Mathematical Studies Analysis

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Contents

1 Advanced topics in metric space theory

1.1 Baire category

Definition. Let X be a metric space.

1. We say that $E \subset X$ is nowhere dense if $(\overline{E})^\circ = \emptyset$.
2. We say that $E \subset X$ is meager in X if
$$E = \bigcup_{\alpha \in A} E_\alpha,$$
where A is a countable set and $E_\alpha \subset X$ is nowhere dense for every $\alpha \in A$.

Theorem. Prove that the following are equivalent for $E \subset X$:

1. E is nowhere dense
2. $\overline{E}$ is nowhere dense
3. $(\overline{E})^c$ is open and dense in X.

Proof. (1) $\implies$ (2). Suppose E is nowhere dense, then $(\overline{E})^\circ = \emptyset$. Note that the closure of $\overline{E}$ is just $\overline{E}$ itself. It follows that $\overline{E}$ is also nowhere dense.

(2) $\implies$ (3). Suppose $\overline{E}$ is nowhere dense. Note that $\overline{E}$ is closed, so $(\overline{E})^c$ is open. Let $x \in X$ be arbitrary. Since $\overline{E}$ is nowhere dense, $x \notin (\overline{E})^\circ$. This implies that for arbitrary $\varepsilon > 0$, we have $B(x, \varepsilon) \not\subset \overline{E}$. This is equivalent to $B(x, \varepsilon) \cap (\overline{E})^c \neq \emptyset$. Hence, $(\overline{E})^c$ is dense in X.

(3) $\implies$ (1). Suppose $(\overline{E})^c$ is dense in X. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. It follows that $B(x, \varepsilon) \cap (\overline{E})^c \neq \emptyset$. This is equivalent to $B(x, \varepsilon) \not\subset \overline{E}$. Therefore, $(\overline{E})^\circ = \emptyset$ and E is nowhere dense. $\square$

Theorem (Baire category thorem)**.** Let X be a complete metric space. Suppose that for each $n \in \mathbb{N}$, $U_n \subset X$ is open and dense in X. Prove that $\bigcap_{n=0}^{\infty} U_n$ is dense in X. Hint: use the shrinking closed set property.

Proof. Consider any $x \in X$ and arbitrary $\varepsilon > 0$, it suffices to show that $U_n \cap B(x, \varepsilon) \neq \emptyset$ for each $n \in \mathbb{N}$. Now inductively choosing a sequence $x_i \in X$ and $\varepsilon_i > 0$ such that for each $i \in \mathbb{N}$, $B[x_i, \varepsilon_i] \subset U_i$, $B[x_{i+1}, \varepsilon_i] \subset B[x_i, \varepsilon_i] \subset B(x, \varepsilon)$, and $\varepsilon_i < 2^{-i}\varepsilon$.

Since U_0 is dense in X, $B(x, \varepsilon) \cap U_0 \neq \emptyset$. Note that both U_0 and $B(x, \varepsilon)$ are open, so we can choose $x_0 \in B(x, \varepsilon) \cap U_0$ and $\varepsilon_0 > 0$ so small that $B[x_0, \varepsilon_0] \subset B(x, \varepsilon) \cap U_0$ and $\varepsilon_0 < \varepsilon$. Now suppose for $0 \leq i \leq n$, we have chosen $x_i \in X$ and $\varepsilon_i > 0$ such that $B[x_i, \varepsilon_i] \subset U_i$ and $\varepsilon_i < 2^{-i}\varepsilon$ for all $0 \leq i \leq n$, $B[x_{i+1}, \varepsilon_{i+1}] \subset B[x_i, \varepsilon_i]$ for all $0 \leq i < n$. Since U_{n+1} is dense in X, $B(x_n, \varepsilon_n) \cap U_{n+1} \neq \emptyset$. Note also both U_{n+1} and $B(x_n, \varepsilon_n)$ are open. Therefore, choose $x_{n+1} \in B(x_n, \varepsilon_n) \cap U_{n+1}$ and $\varepsilon_{n+1} > 0$ so small that $B[x_{n+1}, \varepsilon_{n+1}] \subset B(x_n, \varepsilon_n) \cap U_{n+1}$ and $\varepsilon_{n+1} < \frac{\varepsilon_n}{2}$. It follows that $B[x_{n+1}, \varepsilon_{n+1}] \subset U_{n+1}$ and $B[x_{n+1}, \varepsilon_{n+1}] \subset B[x_n, \varepsilon_n] \subset B(x, \varepsilon)$. Also, $\varepsilon < \frac{\varepsilon_n}{2} < 2^{-n-1}\varepsilon$. Now we have successfully constructing the desired sequence.

