minkowski's inequalities). Let  $p \in [1, \infty]$  and  $f, g \in \text{Leb}_\mu^p(X; V)$ , then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

**Corollary.**  $\text{Leb}_\mu^p(X; V)$  is a vector space for  $p \in [1, \infty]$ , and  $\|\cdot\|_p$  is a semi-norm. Also,  $\|f\|_p = 0$  if and only if  $f = 0$  a.e.

**Theorem.** Let  $f \in \text{Leb}_\mu^p(X; V)$ . For any  $\varepsilon > 0$ , there exists  $\varphi \in S_{\text{fin}}(X; V) \subset \text{Leb}_\mu^p(X; V)$  such that  $\|\varphi - f\|_p < \varepsilon$ .

*Proof sketch.* By Pettis theorem, we can select  $\{\psi_n\}_{n=0}^\infty \subset S(X; V)$  such that  $\|\psi_n\| \leq 2\|f\|$  on  $X$  and  $\psi_n \rightarrow f$  a.e. We then have  $\|\psi_n\|^p \leq 2^p \|f\|^p$ , so  $\psi_n \in S_{\text{fin}}(X; V)$ . Note that  $\|\psi_n - f\| \leq 3\|f\|$  on  $X$ . Dominated convergence theorem then implies

$$\lim_{n \rightarrow \infty} \int_X \|\psi_n - f\| \, d\mu = \int_X \lim_{n \rightarrow \infty} \|\psi_n - f\| \, d\mu = 0.$$

□

Next we define  $L^p$  space by taking the quotient of  $\text{Leb}_\mu^p(X; V)$  under equivalence relation  $\sim$ , where  $f \sim g$  if  $f = g$  a.e. Once we show  $L^p(X; V)$  is complete and  $S_{\text{fin}}(X; V)$  is dense in  $L^p(X; V)$ , we will know the completion of  $S_{\text{fin}}(X; V)$  is  $L^p(X; V)$ .

**Definition** ( $L^p$  spaces). For  $f, g \in \text{Leb}_\mu^p$ , let  $f \sim g$  if  $f = g$  a.e. Then  $\sim$  is an equivalence relation. Let  $[f]$  be the equivalence class of  $f$ . Define

$$L_\mu^p(X; V) = \{[f] : f \in \text{Leb}_\mu^p(X; V)\}$$

with  $\alpha[f] = [\alpha f]$ ,  $[f] + [g] = [f + g]$ , and  $\|[f]\|_{L^p} = \|f\|_p$ . Then it is easy to see that all the operations are well-defined and  $L_\mu^p$  is a vector space.

**Theorem** (Riesz-Fisher).  $L_\mu^p(X; V)$  is a Banach space for  $p \in [1, \infty]$ .

*Proof.* Let  $\{f_k\}_{k=0}^\infty \in L^p$  such that  $\sum_{k=0}^\infty \|f_k\|_p < \infty$ . We want to show  $\sum_{k=0}^\infty f_k$  converges in  $L^p$ .

First suppose  $p < \infty$ . Let

$$g_m = \sum_{k=0}^m \|f_k\|_V \in L^p(X; \mathbb{F}).$$

Then, there exists  $S \in \mathbb{R}$  such that

$$\|g_m\|_{L^p} \leq \sum_{k=0}^m \| \|f_k\|_V \|_p = \sum_{k=0}^m \|f_k\|_p < S.$$

for all  $m \in \mathbb{N}$ . It follows that

$$\int_X |g_m|^p d\mu \leq S^p.$$

Let  $g = \lim_{m \rightarrow \infty} g_m$ . Then  $g(x) \neq \infty$  a.e. and by monotone convergence theorem, we have

$$\int_X |g|^p d\mu = \lim_{n \rightarrow \infty} \int_X |g_n|^p d\mu \leq S^p.$$

Define  $F : X \rightarrow V$  in the following way:

$$F(x) = \begin{cases} 0 & \text{if } g(x) = \infty, \\ \sum_{k=0}^\infty f_k(x) & \text{if } g(x) \neq \infty. \end{cases}$$

