

# Mathematical Studies Analysis

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# 1 Advanced topics in metric space theory

## 1.1 Baire category

**Definition.** Let  $X$  be a metric space.

1. We say that  $E \subset X$  is nowhere dense if  $(\overline{E})^\circ = \emptyset$ .
2. We say that  $E \subset X$  is meager in  $X$  if

$$E = \bigcup_{\alpha \in A} E_\alpha,$$

where  $A$  is a countable set and  $E_\alpha \subset X$  is nowhere dense for every  $\alpha \in A$ .

**Theorem.** Prove that the following are equivalent for  $E \subset X$ :

1.  $E$  is nowhere dense
2.  $\overline{E}$  is nowhere dense
3.  $(\overline{E})^c$  is open and dense in  $X$ .

*Proof.* (1)  $\implies$  (2). Suppose  $E$  is nowhere dense, then  $(\overline{E})^\circ = \emptyset$ . Note that the closure of  $\overline{E}$  is just  $\overline{E}$  itself. It follows that  $\overline{E}$  is also nowhere dense.

(2)  $\implies$  (3). Suppose  $\overline{E}$  is nowhere dense. Note that  $\overline{E}$  is closed, so  $(\overline{E})^c$  is open. Let  $x \in X$  be arbitrary. Since  $\overline{E}$  is nowhere dense,  $x \notin (\overline{E})^\circ$ . This implies that for arbitrary  $\varepsilon > 0$ , we have  $B(x, \varepsilon) \not\subset \overline{E}$ . This is equivalent to  $B(x, \varepsilon) \cap (\overline{E})^c \neq \emptyset$ . Hence,  $(\overline{E})^c$  is dense in  $X$ .

(3)  $\implies$  (1). Suppose  $(\overline{E})^c$  is dense in  $X$ . Let  $x \in X$  and  $\varepsilon > 0$  be arbitrary. It follows that  $B(x, \varepsilon) \cap (\overline{E})^c \neq \emptyset$ . This is equivalent to  $B(x, \varepsilon) \not\subset \overline{E}$ . Therefore,  $(\overline{E})^\circ = \emptyset$  and  $E$  is nowhere dense.  $\square$

**Theorem** (Baire category theorem). Let  $X$  be a complete metric space. Suppose that for each  $n \in \mathbb{N}$ ,  $U_n \subset X$  is open and dense in  $X$ . Prove that  $\bigcap_{n=0}^{\infty} U_n$  is dense in  $X$ . Hint: use the shrinking closed set property.

*Proof.* Consider any  $x \in X$  and arbitrary  $\varepsilon > 0$ , it suffices to show that  $U_n \cap B(x, \varepsilon) \neq \emptyset$  for each  $n \in \mathbb{N}$ . Now inductively choosing a sequence  $x_i \in X$  and  $\varepsilon_i > 0$  such that for each  $i \in \mathbb{N}$ ,  $B[x_i, \varepsilon_i] \subset U_i$ ,  $B[x_{i+1}, \varepsilon_{i+1}] \subset B[x_i, \varepsilon_i] \subset B(x, \varepsilon)$ , and  $\varepsilon_i < 2^{-i}\varepsilon$ .

Since  $U_0$  is dense in  $X$ ,  $B(x, \varepsilon) \cap U_0 \neq \emptyset$ . Note that both  $U_0$  and  $B(x, \varepsilon)$  are open, so we can choose  $x_0 \in B(x, \varepsilon) \cap U_0$  and  $\varepsilon_0 > 0$  so small that  $B[x_0, \varepsilon_0] \subset B(x, \varepsilon) \cap U_0$  and  $\varepsilon_0 < \varepsilon$ . Now suppose for  $0 \leq i \leq n$ , we have chosen  $x_i \in X$  and  $\varepsilon_i > 0$  such that  $B[x_i, \varepsilon_i] \subset U_i$  and  $\varepsilon_i < 2^{-i}\varepsilon$  for all  $0 \leq i \leq n$ ,  $B[x_{i+1}, \varepsilon_{i+1}] \subset B[x_i, \varepsilon_i]$  for all  $0 \leq i < n$ . Since  $U_{n+1}$  is dense in  $X$ ,  $B(x_n, \varepsilon_n) \cap U_{n+1} \neq \emptyset$ . Note also both  $U_{n+1}$  and  $B(x_n, \varepsilon_n)$  are open. Therefore, choose  $x_{n+1} \in B(x_n, \varepsilon_n) \cap U_{n+1}$  and  $\varepsilon_{n+1} > 0$  so small that  $B[x_{n+1}, \varepsilon_{n+1}] \subset B(x_n, \varepsilon_n) \cap U_{n+1}$  and  $\varepsilon_{n+1} < \frac{\varepsilon_n}{2}$ . It follows that  $B[x_{n+1}, \varepsilon_{n+1}] \subset U_{n+1}$  and  $B[x_{n+1}, \varepsilon_{n+1}] \subset B[x_n, \varepsilon_n] \subset B(x, \varepsilon)$ . Also,  $\varepsilon < \frac{\varepsilon_n}{2} < 2^{-n-1}\varepsilon$ . Now we have successfully constructing the desired sequence.

Since  $X$  is complete,  $\bigcap_{n=0}^{\infty} B[x_n, \varepsilon_n] = \{z\}$  for some  $z \in X$ . Note that for each  $n$ , we have  $z \in B[x_n, \varepsilon_n] \subset U_n$ . Also,  $z \in B[x_n, \varepsilon_n] \subset B(x, \varepsilon)$ . Therefore,  $z \in U_n \cap B(x, \varepsilon)$  for each  $n \in \mathbb{N}$  and  $\bigcap_{n=0}^{\infty} U_n$  is dense in  $X$ .  $\square$

**Remark.** An equivalent statement of the theorem is the following:

Let  $X$  be a complete metric space and  $\{C_n\}$  a countable collection of closed subsets of  $X$  such that  $X = \bigcup_{n \in \mathbb{N}} C_n$ . Then at least one of the  $C_n$  contains an open ball.

## 1.2 Open mapping theorem

### Linear surjections

**Theorem** (Open mapping theorem). Let  $X, Y$  be Banach spaces over a common field and assume that  $T \in \mathcal{L}(X; Y)$ . Prove that the following are equivalent.

1.  $T$  is surjective.
2. There exists  $\delta > 0$  such that  $B_Y(0, \delta) \subset \overline{T(B_X(0, 1))}$ .
3. For every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $B_Y(0, \delta) \subset T(B_X(0, \varepsilon))$ .
4.  $T$  is an open map: if  $U \subset X$  is open, then  $T(U) \subset Y$  is open.
5. There exists  $C \geq 0$  such that for each  $y \in Y$  there exists  $x \in X$  such that  $Tx = y$  and

$$\|x\|_X \leq C \|y\|_Y.$$

HINT: Prove that (1)  $\implies$  (2)  $\implies$  (3)  $\implies$  (4)  $\implies$  (5)  $\implies$  (1), keeping in mind the following suggestions.

1. For (1)  $\implies$  (2): Study the sets  $C_n = \overline{T(B_X(0, n))} \subset Y$  for  $n \geq 1$ .
2. For (2)  $\implies$  (3): Prove that  $\overline{T(B_X(0, 1))} \subset T(B_X(0, 3))$  by considering  $y \in \overline{T(B_X(0, 1))}$  and inductively constructing  $\{x_j\}_{j=0}^\infty \subset X$  such that  $\|x_j\|_X < 2^{-j}$  and  $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$  for all  $m \in \mathbb{N}$ .