Since X is complete, $\bigcap_{n=0}^{\infty} B[x_n, \varepsilon_n] = \{z\}$ for some $z \in X$. Note that for each n, we have $z \in B[x_n, \varepsilon_n] \subset U_n$. Also, $z \in B[x_n, \varepsilon_n] \subset B(x, \varepsilon)$. Therefore, $z \in U_n \cap B(x, \varepsilon)$ for each $n \in \mathbb{N}$ and $\bigcap_{n=0}^{\infty} U_n$ is dense in X. $\square$

Remark. An equivalent statement of the theorem is the following:
Let X be a complete metric space and $\{C_n\}$ a countable collection of closed subsets of X such that $X = \bigcup_{n \in \mathbb{N}} C_n$. Then at least one of the C_n contains an open ball.

1.2 Open mapping theorem

Linear surjections

Theorem (Open mapping theorem)**.** Let X, Y be Banach spaces over a common field and assume that $T \in \mathcal{L}(X;Y)$. Prove that the following are equivalent.

1. T is surjective.
2. There exists $\delta > 0$ such that $B_Y(0,\delta) \subset \overline{T(B_X(0,1))}$.
3. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $B_Y(0,\delta) \subset T(B_X(0,\varepsilon))$.
4. T is an open map: if $U \subset X$ is open, then $T(U) \subset Y$ is open.
5. There exists $C \geq 0$ such that for each $y \in Y$ there exists $x \in X$ such that $Tx = y$ and

$$\|x\|_X \leq C \|y\|_Y .$$

HINT: Prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (1), keeping in mind the following suggestions.

1. For (1) $\implies$ (2): Study the sets $C_n = \overline{T(B_X(0,n))} \subset Y$ for $n \geq 1$.
2. For (2) $\implies$ (3): Prove that $\overline{T(B_X(0,1))} \subset T(B_X(0,3))$ by considering $y \in \overline{T(B_X(0,1))}$ and inductively constructing $\{x_j\}_{j=0}^{\infty} \subset X$ such that $\|x_j\|_X < 2^{-j}$ and $y - \sum_{j=0}^{m} Tx_j \in B_Y(0, 2^{-m-1}R)$ for all $m \in \mathbb{N}$.

Proof. (1) $\implies$ (2). Following the hint, for $n \geq 1$ let $C_n = \overline{T(B_X(0,n))}$. Then each of the C_n are closed. Since T is surjective, $Y = \bigcup_{n=1}^{\infty} C_n$. Suppose for contradiction that each C_n are nowhere dense. It then follows that C_n^c are dense in Y. By Baire Category Theorem, $\bigcap_{n=1}^{\infty} C_n^c$ is dense in Y. However, $\bigcap_{n=1}^{\infty} C_n^c = \left(\bigcup_{n=1}^{\infty} C_n\right)^c = \emptyset$, a contradiction. Therefore, at least one C_n is not nowhere dense. That is, there exists some $n \geq 1$, $\overline{T(B_X(0,n))}$ contains an open ball. However, this is the same set as $n\overline{T(B_X(0,1))}$. Therefore, $\overline{T(B_X(0,1))}$ contains an open ball $B_Y(y_0, 4r)$ for some $y_0 \in Y$ and $r > 0$.

Let $y_1 = Tx_1$ for some $x_1 \in B_Y(0,1)$ such that $\|y_0 - y_1\| < 2r$. It follows that $B_Y(y_1, 2r) \subset B_Y(y_0, 4r) \subset \overline{T(B_X(0,1))}$. For any $y \in Y$ such that $\|y\| < r$, we have

$$y = -\frac{1}{2}y_1 + \frac{1}{2}(2y + y_1) = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1).$$

However, notice that

$$\frac{1}{2}(2y + y_1) \subset \frac{1}{2}B_Y(y_1, 2r) \subset \frac{1}{2}\overline{T(B_X(0,1))} = \overline{T(B_X(0,\tfrac{1}{2}))}.$$

It follows that

$$y = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1) \in -T\left(\frac{x_1}{2}\right) + \overline{T(B_X(0,\tfrac{1}{2}))}.$$

Note that $-T(\frac{x_1}{2}) \in T(B_X(0,\frac{1}{2}))$. Therefore, $y \in \overline{T(B_X(0,1))}$. Since y is arbitrary with $\|y\| < r$, we have $B_Y(0,r) \subset \overline{T(B_X(0,1))}$.

(2) $\implies$ (3). Following the hint, we first show $\overline{T(B_X(0,1))} \subset T(B_X(0,3))$. By assumption, we have $B_Y(0,R) \subset \overline{T(B_X(0,1))}$ for some $R > 0$. It follows from homogeneity that for each $m \in \mathbb{N}$, we have

$$2^{-m}B_Y(0,R) = B_Y(0, 2^{-m}R) \subset 2^{-m}\overline{T(B_X(0,1))} = \overline{T(B_X(0,2^{-m}))}.$$