Then  $\|F - \sum_{k=0}^m f_k\|_V^p \leq g^p$  a.e. and  $\lim_{m \rightarrow \infty} \|F - \sum_{k=0}^m f_k\| = 0$  a.e. It then follows that

$$\lim_{m \rightarrow \infty} \left\| F - \sum_{k=0}^m f_k \right\|_{L^p}^p = \lim_{m \rightarrow \infty} \int_X \left\| F - \sum_{k=0}^m f_k \right\|_V^p d\mu = 0,$$

where the last equality is by dominated convergence theorem. □

## 4 Manifolds in $\mathbb{R}^n$ , differential forms, Stokes-Cartan theorem

We know from the area formula that we can integrate on certain  $m$ -dimensional subsets of  $\mathbb{R}^n$  ( $m \leq n$ ). We also know through the fundamental theorem of calculus that there exists a beautiful and useful connection between integral and differential calculus in 1D. The goal is then to develop some form of differential calculus on sets we can integrate over in  $\mathbb{R}^n$  and to generalize FTC.

There are a few caveats:

1. We will only develop the “extrinsic theory” of manifolds in  $\mathbb{R}^n$ . This is opposed to the modern “intrinsic” perspective, which does not rely on containment in  $\mathbb{R}^n$ . However, there are deep theorems in modern manifold theory (Nash<sup>1</sup>, Whitney) that show “intrinsic if and only if extrinsic”
2. A lot of the theory we develop is front-loading for the Stokes-Cartan theorem.
3. We are **NOT** doing differential geometry.
4. All of this works with Banach space replacing  $\mathbb{R}^n$ . We are working in  $\mathbb{R}^n$  to make things easier and avoid some slight subtleties.

### 4.1 Manifolds

**Definition.** Let  $m, n \in \mathbb{N}$  and  $1 \leq m \leq n$  and let  $1 \leq k \leq \infty$ . Let  $M \subset \mathbb{R}^n$ .

1. We say  $M$  admits  $C^k$  local  $m$ -coordinates at  $z \in M$  if the following holds:

- (a) There exists a set  $U \subset \mathbb{R}^n$  open with  $z \in U$ .
- (b) There exists  $\emptyset \neq V \subset \mathbb{R}^m$  open and **homeomorphism**  $\varphi : V \rightarrow \varphi(V) = M \cap U$ .
- (c)  $\varphi \in C^k(V; \mathbb{R}^n)$  and  $D\varphi(x) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n) \simeq \mathbb{R}^{n \times m}$  has rank  $m$  for all  $x \in V$ .

The triple  $(U, V, \varphi)$  is called a  $C^k$   $m$ -coordinate **chart**.

2. We say  $M$  is an  $m$ -dimensional  $C^k$  **manifold** if  $M \neq \emptyset$  and  $M$  admits local  $C^k$   $m$ -coordinates at each  $z \in M$ . If  $M$  is an  $m$ -dimensional manifold, we say  $M$  is an  $m$ -fold.
3. An **atlas** on  $M$  is a collection  $\mathcal{A} = \{(U_\alpha, V_\alpha, \varphi_\alpha)\}_{\alpha \in A}$  for  $A \neq \emptyset$  some index set such that  $(U_\alpha, V_\alpha, \varphi_\alpha)$  is  $C^k$  local  $m$ -coordinates for all  $\alpha \in A$  and  $M \subset \bigcup_\alpha U_\alpha$ .

**Remark.** 1. From time to time it is useful to replace  $\mathbb{R}^n$  and  $\mathbb{R}^m$  with generic real normed vector space of dimension  $n$  and  $m$  respectively. This changes nothing in the definition.

2. The definition fails to define 0-dimensional manifolds. We make the convention that a 0-fold is a nonempty discrete subset of  $\mathbb{R}^n$ , where by discrete we mean every subset of it is both open and closed. We also consider all 0-fold to be  $C^\infty$ .