*Proof.* (1)  $\implies$  (2). Following the hint, for  $n \geq 1$  let  $C_n = \overline{T(B_X(0, n))}$ . Then each of the  $C_n$  are closed. Since  $T$  is surjective,  $Y = \bigcup_{n=1}^\infty C_n$ . Suppose for contradiction that each  $C_n$  are nowhere dense. It then follows that  $C_n^c$  are dense in  $Y$ . By Baire Category Theorem,  $\bigcap_{n=1}^\infty C_n^c$  is dense in  $Y$ . However,  $\bigcap_{n=1}^\infty C_n^c = (\bigcup_{n=1}^\infty C_n)^c = \emptyset$ , a contradiction. Therefore, at least one  $C_n$  is not nowhere dense. That is, there exists some  $n \geq 1$ ,  $\overline{T(B_X(0, n))}$  contains an open ball. However, this is the same set as  $n\overline{T(B_X(0, 1))}$ . Therefore,  $\overline{T(B_X(0, 1))}$  contains an open ball  $B_Y(y_0, 4r)$  for some  $y_0 \in Y$  and  $r > 0$ .

Let  $y_1 = Tx_1$  for some  $x_1 \in B_X(0, 1)$  such that  $\|y_0 - y_1\| < 2r$ . It follows that  $B_Y(y_1, 2r) \subset B_Y(y_0, 4r) \subset \overline{T(B_X(0, 1))}$ . For any  $y \in Y$  such that  $\|y\| < r$ , we have

$$y = -\frac{1}{2}y_1 + \frac{1}{2}(2y + y_1) = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1).$$

However, notice that

$$\frac{1}{2}(2y + y_1) \subset \frac{1}{2}B_Y(y_1, 2r) \subset \frac{1}{2}\overline{T(B_X(0, 1))} = \overline{T(B_X(0, \frac{1}{2}))}.$$

It follows that

$$y = -T\left(\frac{x_1}{2}\right) + \frac{1}{2}(2y + y_1) \in -T\left(\frac{x_1}{2}\right) + \overline{T(B_X(0, \frac{1}{2}))}.$$

Note that  $-T(\frac{x_1}{2}) \in T(B_X(0, \frac{1}{2}))$ . Therefore,  $y \in \overline{T(B_X(0, 1))}$ . Since  $y$  is arbitrary with  $\|y\| < r$ , we have  $B_Y(0, r) \subset \overline{T(B_X(0, 1))}$ .

(2)  $\implies$  (3). Following the hint, we first show  $\overline{T(B_X(0, 1))} \subset T(B_X(0, 3))$ . By assumption, we have  $B_Y(0, R) \subset \overline{T(B_X(0, 1))}$  for some  $R > 0$ . It follows from homogeneity that for each  $m \in \mathbb{N}$ , we have

$$2^{-m}B_Y(0, R) = B_Y(0, 2^{-m}R) \subset 2^{-m}\overline{T(B_X(0, 1))} = \overline{T(B_X(0, 2^{-m}))}.$$

Let  $y \in \overline{T(B_X(0, 1))}$  and pick  $x_0 \in X$  with  $\|x_0\| < 1$  such that  $\|y - Tx_0\| < 2^{-1}R$ . Now suppose we have chosen  $x_j$  for  $0 \leq j \leq m$  such that  $\|x_j\| < 2^{-j}$  and  $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$  for all  $m \in \mathbb{N}$ . By the inclusion above, we can pick  $x_{m+1} \in X$  with  $\|x_{m+1}\| < 2^{-m-1}$  such that

$$\left\|y - \sum_{j=0}^m Tx_j - Tx_{m+1}\right\| = \left\|y - \sum_{j=0}^{m+1} Tx_j\right\| < 2^{-m-2}R.$$

Therefore,  $y - \sum_{j=0}^{m+1} Tx_j \in B_Y(0, 2^{-m-2})R$ . This completes the inductive construction, and we have found a sequence  $\{x_j\}$  such that  $\|x_j\| < 2^{-j}$  and  $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$  for each  $m \in \mathbb{N}$ . Note that

$$\sum_{j=0}^{\infty} \|x_j\| \leq \sum_{j=0}^{\infty} 2^{-j} = 2,$$

so  $\sum_{j=0}^{\infty} x_j$  converges absolutely. Since  $X$  is Banach,  $\sum_{j=0}^{\infty} x_j$  converges to some  $x \in X$  with  $\|x\| \leq 2$ . Also, since  $y - \sum_{j=0}^m Tx_j \in B_Y(0, 2^{-m-1}R)$ , taking the limit where  $m$  approaches infinity we obtain

$$y = \sum_{j=0}^{\infty} Tx_j = T \left( \sum_{j=0}^{\infty} x_j \right) = Tx.$$

Therefore,  $y \in T(B_X(0, 3))$  and thus  $\overline{T(B_X(0, 1))} \subset T(B_X(0, 3))$ .

Now for every  $\varepsilon > 0$ , we have  $\frac{\varepsilon}{3} \overline{T(B_X(0, 1))} \subset \frac{\varepsilon}{3} T(B_X(0, 3)) = T(B_X(0, \varepsilon))$ . By assumption, there exists  $\delta > 0$  such that  $B_Y(0, \delta) \subset \overline{T(B_X(0, 1))}$ . Therefore,

$$B_Y \left( 0, \frac{\delta\varepsilon}{3} \right) = \frac{\varepsilon}{3} B_Y(0, \delta) \subset \frac{\varepsilon}{3} \overline{T(B_X(0, 1))} \subset T(B_X(0, \varepsilon)).$$

(3)  $\implies$  (4). Let  $U \subset X$  be open and  $y \in T(U)$ . There exists  $x \in U$  such that  $Tx = y$ . Since  $U$  is open, there exists  $\varepsilon > 0$  such that  $B_X(x, \varepsilon) \subset U$ . By assumption, there exists  $\delta > 0$  such that  $B_Y(0, \delta) \subset T(B_X(0, \varepsilon))$ . It follows that

$$B_Y(y, \delta) = y + B_Y(0, \delta) \subset Tx + T(B_X(0, \varepsilon)) = T(x + B_X(0, \varepsilon)) \subset T(U).$$

Therefore,  $T(U)$  is open and  $T$  is an open map.

(4)  $\implies$  (5). Since  $T$  is an open map,  $T(B_X(0, 1))$  is open. Also,  $T(0) = 0$  so there exists  $r > 0$  such that  $B_Y(0, r) \subset T(B_X(0, 1))$ . Now let  $y \in Y$ . Then,  $\frac{r}{2\|y\|}y \in B_Y(0, r)$  and there exists  $x \in B_X(0, 1)$  such that  $Tx = \frac{r}{2\|y\|}y$ . It follows that

$$T \left( \frac{2\|y\|}{r}x \right) = y,$$

and since  $x \in B_X(0, 1)$ ,

$$\left\| \frac{2\|y\|}{r}x \right\| = \frac{2\|y\|\|x\|}{r} < \frac{2}{r}\|y\|.$$

Letting  $C = \frac{2}{r}$  completes the proof.