Let $y \in \overline{T(B_X(0,1))}$ and pick $x_0 \in X$ with $\|x\| < 1$ such that $\|y - Tx\| < 2^{-1}R$. Now suppose we have chosen x_j for $0 \leq j \leq m$ such that $\|x_j\| < 2^{-j}$ and $y - \sum_{j=0}^{m} Tx_j \in B_Y(0, 2^{-m-1}R)$ for all $m \in \mathbb{N}$. By the inclusion above, we can pick $x_{m+1} \in X$ with $\|x_{m+1}\| < 2^{-m-1}$ such that

$$\left\| y - \sum_{j=0}^{m} Tx_j - Tx_{m+1} \right\| = \left\| y - \sum_{j=0}^{m+1} Tx_j \right\| < 2^{-m-2}R.$$

Therefore, $y - \sum_{j=0}^{m+1} Tx_j \in B_Y(0, 2^{-m-2})R$. This completes the inductive construction, and we have found a sequence $\{x_j\}$ such that $\|x_j\| < 2^{-j}$ and $y - \sum_{j=0}^{m} Tx_j \in B_Y(0, 2^{-m-1}R)$ for each $m \in \mathbb{N}$. Note that

$$\sum_{j=0}^{\infty} \|x_j\| \leq \sum_{j=0}^{\infty} 2^{-j} = 2,$$

so $\sum_{j=0}^{\infty} x_j$ converges absolutely. Since X is Banach, $\sum_{j=0}^{\infty} x_j$ converges to some $x \in X$ with $\|x\| \leq 2$. Also, since $y - \sum_{j=0}^{m} Tx_j \in B_Y(0, 2^{-m-1}R)$, taking the limit where m approaches infinity we obtain

$$y = \sum_{j=0}^{\infty} Tx_j = T\left(\sum_{j=0}^{\infty} x_j\right) = Tx.$$

Therefore, $y \in T(B_X(0,3))$ and thus $\overline{T(B_X(0,1))} \subset T(B_X(0,3))$.

Now for every $\varepsilon > 0$, we have $\frac{\varepsilon}{3}\overline{T(B_X(0,1))} \subset \frac{\varepsilon}{3}T(B_X(0,3)) = T(B_X(0,\varepsilon))$. By assumption, there exists $\delta > 0$ such that $B_Y(0,\delta) \subset \overline{T(B_X(0,1))}$. Therefore,

$$B_Y\left(0, \frac{\delta\varepsilon}{3}\right) = \frac{\varepsilon}{3}B_Y(0,\delta) \subset \frac{\varepsilon}{3}\overline{T(B_X(0,1))} \subset T(B_X(0,\varepsilon)).$$

(3) $\implies$ (4). Let $U \subset X$ be open and $y \in T(U)$. There exists $x \in U$ such that $Tx = y$. Since U is open, there exists $\varepsilon > 0$ such that $B_X(x,\varepsilon) \subset U$. By assumption, there exists $\delta > 0$ such that $B_Y(0,\delta) \subset T(B_X(0,\varepsilon))$. It follows that

$$B_Y(y,\delta) = y + B_Y(0,\delta) \subset Tx + T(B_X(0,\varepsilon)) = T(x + B_X(0,\varepsilon)) \subset T(U).$$

Therefore, $T(U)$ is open and T is an open map.

(4) $\implies$ (5). Since T is an open map, $T(B_X(0,1))$ is open. Also, $T(0) = 0$ so there exists $r > 0$ such that $B_Y(0,r) \subset T(B_X(0,1))$. Now let $y \in Y$. Then, $\frac{r}{2\|y\|}y \in B_Y(0,r)$ and there exists $x \in B_X(0,1)$ such that $Tx = \frac{r}{2\|y\|}y$. It follows that

$$T\left(\frac{2\|y\|}{r}x\right) = y,$$

and since $x \in B_X(0,1)$,

$$\left\|\frac{2\|y\|}{r}x\right\| = \frac{2\|y\|\|x\|}{r} < \frac{2}{r}\|y\|.$$

Letting $C = \frac{2}{r}$ completes the proof.

(5) $\implies$ (1). Since for each $y \in Y$ there exists $x \in X$ such that $Tx = y$, T is surjective. $\square$

Linear homeomorphisms, norm equivalence, and closed graphs

Theorem. Let X and Y be Banach spaces and suppose that $T \in \mathcal{L}(X,Y)$ is a bijection. Prove that $T^{-1} \in \mathcal{L}(Y,X)$, and in particular T is a linear (and thus bi-Lipschitz) homeomorphism.

Proof. Since $T \in \mathcal{L}(X,Y)$ is a bijection, T is a surjection. It follows that T is an open map. In particular, for any $U \subset X$ open, $T(U) = (T^{-1})^{-1}(U)$ is open. Therfore, T^{-1} is continuous and thus T is a linear homeomorphism. $\square$

Theorem. Let X be a vector space that is complete when equipped with both of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that if there exists a constant $C_1 > 0$ such that $\|x\|_2 \leq C_1\|x\|_1$ for all $x \in X$, then there exists a constant $C_0 > 0$ such that $C_0\|x\|_1 \leq \|x\|_2 \leq C_1\|x\|_1$ for all $x \in X$.