**Example.** 1. Let  $\emptyset \subset U \subset \mathbb{R}^n$  open, then  $U$  is a  $C^\infty$   $n$ -fold.

2. Let  $w + V = \{w + v : v \in V\}$  for  $w \in \mathbb{R}^n$  and  $V \subset \mathbb{R}^n$  a subset of dimension  $m$ . Indeed, let  $\{v_1, \dots, v_m\} \subset V$  be a basis and consider  $\mathbb{R}^m \ni x \mapsto w + \sum_j x_j v_j \in w + V$ .
3. Suppose  $M \subset \mathbb{R}^n$  is a  $C^k$   $m$ -fold. Suppose  $U \subset \mathbb{R}^n$  is open and  $M \cap U \neq \emptyset$ . Then  $M \cap U$  is a  $C^k$   $m$ -fold.
4. Suppose  $M_1, M_2 \subset \mathbb{R}^n$  are  $C^k$   $m$ -folds. Suppose  $|x - y| \geq \varepsilon > 0$  for all  $x \in M_1$  and  $y \in M_2$ . Then  $M_1 \cup M_2$  is a  $C^k$   $m$ -fold.
5. Suppose  $M_1 \subset \mathbb{R}^{n_1}$  and  $M_2 \subset \mathbb{R}^{n_2}$  are  $C^k$   $m_1$ -fold and  $C^k$   $m_2$ -fold respectively. Then  $M_1 \times M_2 \subset \mathbb{R}^{n_1+n_2}$  is a  $C^k$   $(m_1 + m_2)$ -fold.
6. Let  $\varphi : \mathbb{R} \times (-\frac{1}{2}, \frac{1}{2}) \rightarrow \mathbb{R}^3$  via

$$\varphi(\theta, t) = (\cos \theta, \sin \theta, t).$$

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Then the image  $M$  is a cylinder in  $\mathbb{R}^3$ . Note that  $\varphi$  is smooth and  $D\varphi$  has rank 2 everywhere. We can build an atlas on  $M$  via  $\mathcal{A} = \{(U_1, V_2, \varphi), (U_2, V_2, \varphi)\}$  for  $V_1 = (0, 2\pi) \times (-\frac{1}{2}, \frac{1}{2})$  and  $V_2 = (\pi, 3\pi) \times (-\frac{1}{2}, \frac{1}{2})$ . After we figure out the corresponding  $U_1$  and  $U_2$ , we can see that  $M$  is a  $C^\infty$  2-fold.

**Theorem.** Let  $\emptyset \neq U \subset \mathbb{R}^n$  be open,  $f \in C^k(U; \mathbb{R}^r)$  for  $1 \leq r < n$ . Suppose  $\bar{w} \in f(U)$  and

$$M = \{z \in U : f(z) = \bar{w} \text{ and } Df(z) \text{ had rank } r\} \neq \emptyset.$$

Then  $M$  is a  $C^k$   $(n-r)$ -fold.

*Proof.* Let  $z \in M$ . Then,  $Df(z)$  is surjective, so we can apply the right inverse function theorem. This provides open sets  $W \subset \mathbb{R}^r$  and  $V \subset \mathbb{R}^{n-r}$ , and a  $C^k$  diffeomorphism  $F : W \times V \rightarrow F(W \times V) \subset U$ , such that  $f(F(w, v)) = w$  for all  $(w, v) \in W \times V$  and  $\bar{w} \in W$ . Let  $\varphi : V \rightarrow U$  via  $\varphi(v) = F(\bar{w}, v)$ . Then

$$f(\varphi(v)) = f(F(\bar{w}, v)) = \bar{w}.$$

Meanwhile, the right inverse function theorem guarantees that  $D\varphi$  has rank  $n-r$  everywhere. Therefore,  $\varphi : V \rightarrow M \cap U$  is the desired homeomorphism to show that  $M$  is a  $C^k$   $(n-r)$ -fold.  $\square$