(5)  $\implies$  (1). Since for each  $y \in Y$  there exists  $x \in X$  such that  $Tx = y$ ,  $T$  is surjective. □

### Linear homeomorphisms, norm equivalence, and closed graphs

**Theorem.** Let  $X$  and  $Y$  be Banach spaces and suppose that  $T \in \mathcal{L}(X, Y)$  is a bijection. Prove that  $T^{-1} \in \mathcal{L}(Y, X)$ , and in particular  $T$  is a linear (and thus bi-Lipschitz) homeomorphism.

*Proof.* Since  $T \in \mathcal{L}(X, Y)$  is a bijection,  $T$  is a surjection. It follows that  $T$  is an open map. In particular, for any  $U \subset X$  open,  $T(U) = (T^{-1})^{-1}(U)$  is open. Therefore,  $T^{-1}$  is continuous and thus  $T$  is a linear homeomorphism. □

**Theorem.** Let  $X$  be a vector space that is complete when equipped with both of the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ . Prove that if there exists a constant  $C_1 > 0$  such that  $\|x\|_2 \leq C_1 \|x\|_1$  for all  $x \in X$ , then there exists a constant  $C_0 > 0$  such that  $C_0 \|x\|_1 \leq \|x\|_2 \leq C_1 \|x\|_1$  for all  $x \in X$ .

*Proof.* Let  $T : X_1 \rightarrow X_2$ , where  $X_1$  and  $X_2$  are  $X$  equipped with norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ , respectively, be the identity map. Then for any  $x \in X$  with  $\|x\|_1 = 1$ , we have

$$\|Tx\|_2 = \|x\|_2 \leq C_1 \|x\|_1 = C_1.$$

Therefore,  $T \in \mathcal{L}(X_1, X_2)$ .  $T$  is also surjective. Therefore, there exists a constant  $C \geq 0$  such that each  $\|x\|_1 \leq C \|x\|_2$ . Hence, for each  $x \in X$

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C_1 \|x\|_1.$$

Letting  $C_0 = \frac{1}{C}$  completes the proof.  $\square$

**Theorem.** Let  $X$  and  $Y$  be Banach spaces and let  $T : X \rightarrow Y$  be linear (just the algebraic condition). Prove that the following are equivalent

1.  $T$  is continuous, i.e.  $T \in \mathcal{L}(X; Y)$ .
2. The graph of  $T$ ,  $\Gamma(T) = \{(x, Tx) : x \in X\} \subset X \times Y$ , is closed in  $X \times Y$ , where  $X \times Y$  is endowed with any of the usual  $p$ -norms.

*Proof.* (a)  $\implies$  (b). Let  $\{(x_n, Tx_n)\}$  be a convergent sequence in  $\Gamma(T)$ . Since  $X$  is Banach,  $x_n \rightarrow x$  for some  $x \in X$ . Since  $T \in \mathcal{L}(X; Y)$ , we have

$$\lim_{n \rightarrow \infty} Tx_n = T \left( \lim_{n \rightarrow \infty} x_n \right) = Tx.$$

Therefore,  $(x_n, Tx_n) \rightarrow (x, Tx) \in \Gamma(T)$ , and thus  $\Gamma(T)$  is closed.

(b)  $\implies$  (a). Let  $\pi_1 : \Gamma(T) \rightarrow X$  and  $\pi_2 : \Gamma(T) \rightarrow Y$  by  $\pi_1(x, Tx) = x$  and  $\pi_2(x, Tx) = Tx$ . Since  $\Gamma(T)$  is a closed in Banach space  $Y$ ,  $\Gamma(T)$  is Banach space. It is clear that both  $\pi_1$  and  $\pi_2$  are bounded linear maps. Moreover,  $\pi_1$  is a bijection. It follows that  $S = \pi_1^{-1}$  is a bounded linear map. Therefore,  $T = \pi_2 \circ S$  is a bounded linear map.  $\square$

### Linear injections with closed range

**Theorem.** Let  $X$  and  $Y$  be Banach spaces and  $T \in \mathcal{L}(X, Y)$ . Prove the following are equivalent.

1.  $T$  is injective and  $\text{range}(T)$  is closed.
2.  $T : X \rightarrow \text{range}(T)$  is a linear homeomorphism.
3. There exists  $C \geq 0$  such that  $\|x\|_X \leq C \|Tx\|_Y$  for all  $x \in X$ .

HINT: Prove that (1)  $\implies$  (2)  $\implies$  (3)  $\implies$  (1).

*Proof.* (1)  $\implies$  (2). If  $T$  is injective and  $\text{range}(T)$  is closed, then  $\Gamma(T) = \{(x, Tx) : x \in X\}$  is closed in  $X \times Y$ . Therefore,  $T : X \rightarrow \text{range}(T)$  is a bounded linear map. Since  $T$  is injective, this map is actually bijective from  $X$  to  $\text{range}(T)$ . Therefore,  $T$  is a linear homeomorphism.

(2)  $\implies$  (3). Since  $T$  is a bijective bounded linear map, from  $X$  to  $\text{range}(T)$ . There exists a constant  $C \geq 0$  such that for each  $y \in \text{range}(T)$  there exists a unique  $x \in X$  such that  $Tx = y$  and  $\|x\| \leq C \|y\| = C \|Tx\|$ . Since  $T$  is a bijection,  $\|x\| \leq C \|Tx\|$  for all  $x \in X$ .

(3)  $\implies$  (1). Let  $x \in X$  be such that  $Tx = 0$ . It follows that  $\|x\| \leq C \|Tx\| = 0$ . Therefore,  $x = 0$  and  $T$  is injective. To show that  $\text{range}(T)$  is closed, consider a convergent sequence  $\{y_n\} \subset \text{range}(T)$  with  $y_n = Tx_n$ . Since for any  $n, m \in \mathbb{N}$  we have

$$\|x_n - x_m\| \leq C \|T(x_n - x_m)\| = C \|y_n - y_m\|,$$

$\{x_n\}$  is Cauchy. Since  $X$  is Banach,  $x_n \rightarrow x$  for some  $x \in X$ . Therefore, for all  $n \in \mathbb{N}$  we have

$$\|y_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \|x_n - x\|,$$

and  $y_n \rightarrow Tx$ . Hence,  $\text{range}(T)$  is closed and the proof is complete.  $\square$

**Theorem.** Let  $X$  and  $Y$  be Banach spaces over a common field. Then, the following subsets of  $\mathcal{L}(X; Y)$  are open:

1.  $\{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\}$ ,
2.  $\{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\}$ ,
3.  $\mathcal{H}(X; Y) = \{T \in \mathcal{L}(X; Y) : T \text{ is a homeomorphism}\}$ .

*Proof.* 1. Let  $T \in \mathcal{L}(X; Y)$  be surjective. By open mapping theorem, there is  $\delta > 0$  such that  $B_Y(0, \delta) \subset TB_X(0, 1)$ . By homogeneity we have  $B_Y(0, r) \subset TB_X(0, \alpha r)$  for all  $r > 0$  where  $\alpha = \delta^{-1}$ . Now let  $S \in \mathcal{L}(X; Y)$  be such that  $\|T - S\| < \beta < (2\alpha)^{-1}$ . Claim  $S$  is surjective.

