Introduction to Functional Analysis

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1 Banach space theory

1.1 Quotient spaces, Baire category and uniform boundedness

Theorem. Let $\|\cdot\|$ be a **seminorm** on a vector space V. If we define $E = \{v \in V : \|v\| = 0\}$, then E is a subspace of V, and the function on V/E defined by

$$\|v + E\| = \|v\|$$

for any $v + E \in V/E$ defines a **norm**.

Theorem (Baire Category Theorem)**.** Let M be a complete metric space, and let $\{C_n\}_{n=0}^{\infty}$ be a collection of closed subsets of M such that $M = \bigcup_{n=0}^{\infty} C_n$. Then at least one of the C_n contains an open ball $B(x, r) = \{y \in M : d(x, y) < r\}$.

Theorem (Uniform Boundedness Theorem)**.** Let B be Banach space and V a normed vector space. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{B}(B, V)$. Then if for all $b \in B$ we have $\sup_n \|T_n b\| < \infty$ (that is, this sequence is pointwise boudned), then $\sup_n \|T_n\| < \infty$ (the operator norms are bounded).

Proof. For each $k \in \mathbb{N}$, define

$$C_k = \left\{ b \in B : \|b\| \leq 1, \sup_{n \in \mathbb{N}} \|T_n b\| \leq k \right\}.$$

This set is closed for each $k \in \mathbb{N}$, but by assumption, we have

$$\{b \in B : \|b\| \leq 1\} = \bigcup_{k=0}^{\infty} C_k.$$

The left hand side is a closed subset of B, and is thus a complete metric space. By Baire Category Theorem, there exists $k \in \mathbb{N}$ such that C_k contains an open ball $B(b_0, \delta_0)$. Then, if $b \in B(0, \delta_0)$, we have $b_0 + b \in B(b_0, \delta_0)$ and thus

$$\sup_{n \in \mathbb{N}} \|T_n(b_0 + b)\| \leq k.$$

It follows that

$$\sup_{n \in \mathbb{N}} \|T_n b\| \leq \sup_{n \in \mathbb{N}} \|T_n(b_0 + b)\| + \sup_{n \in \mathbb{N}} \|T_n b_0\| \leq 2k.$$

Suppose $\|b\| = 1$, then $\frac{\delta_0}{2} b \in B(0, \delta_0)$ and thus for all $n \in \mathbb{N}$, we have

$$\left\| T_n \left(\frac{\delta_0}{2} b \right) \right\| \leq 2k.$$

Therefore,

$$\sup_{n \in \mathbb{N}} \|T_n\| \leq \frac{4k}{\delta_0}.$$

$\square$

2 Hilbert space theory

2.1 Basic Hilbert space theory

Definition (Pre-Hilbert space)**.** A **pre-Hilbert** space H is a vector space over $\mathbb{C}$ with a **Hermitian inner product**, which is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ satisfying the following properties.

1. For all $\lambda_1, \lambda_2 \in C$ and $v_1, v_2, w \in H$, we have

$$\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, 2 \rangle + \lambda_2 \langle v_2, w \rangle .$$

2. For all $v, w \in H$, we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
3. For all $v \in H$, we have $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$.

Definition. Let H be a pre-Hilbert space. For all $v \in H$, we define

$$\|v\| = \langle v, v \rangle^{\frac{1}{2}} .$$

Theorem (Cauchy-Schwarz inequality)**.** Let H be a pre-Hilbert space. For all $u, v \in H$, we have

$$|\langle u, v \rangle| \leq \|u\| \, \|v\| .$$

Proof. Define $f(t) = \|u + tv\|^2$. Notice that

$$\begin{aligned} f(t) &= \langle u + tv, u + tv \rangle \\ &= \langle u, u \rangle + t^2 \langle v, v \rangle + t \langle u, v \rangle + t \langle v, u \rangle \\ &= \|u\|^2 + t^2 \|v\|^2 + 2t \operatorname{Re}(\langle u, v \rangle). \end{aligned}$$

This implies that

$$0 \leq f(t_{\min}) = \|u\|^2 - \frac{\operatorname{Re}(\langle u, v \rangle)^2}{\|v\|^2}.$$

It follows that

$$|\operatorname{Re}(\langle u, v \rangle)| \leq \|u\| \, \|v\| .$$

This is almost what we want. To finish up, first note that if $\langle u, v \rangle = 0$ then there is nothing to prove, so suppose $\langle u, v \rangle \neq 0$, and define

$$\lambda = \frac{\overline{\langle u, v \rangle}}{|\langle u, v \rangle|}.$$

Note that we have $|\lambda| = 1$ and we have the chain of equalities of real numbers:

$$|\langle u, v \rangle| = \lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \operatorname{Re} \langle \lambda u, v \rangle \leq \|\lambda u\| \, \|v\| .$$

However, $\|\lambda u\| = \|u\|$, so the proof is complete.

□

Theorem. If H is a pre-Hilbert space, then $\|\cdot\|$ is a norm on H.

Proof. Note that

$$\|v\| = 0 \iff \langle v, v \rangle = 0 \iff v = 0.$$

Now if $\lambda \in \mathbb{C}$ and $v \in H$, then

$$\langle \lambda v, \lambda v \rangle = \lambda \overline{\lambda} \langle v, v \rangle = |\lambda|^2 \|v\|^2 .$$

Therefore, $\|\lambda v\| = |\lambda| \, \|v\|$.