**Example.** 1. Let  $S \in \mathbb{R}^{n \times n}$  be symmetric with at least one positive eigenvalue. Consider

$$M = \{x \in \mathbb{R}^n : Sx \cdot x = 1\} \neq \emptyset.$$

Consider  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  via  $f(x) = Sx \cdot x$ , which is  $C^\infty$ . Note that

$$Df(x)h = \nabla f(x) \cdot h = 2Sx \cdot h,$$

and for  $x \in M$ , we have  $Df(x)x = 2 \neq 0$ . Therefore,  $Df(x)$  has rank 1 for all  $x \in M$ . The previous theorem then tells us  $M$  is a  $C^\infty$   $(n-1)$ -fold in  $\mathbb{R}^n$ .

- (a) If  $S = I$ , then  $M = \{x : |x|^2 = 1\} = S^{n-1}$  is a sphere and it is a  $C^\infty$   $(n-1)$ -fold in  $\mathbb{R}^n$ .
- (b) If  $S = \text{diag}(\lambda_1, \dots, \lambda_n)$  with  $\lambda_i > 0$ , then  $M$  is an ellipsoid and it is a  $C^\infty$   $(n-1)$ -fold in  $\mathbb{R}^n$ .
- (c) If  $S = \text{diag}(-\lambda_1, \lambda_2, \dots, \lambda_n)$  with  $\lambda_i > 0$ , then  $M$  is a hyperboloid and it is a  $C^\infty$   $(n-1)$ -fold in  $\mathbb{R}^n$ .

2. Let  $\emptyset \neq U \subset \mathbb{R}^n$  be open,  $f \in C^k(U; \mathbb{R}^m)$ , and

$$\Gamma(f) = \{(X, f(x)) \in \mathbb{R}^{n+m} : x \in U\}.$$

Let  $F : U \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  via  $F(x, y) = y - f(x)$ . Then  $(x, y) \in \Gamma(f)$  if and only if  $F(x, y) = 0$ . However,  $F$  is  $C^k$  and  $DF(x, y)(h, k) = k - Df(x)h$ . The  $k$  term implies that the derivative has rank  $m$ . The previous theorem then implies that  $\Gamma(f)$  is a  $C^k$   $n$ -fold in  $\mathbb{R}^{n+m}$ .

3. Note that  $\text{GL}(n) \subset \mathbb{R}^{n \times n}$  is open and hence a  $C^\infty$   $n^2$ -fold.  
 4. Let  $\text{SL}(n) = \{M \in \text{GL}(n) : \det M = 1\}$ , then  $\det : \text{GL}(n) \rightarrow \mathbb{R}$  is  $C^\infty$  and

$$D\det(M)(N) = (\det M) \text{tr}(M^{-1}N).$$

Therefore, for  $M \in \text{SL}(n)$ ,  $D\det(M)M = \text{tr}(M^{-1}M) = n \neq 0$ . This implies that  $D\det(M)$  has rank 1 at each  $M \in \text{SL}(n)$ . This then shows that  $\text{SL}(n)$  is a  $C^\infty$   $(n^2-1)$ -fold in  $\mathbb{R}^{n \times n}$ .

5. The set of orthogonal matrices  $O(n)$  is a  $C^\infty$   $\frac{n(n-1)}{2}$ -fold in  $\mathbb{R}^{n \times n}$ . Note that

$$O(n) = \{M \in \mathbb{R}^{n \times n} : M^T M = I\}.$$

Consider the map  $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}$  via  $f(M) = M^T M - I$ , which is smooth and

$$Df(M)(N) = M^T N + N^T M = M^T N + (M^T N)^T.$$

Therefore,  $\ker Df(M) = M^{-T} \mathbb{R}_{\text{asym}}^{n \times n} = M \mathbb{R}_{\text{asym}}^{n \times n}$  and hence  $\dim \ker Df(M) = \dim \mathbb{R}_{\text{asym}}^{n \times n} = \frac{n(n-1)}{2}$ . It follows that

$$\text{rank } Df(M) = \frac{n(n+1)}{2} = \dim \mathbb{R}_{\text{sym}}^{n \times n}.$$

Therefore,  $O(n)$  is a  $C^\infty \frac{n(n-1)}{2}$ -fold.