Let  $y \in Y$ , inductively construct sequences  $\{x_n\}$  and  $\{y_n\}$ . First let  $y_0 = y$ . Then,  $\|y_0\| \in B(0, 2\|y_0\|)$ . Select  $x_0 \in X$  be such that  $Tx_0 = y_0$  and  $\|x_0\| \leq 2\alpha\|y_0\|$ . Suppose we have selected  $y_i, x_i$  for  $0 \leq i \leq n$ . Set  $y_{n+1} = y_n - Sx_n$  and select  $x_{n+1}$  be such that  $Tx_{n+1} = y_{n+1}$  and  $\|x_{n+1}\| \leq 2\alpha\|y_{n+1}\|$ . Then, we have

$$\|y_{n+1}\| = \|Tx_n - Sx_n\| \leq \|T - S\| \|x_n\| < 2\alpha\beta\|y_n\|$$

and

$$\|x_{n+1}\| = 2\alpha\|y_{n+1}\| \leq 2\alpha\|T - S\| \|x_n\| < 2\alpha\beta\|x_n\|.$$

Note that  $2\alpha\beta < 1$  and  $X$  is Banach, define

$$x = \sum_{n=0}^{\infty} x_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N x_n.$$

Also note that  $\lim_{n \rightarrow \infty} y_n = 0$ . It follows that

$$Sx = \sum_{n=0}^{\infty} Sx_n = \sum_{n=0}^{\infty} (y_n - y_{n+1}) = y_0 - \lim_{n \rightarrow \infty} y_{n+1} = y.$$

Therefore  $S$  is surjective and the set of surjective bounded linear maps are open.

2. Suppose  $T \in \mathcal{L}(X; Y)$  is injective with closed range. Then, closed range theorem gives  $C > 0$  such that  $\|x\| \leq C\|Tx\|$  for all  $x \in X$ . Now suppose  $S \in \mathcal{L}(X; Y)$  is such that  $\|T - S\| < (2C)^{-1}$ . Claim that  $S$  is also injective with closed range. Indeed,

$$\begin{aligned} \|x\| &\leq C\|Tx\| \leq C\|Sx\| + C\|(T - S)x\| \\ &\leq C\|Sx\| + \frac{1}{2}\|x\|. \end{aligned}$$

This shows that  $\|x\| \leq 2C\|Sx\|$  for all  $x \in X$ . By closed range theorem,  $S$  is injective with closed range. This implies that the set of injective bounded linear operator with closed range is open.

3. This directly follows from

$$\mathcal{H}(X; Y) = \{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\} \cap \{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\}.$$

□

**Theorem.** Let  $X$  and  $Y$  be Banach spaces over a common field. Then the following holds.

1. The sets

$$\mathcal{L}_R(X; Y) = \{T \in \mathcal{L}(X; Y) : \text{there exists } S \in \mathcal{L}(Y; X) \text{ such that } ST = I_X\}$$

and

$$\mathcal{L}_L(X; Y) = \{T \in \mathcal{L}(X; Y) : \text{there exists } S \in \mathcal{L}(Y; X) \text{ such that } TS = I_Y\}$$

are open.

2. The following inclusion holds:

$$\mathcal{L}_L(X; Y) \subset \{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\}$$

and

$$\mathcal{L}_R(X; Y) \subset \{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\}.$$

3. The sets  $\mathcal{L}_L(X; Y) \setminus \mathcal{L}_R(X; Y)$  and  $\mathcal{L}_R(X; Y) \setminus \mathcal{L}_L(X; Y)$  are open.

*Proof.* 1. Let  $T_0 \in \mathcal{L}_R$  and  $S_0 \in \mathcal{L}(Y; X)$  be such that  $T_0 S_0 = I_Y$ . Note that  $I_X \in \mathcal{H}(X)$  and when  $\|P\| < 1$  for  $P \in \mathcal{L}(X)$ , we have  $I_X + P \in \mathcal{H}(X)$ . Suppose now  $T \in \mathcal{L}(X; Y)$  and  $\|T\| < \|S_0\|^{-1}$ . It follows that  $I_X + S_0 T \in \mathcal{H}(X)$ . For such  $T$ , we then have

$$T_0 + T = T_0(I_X + S_0 T).$$

Also,

$$(T_0 + T)(I_X + S_0 T)^{-1} S_0 = T_0(I_X + S_0 T)(I_X + S_0 T)^{-1} S_0 = T_0 S_0 = I_Y.$$

Therefore,  $T_0 + T \in \mathcal{L}_R$  for  $T \in B(T_0, \|S_0\|^{-1})$  and  $\mathcal{L}_R$  is open.

Now let  $T_0 \in \mathcal{L}_L$  and  $S_0 \in \mathcal{L}(Y; X)$  be such that  $S_0 T_0 = I_X$ . Again, for  $T \in \mathcal{L}(X; Y)$  with  $\|T\| < \|S_0\|^{-1}$ , we have

$$T_0 + T = (I_X + T S_0) T_0.$$

and

$$S_0(I_X + T S_0)^{-1}(T_0 + T) = I_X.$$

Therefore,  $\mathcal{L}_L$  is also open.

2. Let  $T \in \mathcal{L}_R$  and  $S \in \mathcal{L}(Y; X)$  be such that  $TS = I_Y$ . Then for any  $y \in Y$  let  $x = Sy$ . It follows that  $Tx = TSy = y$ . Also,  $\|x\| \leq \|S\| \|y\|$  so the 4th item in open mapping theorem guarantees that  $T$  is surjective. Hence,  $\mathcal{L}_L \subset \{T \in \mathcal{L}(X; Y) : T \text{ is surjective}\}$ .

Now let  $T \in \mathcal{L}_L$  and  $S \in \mathcal{L}(Y; X)$  such that  $ST = I_X$ . Now for any  $x \in X$ , we have  $\|x\| = \|STx\| \leq \|S\| \|Tx\|$ . Then the closed range theorem guarantees that  $T$  is injective with closed range. Hence,  $\mathcal{L}_R \subset \{T \in \mathcal{L}(X; Y) : T \text{ is injective with closed range}\}$ .

3. \*\*\*TO-DO\*\*\*

□

### 1.3 Hahn-Banach theorem and duality

**Theorem** (Hahn-Banach theorem in  $\mathbb{R}$ ). Let  $X$  be a real vector space and suppose  $p : X \rightarrow \mathbb{R}$  is such that

$$p(tx + (1-t)y) \leq tp(x) + (1-t)p(y)$$

for all  $t \in [0, 1]$  and  $x, y \in X$ .

Suppose  $Y$  subspace of  $X$  and  $l : Y \rightarrow \mathbb{R}$  is a linear map such that  $l \leq p$  on  $Y$ . Then there exists linear map  $L : X \rightarrow \mathbb{R}$  such that  $L \leq p$  on  $X$  and  $L = l$  on  $Y$ .

*Proof.* Let

$$P = \{(Z, \lambda) : Y \subset Z \subset X, \lambda \text{ linear functional on } Z, \lambda \leq p \text{ on } Z \text{ and } \lambda = l \text{ on } Y\}$$

Define partial order  $(Z_1, \lambda_1) \preceq (Z_2, \lambda_2)$  if and only if  $Z_1 \subset Z_2$  and  $\lambda_1 = \lambda_2$  on  $Z_1$ . It is easy to verify that this is a partial order. Towards using Zorn's Lemma, let  $C \subset P$  be a chain and define

$$U = \bigcup_{(Z, \lambda) \in C} Z, \quad \Lambda = \bigcup_{(Z, \lambda) \in C} \lambda.$$



It is easy to verify that  $(U, \Lambda)$  is an upper bound for the chain. By Zorn's Lemma,  $P$  has a maximal element  $(M, L)$ . It remains to show that  $M = X$ .