Proof. Let $T : X_1 \to X_2$, where X_1 and X_2 are X equipped with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, be the identity map. Then for any $x \in X$ with $\|x\|_1 = 1$, we have

$$\|Tx\|_2 = \|x\|_2 \leq C_1 \|x\|_1 = C_1.$$

Therefore, $T \in \mathcal{L}(X_1, X_2)$. T is also surjective. Therefore, there exists a constant $C \geq 0$ such that each $\|x\|_1 \leq C \|x\|_2$. Hence, for each $x \in X$

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C_1 \|x\|_1 .$$

Letting $C_0 = \frac{1}{C}$ completes the proof. □

Theorem. Let X and Y be Banach spaces and let $T : X \to Y$ be linear (just the algebraic condition). Prove that the following are equivalent

1. T is continuous, i.e. $T \in \mathcal{L}(X; Y)$.
2. The graph of T, $\Gamma(T) = \{(x, Tx) : x \in X\} \subset X \times Y$, is closed in $X \times Y$, where $X \times Y$ is endowed with any of the usual p-norms.

Proof. (a) $\implies$ (b). Let $\{(x_n, Tx_n)\}$ be a convergent sequence in $\Gamma(T)$. Since X is Banach, $x_n \to x$ for some $x \in X$. Since $T \in \mathcal{L}(X; Y)$, we have

$$\lim_{n\to\infty} Tx_n = T\left(\lim_{n\to\infty} x_n\right) = Tx.$$

Therefore, $(x_n, Tx_n) \to (x, Tx) \in \Gamma(T)$, and thus $\Gamma(T)$ is closed.

(b) $\implies$ (a). Let $\pi_1 : \Gamma(T) \to X$ and $\pi_2 : \Gamma(T) \to Y$ by $\pi_1(x, Tx) = x$ and $\pi_2(x, Tx) = Tx$. Since $\Gamma(T)$ is a closed in Banach space Y, $\Gamma(T)$ is Banach space. It is clear that both π_1 and π_2 are bounded linear maps. Moreover, π_1 is a bijection. It follows that $S = \pi_1^{-1}$ is a bounded linear map. Therefore, $T = \pi_2 \circ S$ is a bounded linear map. □

Linear injections with closed range

Theorem. Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Prove the following are equivalent.

1. T is injective and range(T) is closed.
2. $T : X \to \text{range}(T)$ is a linear homeomorphism.
3. There exists $C \geq 0$ such that $\|x\|_X \leq C \|Tx\|_Y$ for all $x \in X$.

HINT: Prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

Proof. (1) $\implies$ (2). If T is injective and range(T) is closed, then $\Gamma(T) = \{(x, Tx) : x \in X\}$ is closed in $X \times Y$. Therefore, $T : X \to \text{range}(T)$ is a bounded linear map. Since T is injective, this map is actually bijective from X to range(T). Therefore, T is a linaer homeomorphism.

(2) $\implies$ (3). Since T is a bijective bounded linear map, from X to range(T). There exists a contant $C \geq 0$ such that for each $y \in \text{range}(T)$ there exists a unique $x \in X$ such that $Tx = y$ and $\|x\| \leq C \|y\| = C \|Tx\|$. Since T is a bijection, $\|x\| \leq C \|Tx\|$ for all $x \in X$.

(3) $\implies$ (1). Let $x \in X$ be such that $Tx = 0$. It follows that $\|x\| \leq C \|Tx\| = 0$. Therefore, $x = 0$ and T is injective. To show that range(T) is closed, consider a convergent sequence $\{y_n\} \subset \text{range}(T)$ with $y_n = Tx_n$. Since for any $n, m \in \mathbb{N}$ we have

$$\|x_n - x_m\| \leq C \|T(x_n - x_m)\| = C \|y_n - y_m\| ,$$

$\{x_n\}$ is Cauchy. Since X is Banach, $x_n \to x$ for some $x \in X$. Therefore, for all $n \in \mathbb{N}$ we have

$$\|y_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \|x_n - x\| ,$$

and $y_n \to Tx$. Hence, range(T) is closed and the proof is complete. □

Theorem. Let X and Y be Banach spaces over a common field. Then, the following subsets of $\mathcal{L}(X;Y)$ are open:

1. $\{T \in \mathcal{L}(X;Y) : T \text{ is surjective}\}$,
2. $\{T \in \mathcal{L}(X;Y) : T \text{ is injective with closed range}\}$,
3. $\mathcal{H}(X;Y) = \{T \in \mathcal{L}(X;Y) : T \text{ is a homeomorphism}\}$.

Proof. 1. Let $T \in \mathcal{L}(X;Y)$ be surjective. By open mapping theorem, there is $\delta > 0$ such that $B_Y(0,\delta) \subset TB_X(0,1)$. By homogeneity we have $B_Y(0,r) \subset TB_X(0,\alpha r)$ for all $r > 0$ where $\alpha = \delta^{-1}$. Now let $S \in \mathcal{L}(X;Y)$ be such that $\|T - S\| < \beta < (2\alpha)^{-1}$. Claim S is surjective.