As a side note,  $\mathbb{R}^{n \times n} = \mathbb{R}_{\text{sym}}^{n \times n} \oplus \mathbb{R}_{\text{asym}}^{n \times n}$  with the column product  $M : N = M_{ij} N_{ij}$  as the inner product.

**Theorem.** Let  $m, n \in \mathbb{N}$  with  $1 \leq m \leq n$ ,  $1 \leq k \leq \infty$ . Let  $M \in \mathbb{R}^n$ ,  $z \in M$ , then the following are equivalent:

1.  $M$  admits  $C^k$  local  $m$ -coordinates at  $z$ , say  $(U, V, \varphi)$ .
2. There exists an open  $W \in \mathbb{R}^n$  and  $C^k$  diffeomorphism  $F : W \rightarrow F(W) \in \mathbb{R}^n$  such that

$$F(M \cap W) = F(W) \cap \{x \in \mathbb{R}^n : x_l = 0 \text{ for } m+1 \leq l \leq n\}.$$

In either case,  $\varphi$  and  $F$  are related via  $\varphi^{-1} = (F_1, \dots, F_n)$ .

This is in fact the flattening map we encountered in HW.

Why is this important? There are two reasons.

1. This unifies the perspective on  $m$ -folds. We saw

$$M = \{x : f(x) = y \text{ and } Df(x) \text{ had full rank}\}.$$

is a manifold. Locally, this is the only example we need. That is, all manifolds locally are locally a level set of some functions.

2. This theorem is very useful for certain technical points...

\*\*\* TO-DO \*\*\*add picture for the next theorem

**Theorem.** Suppose  $(U_0, V_0, \varphi_0)$  and  $(U_1, V_1, \varphi_1)$  are  $C^k$  local  $m$ -coordinates at  $z \in M$ . Suppose also  $U_0 \cap U_1 \neq \emptyset$  and  $z \in U_0 \cap U_1 \cap M$ . Then the transition map  $\zeta : \varphi_0^{-1}(U_0 \cap U_1 \cap M) \rightarrow \varphi_1^{-1}(U_0 \cap U_1 \cap M)$  is a  $C^k$  diffeomorphism.

*Proof.* Both sets are open because  $\varphi_i : V_i \rightarrow \varphi_i(V_i) = M \cap U_i$  is homeomorphism for each  $i \in \{0, 1\}$ . Let  $\tilde{V}_i = \varphi_i^{-1}(U_0 \cap U_1 \cap M)$  for each  $i \in \{0, 1\}$  and fix  $x \in \tilde{V}_0$ . By the previous theorem, we can find a flattening map  $F : W \rightarrow F(W)$  such that  $F$  is a  $C^k$  diffeomorphism and  $\varphi_1^{-1} = (F_1, \dots, F_m) = \tilde{F}$ . Therefore,  $\zeta = \varphi_1^{-1} \circ \varphi_0 = \tilde{F} \circ \varphi_0$  near  $x$ . However  $x \in \tilde{V}_0$  is arbitrary, so  $\zeta$  is a  $C^k$  diffeomorphism.  $\square$

**Lemma.** Suppose  $(U_0, V_0, \varphi_0)$  and  $(U_1, V_1, \varphi_1)$  are  $C^k$  local  $m$ -coordinates at  $z \in M$ . Suppose also  $U_0 \cap U_1 \neq \emptyset$  and  $z \in U_0 \cap U_1 \cap M$ . Then the following holds:

1.  $D\varphi_0(\varphi_0^{-1}(z)) = D\varphi_1(\varphi_1^{-1}(z))D[\varphi_1^{-1} \circ \varphi_0](\varphi_0^{-1}(z))$ .
2.  $\text{range } D\varphi_0(\varphi_0^{-1}(z)) = \text{range } D\varphi_1(\varphi_1^{-1}(z))$