Suppose for contradiction that  $M \neq X$ . Pick  $x_0 \in X \setminus M$ . For any  $x, y \in M$ , we have

$$\begin{aligned} \beta L(x) + \alpha L(y) &= L(\beta x + \alpha y) \\ &= \frac{1}{\alpha + \beta} L\left(\frac{\beta}{\alpha + \beta}x + \frac{\alpha}{\alpha + \beta}y\right) \\ &\leq (\alpha + \beta)p\left(\frac{\beta}{\alpha + \beta}x + \frac{\alpha}{\alpha + \beta}y\right) \\ &= (\alpha + \beta)p\left(\frac{\beta}{\alpha + \beta}(x - \alpha x_0) + \frac{\alpha}{\alpha + \beta}(y + \beta x_0)\right) \\ &\leq \beta p(x - \alpha x_0) + \alpha p(y + \beta x_0). \end{aligned}$$

This implies that

$$\sup_{\substack{x \in M \\ \alpha > 0}} \frac{1}{\alpha} [L(x) - p(x - \alpha x_0)] \leq \inf_{\substack{y \in M \\ \beta > 0}} \frac{1}{\beta} [p(y + \beta x_0) - L(y)].$$

Note that  $-p(-x_0) \leq \text{LHS}$  and  $\text{RHS} \leq p(x_0)$ , so  $\text{LHS}, \text{RHS} < \infty$ . Now pick  $v \in \mathbb{R}$  such that  $\text{LHS} \leq v \leq \text{RHS}$ . For  $x \in M$  and  $0 < t \in \mathbb{R}$  we have

$$L(x) - tv \leq p(x - tv_0), \quad L(x) + tv \leq p(x + tv_0).$$

Now define  $\hat{L} : M \oplus \mathbb{R}x_0 \rightarrow \mathbb{R}$  by  $\hat{L}(x + \alpha x_0) = L(x) + \alpha v$ . It follows that  $(M \oplus \mathbb{R}x_0, \hat{L}) \in P$ . However,  $(M, L) \prec (M \oplus \mathbb{R}, \hat{L})$ , a contradiction. Therefore,  $M = X$  and the proof is complete.  $\square$

**Theorem** (Hahn-Banach theorem in  $\mathbb{C}$ ). Let  $X$  be complex vector space and suppose  $p : X \rightarrow \mathbb{R}$  is such that

$$p(\alpha x + \beta y) \leq |\alpha|p(x) + |\beta|p(y)$$

for all  $\alpha, \beta \in \mathbb{C}$  such that  $|\alpha| + |\beta| = 1$  and  $x, y \in X$ .

Suppose  $Y$  subspace of  $X$  and  $l : Y \rightarrow \mathbb{C}$  is a linear map such that  $|l| \leq p$  on  $Y$ . Then there exists linear map  $L : X \rightarrow \mathbb{C}$  such that  $|L| \leq p$  on  $X$  and  $L = l$  on  $Y$ .

*Proof.* Define  $\lambda : Y \rightarrow \mathbb{R}$  by  $\lambda(x) = \text{Re}(l(x))$ . Note that

$$\lambda(ix) = \text{Re}(il(x)) = -\text{Im}(l(x)).$$

This implies that  $l(x) = \lambda(x) - i\lambda(ix)$ . Now treat  $X$  and  $Y$  as vector space over  $\mathbb{R}$  and apply Hahn-Banach theorem in  $\mathbb{R}$  to extend  $\lambda$  to  $\Lambda : X \rightarrow \mathbb{R}$  that agrees with  $\lambda$  on  $Y$ .

Define  $L : X \rightarrow \mathbb{C}$  by  $L(x) = \Lambda(x) - i\Lambda(ix)$ . It remains to show that  $|L| \leq p$ . For  $x \in X$ , write  $L(x) = |L(x)|e^{i\theta}$  for some  $\theta \in \mathbb{R}$ . It follows that

$$\begin{aligned} |L(x)| &= L(x)e^{-i\theta} \\ &= \Lambda(e^{-i\theta}) - i\Lambda(ie^{-i\theta}x) \\ &= \Lambda(e^{-i\theta}x) \\ &\leq p(e^{-i\theta}x) \\ &\leq |e^{-i\theta}|p(x) \\ &= p(x), \end{aligned}$$

as desired.  $\square$

**Theorem** (Hahn-Banach theorem for bounded linear functionals). Let  $X$  be a normed vector space over  $\mathbb{F}$  and  $Y$  a subspace of  $X$ . If  $\lambda \in Y^*$  then there exists  $\Lambda \in X^*$  such that  $\Lambda = \lambda$  on  $Y$  and the operator norm  $\|\lambda\|_{Y^*} = \|\Lambda\|_{X^*}$ .

*Proof.* Consider  $p : X \rightarrow \mathbb{R}$  where  $p(x) = \|\lambda\|_{Y^*} \|x\|$ . Apply Hahn-Banach theorem. □

Next we show some useful implications of Hahn-Banach theorem.

**Theorem.** Let  $X$  be a normed vector space and fix  $x \in X$ . Then the following holds:

1. There exists  $\lambda \in X^*$  such that  $\|\lambda\| = \|x\|$  and

$$\lambda(x) = \|\lambda\| \|x\| = \|x\|^2.$$

2. We have

$$\|x\| = \max_{\substack{w \in X^* \\ \|w\|=1}} |w(x)|.$$

3.  $x = 0$  if and only if  $w(x) = 0$  for all  $w \in X^*$ .

*Proof.* 1. Let  $Y = \mathbb{F}x$  and define  $\lambda \in Y^*$  by  $\lambda(ax) = a \|x\|^2$ . Apply Hahn-Banach theorem.

2. Suppose  $x \neq 0$ . Define  $w = \frac{\lambda}{\|x\|}$  then it follows that  $|w(x)| = \|x\|$ .

3. Follows directly from (2). □

**Proposition.** Let  $X$  be normed vector space. Then the mapping  $\langle \cdot, \cdot \rangle : X^* \times X \rightarrow \mathbb{F}$  by  $(w, x) \mapsto w(x)$  is a bilinear map. That is,  $\langle \cdot, \cdot \rangle \in \mathcal{L}(X^*, X; \mathbb{F})$ . Moreover, if  $X \neq \{0\}$ , then  $\|\langle \cdot, \cdot \rangle\| = 1$ .

*Proof.* It is easy to see that  $\langle \cdot, \cdot \rangle$  is bilinear. For boundedness,

$$|\langle w, x \rangle| = |w(x)| \leq \|w\| \|x\|.$$

Hence,  $\|\langle \cdot, \cdot \rangle\| \leq 1$ . Meanwhile, pick some  $x \in X$  with  $\|x\| = 1$ . It follows that

$$1 = \|x\| = \max_{\substack{w \in X^* \\ \|w\|=1}} |w(x)| \leq \|\langle \cdot, \cdot \rangle\|.$$

Therefore,  $\|\langle \cdot, \cdot \rangle\| = 1$ . □

**Definition.** Let  $X$  be normed vector space and  $E \subset X$ ,  $W \subset X^*$ . Say  $W$  is a *norming set* for  $E$  if

$$\|x\| = \sup_{\substack{w \in W \\ \|w\|=1}} |\langle w, x \rangle|$$

for all  $x \in E$ .