   Let $y \in Y$, inductively construct sequences $\{x_n\}$ and $\{y_n\}$. First let $y_0 = y$. Then, $\|y_0\| \in B(0, 2\|y_0\|)$. Select $x_0 \in X$ be such that $Tx_0 = y_0$ and $\|x_0\| \leq 2\alpha \|y_0\|$. Suppose we have selected y_i, x_i for $0 \leq i \leq n$. Set $y_{n+1} = y_n - Sx_n$ and select x_{n+1} be such that $Tx_{n+1} = y_{n+1}$ and $\|x_{n+1}\| \leq 2\alpha \|y_{n+1}\|$. Then, we have

   $$\|y_{n+1}\| = \|Tx_n - Sx_n\| \leq \|T - S\| \|x_n\| < 2\alpha\beta \|y_n\|$$

   and

   $$\|x_{n+1}\| = 2\alpha \|y_{n+1}\| \leq 2\alpha \|T - S\| \|x_n\| < 2\alpha\beta \|x_n\|.$$

   Note that $2\alpha\beta < 1$ and X is Banach, define

   $$x = \sum_{n=0}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=0}^{N} x_n.$$

   Also note that $\lim_{n\to\infty} y_n = 0$. It follows that

   $$Sx = \sum_{n=0}^{\infty} Sx_n = \sum_{n=0}^{\infty} (y_n - y_{n+1}) = y_0 - \lim_{n\to\infty} y_{n+1} = y.$$

   Therefore S is surjective and the set of surjective bounded linear maps are open.

2. Suppose $T \in \mathcal{L}(X;Y)$ is injective with closed range. Then, closed range theorem gives $C > 0$ such that $\|x\| \leq C\|Tx\|$ for all $x \in X$. Now supose $S \in \mathcal{L}(X;Y)$ is such that $\|T - S\| < (2C)^{-1}$. Claim that S is also injective with closed range. Indeed,

   $$\begin{aligned} \|x\| &\leq C\|Tx\| \leq C\|Sx\| + C\|(T-S)x\| \\ &\leq C\|Sx\| + \frac{1}{2}\|x\|. \end{aligned}$$

   This shows that $\|x\| \leq 2C\|Sx\|$ for all $x \in X$. By closed range theorem, S is injective with closed range. This implies that the set of injective bounded linear operator with closed range is open.

3. This directly follows from

   $$\mathcal{H}(X;Y) = \{T \in \mathcal{L}(X;Y) : T \text{ is surjective}\} \cap \{T \in \mathcal{L}(X;Y) : T \text{ is injective with closed range}\}.$$

□

Theorem. Let X and Y be Banach spaces over a common field. Then the following holds.

1. The sets

   $$\mathcal{L}_R(X;Y) = \{T \in \mathcal{L}(X;Y) : \text{there exists } S \in \mathcal{L}(Y;X) \text{ such that } ST = I_X\}$$

   and

   $$\mathcal{L}_L(X;Y) = \{T \in \mathcal{L}(X;Y) : \text{there exists } S \in \mathcal{L}(Y;X) \text{ such that } TS = I_Y\}$$

   are open.

2. The following inclusion holds:

$$\mathcal{L}_L(X;Y) \subset \{T \in \mathcal{L}(X;Y) : T \text{ is surjective}\}$$

and

$$\mathcal{L}_R(X;Y) \subset \{T \in \mathcal{L}(X;Y) : T \text{ is injective with closed range}\}.$$

3. The sets $\mathcal{L}_L(X;Y) \setminus \mathcal{L}_R(X;Y)$ and $\mathcal{L}_R(X;Y) \setminus \mathcal{L}_L(X;Y)$ are open.

Proof.
1. Let $T_0 \in \mathcal{L}_R$ and $S_0 \in \mathcal{L}(Y;X)$ be such that $T_0S_0 = I_Y$. Note that $I_X \in \mathcal{H}(X)$ and when $\|P\| < 1$ for $P \in \mathcal{L}(X)$, we have $I_X + P \in \mathcal{H}(X)$. Suppose now $T \in \mathcal{L}(X;Y)$ and $\|T\| < \|S_0\|^{-1}$. It follows that $I_X + S_0T \in \mathcal{H}(X)$. For such T, we then have

$$T_0 + T = T_0(I_X + S_0T).$$

Also,

$$(T_0 + T)(I_X + S_0T)^{-1}S_0 = T_0(I_X + S_0T)(I_X + S_0T)^{-1}S_0 = T_0S_0 = I_Y.$$

Therefore, $T_0 + T \in \mathcal{L}_R$ for $T \in B(T_0, \|S_0\|^{-1})$ and $\mathcal{L}_R$ is open.

Now let $T_0 \in \mathcal{L}_L$ and $S_0 \in \mathcal{L}(Y;X)$ be such that $S_0T_0 = I_X$. Again, for $T \in \mathcal{L}(X;Y)$ with $\|T\| < \|S_0\|^{-1}$, we have

$$T_0 + T = (I_X + TS_0)T_0.$$

and

$$S_0(I_X + TS_0)^{-1}(T_0 + T) = I_X.$$

Therefore, $\mathcal{L}_R$ is also open.