*Proof.* Let  $a = \varphi_0^{-1}(z)$  and  $b = \varphi_1^{-1}(z)$ , then

$$\varphi_0 = \varphi_1 \circ \varphi_1^{-1} \circ \varphi_0 = \varphi_1 \circ \zeta.$$

It follows that  $D\varphi_0 = [D\varphi_1 \circ \zeta] \circ D\zeta$ , and

$$D\varphi_0(a) = [D\varphi_1 \circ \zeta(a)]D\zeta(a) = [D\varphi_1(b)]D\zeta(a).$$

This proves item 1. However,  $\zeta$  is a diffeomorphism, so

$$\text{range } D\varphi_0(\varphi_0^{-1}(z)) = \text{range } D\varphi_1(\varphi_1^{-1}(z)).$$

$\square$

**Definition** (tangent space). Define the **tangent space** to  $M$  at  $z$  to be  $T_z M = \text{range } D\varphi(\varphi^{-1}(z))$  for any  $C^k$   $m$ -local coordinate chart  $(U, V, \varphi)$ .

**Example.** Let  $(U, V, \varphi)$  be  $m$ -coordinates at  $z \in M$  and let  $x = \varphi^{-1}(z)$ . Then,

$$T_z M = \text{range } D\varphi(x) = \text{span}\{D\varphi(x)e_1, \dots, D\varphi(x)e_m\} = \text{span}\{\partial_1\varphi(x), \dots, \partial_m\varphi(x)\}.$$

**Theorem.** Let  $\emptyset \neq U \subset \mathbb{R}^n$  be open,  $f : U \rightarrow \mathbb{R}^r$  be  $C^k$  for  $1 \leq r < n$ . We saw before that if  $z_0 \in U$ , and

$$M = \{z \in U : f(z) \neq f(z_0) \text{ and } Df(z) \text{ has rank } r\} \neq \emptyset,$$

then  $M$  is a  $C^k$   $(n - r)$ -fold. If  $z \in M$ , then  $T_z M = \ker Df(z)$ .

*Proof.* Let  $(\tilde{U}, V, \varphi)$  be local coordinate at  $z \in M$ . Then  $f(\varphi(x)) = f(z_0)$  for all  $x \in C$ . Then,

$$0 = Df(\varphi(x))D\varphi(x).$$

Therefore, for  $x_0 \in V$  such that  $\varphi(x_0) = z$ , we have  $0 = Df(z)D\varphi(x_0)$ . This implies that  $T_z M = \text{range } D\varphi(x_0) \subset \ker Df(z)$ . However,

$$\dim \text{range } D\varphi(x_0) = \dim \ker Df(z) = n - r,$$

so  $T_z M = \ker Df(z)$ . □

**Example.** 1. Let  $M = \{x \in \mathbb{R}^n : Sx \cdot x = 1\}$  for  $S \in \mathbb{R}_{\text{sym}}^{n \times n}$  with at least one positive eigenvalue. Define  $f(x) = Sx \cdot x$ , then  $Df(x)h = 2Sx \cdot h$ . This then implies that

$$T_x M = \{h : Sx \cdot h = 0\}.$$

2. Let  $\Gamma(f) = \{(x, f(x)) : x \in U\}$  for  $f : U \rightarrow \mathbb{R}^m$  and  $U \subset \mathbb{R}^n$  open. Define  $F(x, y) = f(x) - y$ , then  $DF(x, y)(h, k) = Df(x)h - k$ . This then implies that

$$T_{(x, f(x))}\Gamma(f) = \{(h, Df(x)h) : h \in \mathbb{R}^n\}.$$

3. Recall that  $\text{SL}(n) = \{M \in \mathbb{R}^{n \times n} : \det M = 1\}$ . Define  $f(x) = \det M$ , then  $Df(M)N = \det M \text{tr}(M^{-1}N)$ . This then implies that

$$T_M \text{SL}(n) = M\mathbb{R}_{\text{tr}=0}^{n \times n},$$

where  $\mathbb{R}_{\text{tr}=0}^{n \times n} = \{N \in \mathbb{R}^{n \times n} : \text{tr } N = 0\}$