**Proposition.** Let  $X$  be normed vector space and  $S \subset X$  be a separable set. Let  $W$  be a norming set for  $S$ . Then, there exists  $\{w_n\}_{n=0}^\infty \subset W$  such that  $\|w_n\| = 1$ , and the sequence is norming for  $S$ . That is,

$$\|x\| = \sup_{n \in \mathbb{N}} |\langle w_n, x \rangle|.$$

*Proof.* Let  $\{v_n\}_{n=0}^\infty \subset S$  be dense. For any  $n, k \in \mathbb{N}$ , choose  $w_{n,k} \in W$  with  $\|w_{n,k}\| = 1$  such that

$$(1 - 2^{-k}) \|v_n\| \leq |w_{n,k}, v_n|.$$

Let  $x \in S$  and  $0 < \varepsilon < 1$  be arbitrary. Pick  $v_n \in S$  such that  $\|v_n - x\| < \varepsilon$  and pick  $j \in \mathbb{N}$  such that  $2^{-j} < \varepsilon$ . Then,

$$\begin{aligned} (1 - \varepsilon) \|x\| &\leq (1 - 2^{-j}) \|x\| \\ &\leq (1 - 2^{-j}) \|v_n\| + (1 - 2^{-j}) \|v_n - x\| \\ &\leq |\langle w_{n,j}, v_j \rangle| + \varepsilon \\ &\leq |\langle w_{n,j}, x \rangle| + |\langle w_{n,j}, x - v_n \rangle| + \varepsilon \\ &\leq |\langle w_{n,j}, x \rangle| + 2\varepsilon. \end{aligned}$$

This shows that  $\{w_{n,k}\}_{n,k=0}^\infty$  is a norming sequence. □

**Theorem.** Let  $X$  be normed vector space and define  $J : X \rightarrow X^{**}$  by  $\langle Jx, w \rangle = \langle w, x \rangle = w(x)$ . Then the following holds:

1.  $J \in \mathcal{L}(X, X^{**})$ .
2.  $J$  is an isometric embedding. In particular, it is injective.
3.  $\text{range}(J) \subset X^{**}$  is a norming set for  $X^*$ .
4.  $X$  is Banach if and only if  $\text{range}(J)$  is closed.

*Proof.* Note that we have

$$\begin{aligned} \|Jx\|_{X^{**}} &= \sup \{ |\langle Jx, w \rangle| : w \in X^* \text{ and } \|w\| \leq 1 \} \\ &= \sup \{ |\langle w, x \rangle| : w \in X^* \text{ and } \|w\| \leq 1 \} \\ &= \|x\|, \end{aligned}$$

where the last step is by a previous theorem that shows the existence of  $w \in X^*$  such that  $\|w\| = 1$  and  $|w(x)| = \|x\|$ . This implies (1) and (2). Now we know  $X$  is isometrically isomorphic to  $\text{range}(J) \subset X^{**}$ . Therefore,  $X$  is Banach if and only if  $\text{range}(J)$  is Banach. However,  $X^{**} = \mathcal{L}(X^*, \mathbb{F})$  is Banach, so  $\text{range}(J)$  is Banach if and only if  $\text{range}(J)$  is closed. This implies (4).

To show (3), note that we have

$$\begin{aligned} \|w\|_{X^*} &= \sup \{ |\langle w, x \rangle| : x \in X \text{ and } \|x\| \leq 1 \} \\ &= \sup \{ |\langle Jx, w \rangle| : x \in X \text{ and } \|x\| \leq 1 \} \\ &= \sup \{ |\langle v, w \rangle| : v \in \text{range}(J) \text{ and } \|v\|_{X^{**}} \leq 1 \}. \end{aligned}$$

This shows (3), completing the proof. □

## 2 Differential Calculus

**Theorem** (Local injectivity theorem). Let  $X$  and  $Y$  be Banach spaces,  $z \in U \subset X$  with  $U$  open. Let  $f : U \rightarrow Y$  differentiable with  $Df$  continuous at  $z$ . Suppose  $Df(z) \in \mathcal{L}(X; Y)$  injective with closed range. Then for any  $0 < \varepsilon < 1$ , there exists  $r > 0$  such that

1.  $B[z, r] \subset U$ .
2.  $Df(x)$  injective with closed range for all  $x \in B[z, r]$ .
3. If  $x, y \in B(z, r)$ , then

$$(1 - \varepsilon) \|Df(z)(x - y)\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|Df(z)(x - y)\|.$$

4. The restriction  $f : B(z, r) \rightarrow f(B(z, r))$  is bi-Lipschitz homeomorphism.

**Theorem** (Local surjectivity theorem). Let  $X$  and  $Y$  be Banach spaces,  $z \in U \subset X$  with  $U$  open. Let  $f : U \rightarrow Y$  differentiable with  $Df$  continuous at  $z$ . Suppose  $Df(z) \in \mathcal{L}(X; Y)$  surjective. Then there exists  $r_0, \gamma > 0$  such that

1.  $B_X[z, r_0] \subset U$ .
2.  $Df(x)$  surjective for all  $x \in B_X[z, r_0]$ .
3.  $B_Y[f(z), \gamma r] \subset f(B_X[z, r])$  for all  $0 \leq r \leq r_0$ .

**Definition.** Let  $X$  and  $Y$  be normed vector spaces and suppose that  $\emptyset \neq U \subset X$  is open. Let  $f : U \rightarrow Y$ . For  $k \geq 1$ , say  $f$  is a  $C^k$  diffeomorphism if

1.  $f : U \rightarrow f(U)$  homeomorphism with  $f(U) \subset Y$  open.
2.  $f \in C^k(U; Y)$ .
3.  $f^{-1} \in C^k(f(U); X)$ .

If  $f$  is a  $C^k$  diffeomorphism for all  $k \geq 1$ , say  $f$  is a smooth diffeomorphism.

**Theorem** (Inverse function theorem). Let  $X$  and  $Y$  be Banach spaces,  $U \subset X$  open and  $x_0 \in U$ . Suppose  $f : U \rightarrow Y$  differentiable,  $Df$  continuous at  $x_0$ ,  $Df(x_0)$  linear homeomorphism. Then there exists bounded and open  $V \subset U$  with  $x_0 \in V$  such that

1.  $f : V \rightarrow f(V)$  is bi-Lipschitz homeomorphism,  $Df(x)$  linear homeomorphism for all  $x \in V$ ,  $f(V) \subset Y$  bounded and open,  $f^{-1} : f(V) \rightarrow V$  differentiable with

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$$

for all  $y \in f(V)$  and  $Df^{-1}$  is continuous at  $f(x_0)$ . Also, there exists  $C_0, C_1 > 0$  such that

$$C_0 \leq \|Df(x)\| \leq C_1$$

for all  $x \in V$ , and

$$\frac{1}{C_1} \leq \|Df^{-1}(y)\| \leq \frac{1}{C_0}$$

for all  $y \in f(V)$ .

2. If  $f \in C^k(U; Y)$  for some  $1 \leq k \leq \infty$ , then  $f^{-1} \in C^k(f(V); X)$ .
3. If  $f \in C^k(U; Y)$  for  $1 \leq k \in \mathbb{N}$ , then there exists open  $V_k \subset V$  such that  $x_0 \in V_k$ ,  $f \in C_b^k(V_k; Y)$  and  $f^{-1} \in C_b^k(f(V_k); X)$ .