2. Let $T \in \mathcal{L}_R$ and $S \in \mathcal{L}(Y;X)$ be such that $TS = I_Y$. Then for any $y \in Y$ let $x = Sy$. It follows that $Tx = TSy = y$. Also, $\|x\| \leq \|S\| \|y\|$ so the 4th item in open mapping theorem guarantees that T is surjective. Hence, $\mathcal{L}_L \subset \{T \in \mathcal{L}(X;Y) : T \text{ is surjective}\}$.

Now let $T \in \mathcal{L}_L$ and $S \in \mathcal{L}(Y;X)$ such that $ST = I_X$. Now for any $x \in X$, we have $\|x\| = \|STx\| \leq \|S\| \|Tx\|$. Then the closed range theorem guarantees that T is injective with closed range. Hence, $\mathcal{L}_R \subset \{T \in \mathcal{L}_R(X;Y) : T \text{ is injective with closed range}\}$.

3. ***TO-DO***

□

1.3 Hahn-Banach theorem and duality

Theorem (Hahn-Banach theorem in $\mathbb{R}$)**.** Let X be a real vector space and suppose $p : X \to \mathbb{R}$ is such that

$$p(tx + (1-t)y) \leq tp(x) + (1-t)p(y)$$

for all $t \in [0,1]$ and $x, y \in X$.

Suppose Y subspace of X and $l : Y \to \mathbb{R}$ is a linear map such that $l \leq p$ on Y. Then there exists linear map $L : X \to \mathbb{R}$ such that $L \leq p$ on X and $L = l$ on Y.

Proof. *** TO-DO ***

□

Theorem (Hahn-Banach theorem in $\mathbb{C}$)**.** Let X be complex vector space and suppose $p : X \to \mathbb{R}$ is such that

$$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y)$$

for all $\alpha, \beta \in \mathbb{C}$ such that $|\alpha| + |\beta| = 1$ and $x, y \in X$.

Suppose Y subspace of X and $l : Y \to \mathbb{C}$ is a linear map such that $|l| \leq p$ on Y. Then there exsits linear map $L : X \to \mathbb{C}$ such that $|L| \leq p$ on X and $L = l$ on Y.

Proof. *** TO-DO ***

□

Theorem (Hahn-Banach theorem for bounded linear functionals)**.** Let X be a normed vector space over $\mathbb{F}$ and Y a subspace of X. If $\lambda \in Y^*$ then there exists $\Lambda \in X^*$ such that $\Lambda = \lambda$ on Y and the operator norm $\|\lambda\|_{Y^*} = \|\Lambda\|_{X^*}$.

Proof. *** TO-DO ***

□

Next we show some useful implications of Hahn-Banach.

Theorem. Let X be a normed vector space and fix $x \in X$. Then the following holds:

1. There exists $\lambda \in X^*$ such that $\|\lambda\| = \|x\|$ and
$$\lambda(x) = \|\lambda\| \|x\| = \|x\|^2 .$$
2. We have
$$\|x\| = \max_{\substack{w \in X^* \\ \|w\|=1}} |w(x)| .$$
3. $x = 0$ if and only if $w(x) = 0$ for all $w \in X^*$.

Proof. *** TO-DO ***

□

2 Measure and integration

2.1 Constructing outer measures

Lemma. Let X be a set with gauge $(\mathcal{E}, \gamma)$ that covers X. Let $A \subset X$, then the following holds:

1. Let μ^* be the outer measure generated by $(\mathcal{E}, \gamma)$. Then there exists collection $\{E_{m,n}\}_{m,n=0}^{\infty} \subset \mathcal{E}$ such that $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$ such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$.
2. Suppose (X, d) is metric space and the gauge is fine. Let μ_d^* be the metric outer measure. Then there exists collection $\{E_{m,n}\}_{m,n=0}^{\infty} \subset \mathcal{E}$ such that $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$ such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$.

Proof. We only prove (2) since (1) is similar. Since the gauge is fine, $(\mathcal{E}_\delta, \gamma_\delta)$ covers X for all $\delta > 0$. Then, for any $m \in \mathbb{N}$, there exists $\{E_{m,n}\}_n \subset \mathcal{E}_{2^{-m}}$ such that $A \subset \bigcup_{n=0}^{\infty} E_{m,n}$ and $\sum_{n=0}^{\infty} \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}$. Now let $E = \bigcap_{m=0}^{\infty} \bigcup_{n=0}^{\infty} E_{m,n}$. Note that $A \subset E$ and for any $m \in \mathbb{N}$, we have

$$\mu_{2^{-m}}^*(E) \leq \mu_{2^{-m}}^* \left(\bigcup_{n=0}^{\infty} E_{m,n} \right) \leq \sum_{n=0}^{\infty} \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}.$$

Taking the limit as $m \to \infty$, we have

$$\mu_d^*(E) \leq \mu_d^*(A) \leq \mu_d^*(E),$$

as desired. □

Theorem. Let (X, d) be metric space with $(\mathcal{E}, \gamma)$ such that all sets in $\mathcal{E}$ are open. Assume that μ^* is a metric outer measure on X such that either

1. μ^* is generated by $(\mathcal{E}, \gamma)$, or
2. $\mu^* = \mu_d^*$ is generated by $(\mathcal{E}_\delta, \gamma_\delta)$.

