Practice problems A

Problem 1

Suppose $\omega : [0, \infty) \to [0, \infty]$ any function such that $\omega(x) = 0$ if and only if $x = 0$, ω continuous at 0, and ω is nondecreasing. For $f : X \to Z$ define

$$[f]_\omega = \sup \left\{ \frac{d(f(x), f(y))}{\omega(d(x,y))} : x, y \in X, x \neq y \right\}$$

and the space

$$C^{0,\omega}(X; Z) = \{ f : X \to Z \mid [f]_\omega < \infty \}.$$

1. Prove that $C^{0,\omega}(X; Z) \subset C^0(X; Z)$.
2. Suppose Z Banach. Show that $\|f\|_{C^{0,\omega}} = \|f\|_{C^0} + [f]_\omega$ is a norm on $C_b^{0,\omega}(X; Z) = C_b^0(X; Z) \cap C^{0,\omega}(X; Z)$, and that $C_b^{0,\omega}(X; Z)$ is complete with respect to this norm.
3. Suppose that X is compact and $d \in \mathbb{N}$, show that $B_{C^{0,\omega}(X;\mathbb{R}^d)}[0, 1] \subset C^0(X; \mathbb{R}^d)$ is compact.
4. Suppose X compact and infinte, and $d \in \mathbb{N}$. Show that $B_{C^{0,\omega}}[0, 1] \subset C^{0,\omega}$ is not compact. Conclude that $\mathrm{id} : (C^{0,\omega}, \|\cdot\|_{C^0}) \to (C^{0,\omega}, \|\cdot\|_{C^{0,\omega}})$ is not continuous. Also conclude that $(C^{0,\omega}, \|\cdot\|_{C^0})$ is not complete.
5. Another way to see this last fact is to first prove $C^{0,\omega}(X; \mathbb{R}^d)$ is a strict subset of $C^0(X; \mathbb{R}^d)$. It is helpful to study the sets $E_n = \left\{ f \in C^0(X; \mathbb{R}^d) : [f]_\omega \leq n \right\}$. Show that $C^{0,\omega}(X; \mathbb{R})$ is dense in $C^0(X; \mathbb{R})$. Use this to show that $C^{0,\omega}(X; \mathbb{R}^d)$ is dense in $C^0(X; \mathbb{R})$, and conclude $(C^{0,\omega}(X; \mathbb{R}^d), \|\cdot\|_{C^0})$ is not complete.

Proof.

1. Let $x \in X$ and $\varepsilon > 0$. It follws that for any $x \neq y$ we have

$$d(f(x), f(y)) \leq [f]_\omega \omega(d(x, y)).$$

Since $\omega(0) = 0$, ω continuous at 0, and ω is nondecreasing, we can find $\delta > 0$ such that $0 \leq t < \delta$ implies $0 \leq \omega(t) < \varepsilon$. Therefore, $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon [f]_\omega$. Since $[f]_\omega < \infty$, f is continuous and $C^{0,\omega}(X; Z) \subset C^0(X; Z)$.

2. It is easy to show that $\|\cdot\|_{C^{0,\omega}}$ is indeed a norm on $C_b^{0,\omega}(X; Z)$. Now we show that $C_b^{0,\omega}(X; Z)$ is complete with respect to this norm. Suppose $\{f_n\} \subset C_b^{0,\omega}$ Cauchy. Then it is also Cauchy in C_b^0. Therefore there is $f \in C_b^0$ such that $f_n \to f$ under C^0 norm. Remain to show $[f - f_n]_\omega \to 0$. Let $x, y \in X$ and $x \neq y$ and $m, n \geq N$ implies $[f_m - f_n]_\omega < \varepsilon$. Then,

$$\frac{\|f_m(x) - f_m(y) - f_n(x) + f_n(y)\|_Z}{\omega(d(x, y))} < \varepsilon.$$

Take $m \to \infty$ and take supremum of all $x, y \in X$ with $x \neq y$ completes the proof.

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