**Theorem** (Implicit function theorem). Let  $X$  and  $Y$  be Banach spaces,  $U \subset X \times Y$  be open with  $(x_0, y_0) \in U$ , and suppose  $f : U \rightarrow Z$  is differentiable in  $U$  with  $Df$  continuous at  $(x_0, y_0)$ . Further suppose  $z_0 = f(x_0, y_0)$  and  $D_2f(x_0, y_0) \in \mathcal{L}(Y; Z)$  is an isomorphism. Then there exists open sets  $x_0 \in V \subset X$ ,  $z_0 \in W \subset Z$ ,  $y_0 \in S \subset Y$ , and  $g \in C_b^{0,1}(V \times W; Y)$  such that the following holds:

1.  $g(x_0, z_0) = y_0$  and  $(x, g(x, z)) \in V \times S \subset U$  for all  $(x, z) \in V \times W$ . Also,  $g$  is differentiable on  $V \times W$  and  $Dg$  continuous at  $(x_0, z_0)$ .
2.  $f(x, g(x, z)) = z$  for all  $(x, z) \in V \times W$ . Moreover, if  $(x, y) \in V \times S$  such that  $f(x, y) = z$  for some  $z \in W$ , then  $y = g(x, z)$ .
3.  $D_2f(x, g(x, z))$  is an isomorphism for all  $(x, z) \in V \times W$ , and

$$\begin{aligned} D_1g(x, z) &= -[D_2f(x, g(x, z))]^{-1} D_1f(x, g(x, z)), \\ D_2g(x, z) &= [D_2f(x, g(x, z))]^{-1}. \end{aligned}$$

4. If  $f \in C^k$  then  $g \in C^k$  too for  $1 \leq k \leq \infty$ . If  $k$  finite and  $f \in C_b^k$  then the sets can be picked such that  $g \in C_b^k$ .

### 3 Measure and integration

#### 3.1 Constructing outer measures

**Lemma.** Let  $X$  be a set with gauge  $(\mathcal{E}, \gamma)$  that covers  $X$ . Let  $A \subset X$ , then the following holds:

1. Let  $\mu^*$  be the outer measure generated by  $(\mathcal{E}, \gamma)$ . Then there exists collection  $\{E_{m,n}\}_{m,n=0}^\infty \subset \mathcal{E}$  such that  $E = \bigcap_{m=0}^\infty \bigcup_{n=0}^\infty E_{m,n}$  such that  $A \subset E$  and  $\mu^*(A) = \mu^*(E)$ .
2. Suppose  $(X, d)$  is metric space and the gauge is fine. Let  $\mu_d^*$  be the metric outer measure. Then there exists collection  $\{E_{m,n}\}_{m,n=0}^\infty \subset \mathcal{E}$  such that  $E = \bigcap_{m=0}^\infty \bigcup_{n=0}^\infty E_{m,n}$  such that  $A \subset E$  and  $\mu^*(A) = \mu^*(E)$ .

*Proof.* We only prove (2) since (1) is similar. Since the gauge is fine,  $(\mathcal{E}_\delta, \gamma_\delta)$  covers  $X$  for all  $\delta > 0$ . Then, for any  $m \in \mathbb{N}$ , there exists  $\{E_{m,n}\}_n \subset \mathcal{E}_{2^{-m}}$  such that  $A \subset \bigcup_{n=0}^\infty E_{m,n}$  and  $\sum_{n=0}^\infty \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}$ . Now let  $E = \bigcap_{m=0}^\infty \bigcup_{n=0}^\infty E_{m,n}$ . Note that  $A \subset E$  and for any  $m \in \mathbb{N}$ , we have

$$\mu_{2^{-m}}^*(E) \leq \mu_{2^{-m}}^* \left( \bigcup_{n=0}^\infty E_{m,n} \right) \leq \sum_{n=0}^\infty \gamma(E_{m,n}) \leq \mu_{2^{-m}}^*(A) + 2^{-m}.$$

Taking the limit as  $m \rightarrow \infty$ , we have

$$\mu_d^*(E) \leq \mu_d^*(A) \leq \mu_d^*(E),$$

as desired.  $\square$

**Theorem.** Let  $(X, d)$  be metric space with  $(\mathcal{E}, \gamma)$  such that all sets in  $\mathcal{E}$  are open. Assume that  $\mu^*$  is a metric outer measure on  $X$  such that either

1.  $\mu^*$  is generated by  $(\mathcal{E}, \gamma)$ , or
2.  $\mu^* = \mu_d^*$  is generated by  $(\mathcal{E}_\delta, \gamma_\delta)$ .

Further suppose that  $X = \bigcup_{n=0}^\infty A_n$  where  $A_n \subset X$  is such that  $\mu^*(A_n) < \infty$ . Then the following holds:

1. The gauge covers  $X$  in case 1 and is fine in case 2.
2. In both cases,  $\mu^*$  is cover-regular. More precisely, for each  $A \subset X$ , there is  $G \in G_\delta(X) \subset \mathcal{B}(X) \subset \mathfrak{M}$  such that  $A \subset G$  and  $\mu^*(A) = \mu^*(G)$ .
3. In both cases, the following are equivalent for  $E \subset X$ :
  - (a)  $E \in \mathfrak{M}$ , i.e.  $E$  is measurable.
  - (b) there exists  $G \in G_\delta(X)$  such that  $E \subset G$  and  $\mu^*(G \setminus E) = 0$ .
  - (c) there exists  $F \in F_\sigma(X)$  such that  $F \subset E$  and  $\mu^*(E \setminus F) = 0$ .

*Proof. Step 0: proof for (1) and (2).*

We know  $X = \bigcup_{n=0}^\infty A_n$  for some  $\mu^*(A_n) < \infty$ . For case (1), we can pick  $\{E_{n,m}\} \subset \mathcal{E}$  such that  $A_n \subset \bigcup_{m=0}^\infty E_{n,m}$ . Then  $X = \bigcup_{n=0}^\infty A_n = \bigcup_{n,m} E_{n,m}$ . Therefore,  $\mathcal{E}$  covers  $X$ . For case (2), note that  $\mu^*(A_n) < \infty$  and  $\mu_d^*(A_n) = \sup_{\delta > 0} \mu_\delta^*(A_n)$  for each  $\delta > 0$  and  $n \in \mathbb{N}$ . Then for each  $\delta > 0$ , there exists  $\{E_{n,m}\} \subset \mathcal{E}_\delta$  such that  $A_n \subset \bigcup_{m=0}^\infty E_{n,m}$ . Then,  $X = \bigcup_{n=0}^\infty A_n = \bigcup_{n,m} E_{n,m}$ . Therefore,  $(\mathcal{E}, \gamma)$  is fine.

We have the following observations:

1.  $\mu^*$  is a metric outer measure. This implies that  $\mathcal{B}(X) \subset \mathfrak{M}$ .
2.  $G_\delta(X) \cup F_\sigma(X) \subset \mathcal{B}(X) \subset \mathfrak{M}$  and  $\mu^*(A) = 0$  implies  $A \in \mathfrak{M}$ .
3. By previous lemma and all sets in  $\mathcal{E}$  are open, we know for each  $A \subset X$  there is  $E \in G_\delta(X)$  such that  $A \subset E$  and  $\mu^*(A) = \mu^*(E)$ . In particular,  $\mu^*$  is cover regular.