Further suppose that $X = \bigcup_{n=0}^{\infty} A_n$ where $A_n \subset X$ is such that $\mu^*(A_n) < \infty$. Then the following holds:

1. The gauge covers X in case 1 and is fine in case 2.
2. In both cases, μ^* is cover-regular. More precisely, for each $A \subset X$, there is $G \in G_\delta(X) \subset \mathcal{B}(X) \subset \mathfrak{M}$ such that $A \subset G$ and $\mu^*(A) = \mu^*(G)$.
3. In both cases, the following are equivalent for $E \subset X$:
   (a) $E \in \mathfrak{M}$, i.e. E is measurable.
   (b) there exists $G \in G_\delta(X)$ such that $E \subset G$ and $\mu^*(G \setminus E) = 0$.
   (c) there exists $F \in F_\sigma(X)$ such that $F \subset E$ and $\mu^*(E \setminus F) = 0$.

Proof. **Step 0: proof for (1) and (2).**

We know $X = \bigcup_{n=0}^{\infty} A_n$ for some $\mu^*(A_n) < \infty$. For case (1), we can pick $\{E_{n,m}\} \subset \mathcal{E}$ such that $A_n \subset \bigcup_{m=0}^{\infty} E_{n,m}$. Then $X = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n,m} E_{n,m}$. Therefore, $\mathcal{E}$ covers X. For case (2), note that $\mu^*(A_n) < \infty$ and $\mu_d^*(A_n) = \sup_{\delta > 0} \mu_\delta^*(A_n)$ for each $\delta > 0$ and $n \in \mathbb{N}$. Then for each $\delta > 0$, there exists $\{E_{n,m}\} \subset \mathcal{E}_\delta$ such that $A_n \subset \bigcup_{m=0}^{\infty} E_{n,m}$. Then, $X = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n,m} E_{n,m}$. Therefore, $(\mathcal{E}, \gamma)$ is fine.

We have the following observations:

1. μ^* is a metric outer measure. This miplies that $\mathcal{B}(X) \subset \mathfrak{M}$.
2. $G_\delta(X) \cup F_\sigma(X) \subset \mathcal{B}(X) \subset \mathfrak{M}$ and $\mu^*(A) = 0$ implies $A \in \mathfrak{M}$.
3. By previous lemma and all sets in $\mathcal{E}$ are open, we know for each $A \subset X$ there is $E \in G_\delta(X)$ such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$. In particular, μ^* is cover regular.

Step 1: starting on (3).

For (b) $\implies$ (a), suppose (b) holds for $E \subset X$. Then $E = G \setminus (G \setminus E) \in \mathfrak{M}$ since $\mu^*(G \setminus E) = 0$.

For (c) $\implies$ (a), suppose (c) holds for $E \subset X$. Then $E = F \cup (E \setminus F) \in \mathfrak{M}$ since $\mu^*(E \setminus F) = 0$.

Next we show "(a) $\implies$ (c)" implies "(a) $\implies$ (b)". Suppose $E \in \mathfrak{M}$, then $E^c \in \mathfrak{M}$. By (a) $\implies$ (b) we know there exists $F \in F_\sigma$ such that $F \subset E^c$ and $\mu^*(E^c \setminus F) = 0$. Let $G = F^c \in G_\delta$ then $E \subset G$ and $G \subset E = E^c \subset F$.

Therefore, it remains to show (a) $\implies$ (c) to complete the proof for the theorem.

Step 2: reduction for (a) $\implies$ (c).

Claim it suffices to show it for E such that $\mu^*(E) < \infty$. Suppose we did this and $\mu^*(E) = \infty$. Using observation there exists $B_n \in \mathfrak{M}$ such that $A_n \subset B_n$ and $\mu^*(B_n) = \mu^*(A_n) < \infty$. Then $E_n = E \cap B_n \in \mathfrak{M}$ and $\mu^*(E_n) < \infty$. Then by special case there is $F_n \in F_\sigma(X)$ such that $F_n \subset E_n$ and $\mu^*(F_n \setminus E_n) = 0$. Let $F = \bigcup_{n=0}^\infty F_n \in F_\sigma$ then $F \subset \bigcup_{n=0}^\infty E_n = E$ and

$$\mu^*(E \setminus F) \leq \sum_{n=0}^\infty \mu^*(E_n \setminus F_n) = 0.$$

Step 3: further reduction.