4. Let  $O(n) = \{M \in \mathbb{R}^{n \times n} : M^T M = I\}$ . Define  $f(M) = M^T M - I$ , then  $Df(M)N = M^T N + N^T M$ . This then implies that

$$T_M O(n) = M\mathbb{R}_{\text{asym}}^{n \times n}.$$

## 4.2 Mappings between manifolds

\*\*\* TO-DO \*\*\*: Add picture

**Lemma.** Suppose  $M \subset \mathbb{R}^{n_1}$  and  $N \subset \mathbb{R}^{n_2}$  are  $C^k$   $m_1$ -fold and  $m_2$ -fold respectively. Suppose also that  $f : M \rightarrow N$ . Let  $x$  be such that  $\varphi_1(x) = z \in U_1 \cap U_2 \cap M$  and  $f(z) \in \tilde{U}_1 \cap \tilde{U}_2 \cap N$ . Then the following holds:

1.  $\psi_1^{-1} \circ f \circ \varphi_1$  is differentiable at  $x$  if and only if  $\psi_2^{-1} \circ f \circ \varphi_2$  is differentiable at  $\zeta(x)$ . The same holds when differentiable is replaced with  $C^j$  for  $1 \leq j \leq k$ .
2. Given  $v \in T_z M = \text{range } D\varphi_1(\varphi_1^{-1}(z))$ , there exists a unique  $w \in T_{f(z)} N$  such that  $v = D\varphi_1(\varphi_1^{-1})h$  and  $w = D\psi_1(\psi_1^{-1}(f(z)))D[\psi_1^{-1} \circ f \circ \varphi_1](\varphi_1^{-1}(z))h$ .

*Proof.* Note that  $\varphi_1 = \varphi_2 \circ \varphi_2^{-1} \circ \varphi_1 = \varphi_2 \circ \zeta$  and  $\psi_1 = \psi_2 \circ \psi_2^{-1} \circ \psi_1 = \psi_2 \circ \theta$  and use chain rule.  $\square$

**Definition.** Suppose  $M_1 \subset \mathbb{R}^{n_1}$  and  $M_2 \subset \mathbb{R}^{n_2}$  are  $C^k$   $m_1$ -fold and  $m_2$ -fold respectively. Suppose also that  $f : M_1 \rightarrow M_2$ .

1. We say  $f$  is differentiable at  $z \in M$  if  $\psi^{-1} \circ f \circ \varphi$  is differentiable at  $\varphi^{-1}(z)$  for any choice of  $C^k$  local coordinates at  $z$  and  $f(z)$ . We define  $Df(z) \in \mathcal{L}(T_z M_1, T_{f(z)} M_2)$  via  $v \mapsto w$  from the previous lemma.
2. We say  $f \in C^j(M_1, M_2)$  for  $1 \leq j \leq k$  if  $\psi^{-1} \circ f \circ \varphi$  is  $C^j$ .

Define flattening map  $F$  and  $G$ . WLOG we can assume  $F(W) = V \times (-\varepsilon, \varepsilon)$  and  $G(\widetilde{W}) = \widetilde{V} \times (-\delta, \delta)$ . Given  $\theta : V_1 \rightarrow X$ , define  $\widehat{\theta} : V_1 \times (-\varepsilon, \varepsilon) \rightarrow X$  via  $\widehat{\theta}(x, y) = \theta(x)$  and similarly for  $V_2$ . Now suppose  $f \in C^j(M_1; M_2)$ , then  $\theta := f \circ \varphi$  is  $C^j(V; \mathbb{R}^{n_2})$ . Let  $\widehat{\theta} \in C^j(F(W); \mathbb{R}^{n_2})$  and then  $\widehat{f} = \widehat{\theta} \circ F$ . Then  $\widehat{f}|_{W \cap M} = \widehat{\theta} \circ \varphi^{-1} = f$ . Also,  $\widehat{f}$  is  $C^j$ .