**Step 1: starting on (3).**

For (b)  $\implies$  (a), suppose (b) holds for  $E \subset X$ . Then  $E = G \setminus (G \setminus E) \in \mathfrak{M}$  since  $\mu^*(G \setminus E) = 0$ .

For (c)  $\implies$  (a), suppose (c) holds for  $E \subset X$ . Then  $E = F \cup (E \setminus F) \in \mathfrak{M}$  since  $\mu^*(E \setminus F) = 0$ .

Next we show “(a)  $\implies$  (c)” implies “(a)  $\implies$  (b)”. Suppose  $E \in \mathfrak{M}$ , then  $E^c \in \mathfrak{M}$ . By (a)  $\implies$  (b) we know there exists  $F \in F_\sigma$  such that  $F \subset E^c$  and  $\mu^*(E^c \setminus F) = 0$ . Let  $G = F^c \in G_\delta$  then  $E \subset G$  and  $G \subset E = E^c \subset F$ .

Therefore, it remains to show (a)  $\implies$  (c) to complete the proof for the theorem.

**Step 2: reduction for (a)  $\implies$  (c).**

Claim it suffices to show it for  $E$  such that  $\mu^*(E) < \infty$ . Suppose we did this and  $\mu^*(E) = \infty$ . Using observation there exists  $B_n \in \mathfrak{M}$  such that  $A_n \subset B_n$  and  $\mu^*(B_n) = \mu^*(A_n) < \infty$ . Then  $E_n = E \cap B_n \in \mathfrak{M}$  and  $\mu^*(E_n) < \infty$ . Then by special case there is  $F_n \in F_\sigma(X)$  such that  $F_n \subset E_n$  and  $\mu^*(F_n \setminus E_n) = 0$ . Let  $F = \bigcup_{n=0}^{\infty} F_n \in F_\sigma$  then  $F \subset \bigcup_{n=0}^{\infty} E_n = E$  and

$$\mu^*(E \setminus F) \leq \sum_{n=0}^{\infty} \mu^*(E_n \setminus F_n) = 0.$$

**Step 3: further reduction.**

Claim it suffices to show it for the case where  $\mu^*(E) < \infty$  and  $E \in G_\delta(X)$ . Suppose we have proved this and consider  $E \subset X$  such that  $\mu^*(E) < \infty$ . Observation 3 allows us to pick  $G \in G_\delta(X)$  such that  $E \subset G$  and  $\mu^*(E) = \mu^*(G)$ . Now pick  $H \in G_\delta$  such that  $G \setminus E \subset H$  and  $\mu^*(H) = \mu^*(G \setminus E)$ .

Now apply special case. This gives  $F \in F_\sigma$  such that  $F \subset G$  and  $\mu^*(G \setminus F) = 0$ . Let  $K = F \setminus H = F \cap H^c \in F_\sigma$  and  $K = F \cap H^c \subset G \cap (G \setminus E)^c \subset E$ .

Note that  $E, F, G, H, K \in \mathfrak{M}$ , so

$$\begin{aligned} \mu^*(E \setminus K) &= \mu^*(E) - \mu^*(K) \\ &= \mu^*(G) - \mu^*(F \setminus H) \\ &= \mu^*(G) - \mu^*(F) + \mu^*(F \cap H) \\ &\leq \mu^*(G) - \mu^*(F) + \mu^*(H) \\ &= \mu^*(G \setminus F) + \mu^*(H) \\ &= \mu^*(G \setminus E) \\ &= \mu^*(G) - \mu^*(E) \\ &= 0. \end{aligned}$$

Therefore,  $K$  is the desired  $F_\sigma$  set.

**Step 4: finishing (a)  $\implies$  (c).**

Suppose  $E \in G_\delta(X)$  and  $\mu^*(E) < \infty$ . Write  $E = \bigcup_{n=0}^{\infty} V_n$  where  $V_n \subset X$  open. For  $m, n \in \mathbb{N}$ , let

$$C_{n,m} = \{x \in V_n : \text{dist}(x, V_n^c) \geq 2^{-m}\} \subset V_n.$$

Note that  $C_{n,m}$  is closed,  $C_{n,m} \subset C_{n,m+1}$ ,  $V_n = \bigcup_m C_{n,m}$ . Since  $E, C_{n,m}, V_n \in \mathfrak{M}$ , we have

$$\mu^*(E) = \mu^*(E \cap V_n) = \lim_{m \rightarrow \infty} \mu^*(E \cap C_{n,m}).$$

Thus, there exists  $M(n, k)$  such that  $\mu^*(E \setminus C_{n, M(n, k)}) < 2^{-n-k}$ . Now let  $D_k = \bigcup_{n=0}^{\infty} C_{n, M(n, k)}$  closed. Also,  $D_k \subset \bigcup_{n=0}^{\infty} V_n = E$  and

$$\mu^*(E) - \mu^*(D_k) = \mu^*(E \setminus D_k) \leq \sum_{n=0}^{\infty} \mu^*(E \setminus C_{n, M(n, k)}) \leq 2^{-k+1}.$$

Let  $F = \bigcup_{k=0}^{\infty} D_k \subset E$  and note that  $F \in F_{\sigma}$ . Then

$$\mu^*(E \setminus F) = \mu^*(E) - \mu^*(F) \leq \mu^*(E) - \mu^*(D_k) < 2^{-k+1}$$

for all  $k \in \mathbb{N}$ . Therefore,  $\mu^*(E \setminus F) = 0$ .

□

**Lemma.** Suppose  $(X, d)$  metric space with metric outer measure  $\mu^*$ . Suppose  $X = \bigcup_{n=0}^{\infty} V_n$  for  $V_n \subset X$  open and  $\mu^*(V_n) < \infty$ . Suppose  $E \subset G \in G_{\delta}(X)$  such that  $\mu^*(G \setminus E) = 0$ . Then for each  $\varepsilon > 0$ , there exists open  $U \subset X$  such that  $E \subset U$  and  $\mu^*(U \setminus E) < \varepsilon$ .

*Proof.* Let  $E_n = E \cap V_n$  and  $G = G \cap V_n$ . Write  $G = \bigcap_{j=0}^{\infty} W_j$  where  $W_j$  open. Now set

$$Z_{n,m} = V_n \cap \bigcap_{j=0}^m W_j,$$

which are open for all  $n, m \in \mathbb{N}$ . Now notice that  $G_n \subset Z_{n,m+1} \subset Z_{n,m} \subset V_n$ . Note that  $\mu^*(V_n) < \infty$ , so  $\mu^*(G_n) = \lim_{m \rightarrow \infty} \mu^*(Z_{n,m})$ . Therefore, for all  $\varepsilon > 0$ , there exists  $M(n)$  such that

$$\mu^*(Z_{n,M(n)} \setminus G_n) < \varepsilon 2^{-n-2}.$$

Then set  $U = \bigcup_{n=0}^{\infty} Z_{n,M(n)} \supset \bigcup_{n=0}^{\infty} G_n = G \supset E$  open, then we have

$$\begin{aligned} \mu^*(U \setminus E) &= \mu^*(U \setminus G) + \mu^*(G \setminus E) \\ &= \mu^*\left(\bigcup_{n=0}^{\infty} Z_{n,M(n)} \cap \bigcap_{n=0}^{\infty} G_n^c\right) \\ &\leq \sum_{n=0}^{\infty} \mu^*(Z_{n,M(n)} \setminus G_n) \\ &< \varepsilon, \end{aligned}$$

as desired.

□