Claim it suffices to show it for the case where $\mu^*(E) < \infty$ and $E \in G_\delta(X)$. Suppose we have proved this and consider $E \subset X$ such that $\mu^*(E) < \infty$. Observation 3 allows us to pick $G \in G_\delta(X)$ such that $E \subset G$ and $\mu^*(E) = \mu^*(G)$. Now pick $H \in G_\delta$ such that $G \setminus E \subset H$ and $\mu^*(H) = \mu^*(G \setminus E)$.

Now apply special case. This gives $F \in F_\sigma$ such that $F \subset G$ and $\mu^*(G \setminus F) = 0$. Let $K = F \setminus H = F \cap H^c \in F_\sigma$ and $K = F \cap H^c \subset G \cap (G \setminus E)^c \subset E$.

Note that $E, F, G, H, K \in \mathfrak{M}$, so

$$\begin{aligned}
\mu^*(E \setminus K) &= \mu^*(E) - \mu^*(K) \\
&= \mu^*(G) - \mu^*(F \setminus H) \\
&= \mu^*(G) - \mu^*(F) + \mu^*(F \cap H) \\
&\leq \mu^*(G) - \mu^*(F) + \mu^*(H) \\
&= \mu^*(G \setminus F) + \mu^*(H) \\
&= \mu^*(G \setminus E) \\
&= \mu^*(G) - \mu^*(E) \\
&= 0.
\end{aligned}$$

Therefore, K is the desired F_σ set.

Step 4: finishing (a) $\implies$ (c).

Suppose $E \in G_\delta(X)$ and $\mu^*(E) < \infty$. Write $E = \bigcup_{n=0}^\infty V_n$ where $V_n \subset X$ open. For $m, n \in \mathbb{N}$, let

$$C_{n,m} = \left\{x \in V_n : \operatorname{dist}(x, V_n^c) \geq 2^{-m}\right\} \subset V_n.$$

Note that $C_{n,m}$ is closed, $C_{n,m} \subset C_{n,m+1}$, $V_n = \bigcup_m C_{n,m}$. Since $E, C_{n,m}, V_n \in \mathfrak{M}$, we have

$$\mu^*(E) = \mu^*(E \cap V_n) = \lim_{m \to \infty} \mu^*(E \cap C_{n,m}).$$

Thus, there exists $M(n,k)$ such that $\mu^*(E \setminus C_{n,M(n,k)}) < 2^{-n-k}$. Now let $D_k = \bigcup_{n=0}^\infty C_{n,M(n,k)}$ closed. Also, $D_k \subset \bigcup_{n=0}^\infty V_n = E$ and

$$\mu^*(E) - \mu^*(D_k) = \mu^*(E \setminus D_k) \leq \sum_{n=0}^\infty \mu^*(E \setminus C_{n,M(n,k)}) \leq 2^{-k+1}.$$

Let $F = \bigcup_{k=0}^{\infty} D_k \subset E$ and note that $F \in F_\sigma$. Then

$$\mu^*(E \setminus F) = \mu^*(E) - \mu^*(F) \leq \mu^*(E) - \mu^*(D_k) < 2^{-k+1}$$

for all $k \in \mathbb{N}$. Therefore, $\mu^*(E \setminus F) = 0$.

□

Lemma. Suppose (X, d) metric space with metric outer measure μ^*. Suppose $X = \bigcup_{n=0}^{\infty} V_n$ for $V_n \subset X$ open and $\mu^*(V_n) < \infty$. Suppose $E \subset G \in G_\delta(X)$ such that $\mu^*(G \setminus E) = 0$. Then for each $\varepsilon > 0$, there exists open $U \subset X$ such that $E \subset U$ and $\mu^*(U \setminus E) < \varepsilon$.

Proof. Let $E_n = E \cap V_n$ and $G = G \cap V_n$. Write $G = \bigcap_{j=0}^{\infty} W_j$ where W_j open. Now set

$$Z_{n,m} = V_n \cap \bigcap_{j=0}^{m} W_j,$$

which are open for all $n, m \in \mathbb{N}$. Now notice that $G_n \subset Z_{n,m+1} \subset Z_{n,m} \subset V_n$. Note that $\mu^*(V_n) < \infty$, so $\mu^*(G_n) = \lim_{m\to\infty} \mu^*(Z_{n,m})$. Therefore, for all $\varepsilon > 0$, there exists $M(n)$ such that

$$\mu^*(Z_{n,M(n)} \setminus G_n) < \varepsilon 2^{-n-2}.$$

Then set $U = \bigcup_{n=0}^{\infty} Z_{n,M(n)} \supset \bigcup_{n=0}^{\infty} G_n = G \supset E$ open, then we have

$$\begin{aligned}
\mu^*(U \setminus E) &= \mu^*(U \setminus G) + \mu^*(G \setminus E) \\
&= \mu^*\left(\bigcup_{n=0}^{\infty} Z_{n,M(n)} \cap \bigcap_{n=0}^{\infty} G_n^c\right) \\
&\leq \sum_{n=0}^{\infty} \mu^*(Z_{n,M(n)} \setminus G_n) \\
&< \varepsilon,
\end{aligned}$$

as desired.

□