Finally, let $u, v \in H$, then

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v\rangle \\
&\le \|u\|^2 + \|v\|^2 + 2\left|\langle u, v\rangle\right| \\
&\le \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\| \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

This completes the proof. □

Theorem. If $u_n \to u$ and $v_n \to v$ in a pre-Hilbert space H, then $\langle u_n, v_n\rangle \to \langle u, v\rangle$.

Proof. If $u_n \to u$ and $v_n \to v$, then $\|u_n - u\| \to 0$ and $\|v_n - v\| \to 0$. It follows that

$$\begin{aligned}
|\langle u_n, v_n\rangle - \langle u, v\rangle| &= |\langle u_n - u, v_n\rangle - \langle u, v - v_n\rangle| \\
&\le |\langle u_n - u, v_n\rangle| + |\langle u, v - v_n\rangle| \\
&\le \|u_n - u\|\,\|v_n\| + \|u\|\,\|v - v_n\| \\
&\le \|u_n - u\| \sup_{k\in\mathbb{N}} \|v_k\| + \|u\|\,\|v - v_n\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. This completes the proof. □

Definition (Hilbert space)**.** A **Hilbert space** is a pre-Hilbert space that is complete with repsect to the norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.

Example. Some examples of Hilbert spaces:

- $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) : z_j \in \mathbb{C}\}$ with $\langle z, w\rangle = \sum_j z_j \overline{w_j}$ is a Hilbert space.
- $\ell^2 = \left\{a = \{a_k\}_{k=0}^\infty : a_k \in \mathbb{C}, \sum_{k=0}^\infty |a_k|^2 < \infty\right\}$ with $\langle a, b\rangle = \sum_{k=0}^\infty a_k \overline{b_k}$ is a Hilbert space.
- If $E \subset \mathbb{R}$ is measurable, then $L^2(E) = \left\{f : E \to \mathbb{C}, \int_E |f|^2 < \infty\right\}$ with $\langle f, g\rangle = \int_E f\overline{g}$ is a Hilbert space.

We will show that each separable Hilbert spaces is isometrically isomorphic to either $\mathbb{C}^n$ or ℓ^2.

Now we have seen that ℓ^2 and L^2 spaces are Hilbert spaces. This is expected since the definition of the inner product in these spaces uses the fact that they are ℓ^2 or L^2. A natural question then is whether other ℓ^p or L^p spaces are also Hilbert spaces with respect to some inner product? It turns out there is a simple way to decide whether a norm come from a inner-product, and thus whether a Banach space is a Hilbert space.

Theorem (Parallelogram Law)**.** If H is a pre-Hilbert space, then for all $u, v \in H$, we have

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right).$$

In addition, if H is a normed vector space satisfying this equality, then H is a pre-Hilbert space.

Using the previous theorem, we can verify that ℓ^p and L^p with $p \neq 2$ are **not** Hilbert spaces.

Definition (Orthogonal)**.** If H is a pre-Hilbert space, $u, v \in H$ are **orthogonal** if $\langle u, v\rangle = 0$. We denote this as $u \perp v$.

Definition (Orthonormal sets)**.** If H is a pre-Hilbert space, a subset $\{e_\lambda\}_{\lambda\in\Lambda} \subset H$ is **orthonormal** if for all $\lambda \in \Lambda$, we have $\|e_\lambda\| = 1$ and $\lambda_1 \neq \lambda_2$ implies $e_{\lambda_1} \perp e_{\lambda_2}$.

Remark. we will mainly be interested in the case where we have a countable orthonormal set.

Example. The set $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ as elements in $L^2([-\pi,\pi])$ is an orthonormal subset of $L^2([-\pi,\pi])$. Indeed, for any $m,n\in\mathbb{Z}$, we have

$$\int_{-\pi}^{\pi} e^{imx}\overline{e^{inx}} = \int_{-\pi}^{\pi} e^{i(m-n)x} = \begin{cases} 2\pi & (m=n), \\ 0 & (m\neq n). \end{cases}$$

Therefore, $\left\langle \frac{e^{inx}}{\sqrt{2\pi}}, \frac{e^{imx}}{\sqrt{2\pi}} \right\rangle = \delta_{mn}$, and $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ is an orthonormal subset of $L^2([-\pi,\pi])$.

Theorem (Bessel)**.** If $\{e_n\}_{n=0}^{\infty}$ is countable orthonormal subset of a pre-Hilbert space H, then for all $u\in H$, we have

$$\sum_{n=0}^{\infty} |\langle u, e_n\rangle|^2 \leq \|u\|^2 .$$

Proof. We first do the finite case. Suppose $\{e_n\}_{n=1}^{N}$ is an orthonormal subset of H. Then,

$$\begin{aligned}
\left\|\sum_{n=0}^{N} \langle u, e_n\rangle e_n\right\|^2 &= \left\langle \sum_{n=0}^{N} \langle u, e_n\rangle e_n, \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\rangle \\
&= \sum_{n=0}^{N}\sum_{m=1}^{N} \langle u, e_n\rangle \overline{\langle u, e_m\rangle} \langle e_n, e_m\rangle \\
&= \sum_{n=0}^{N} |\langle u, e_n\rangle|^2 .
\end{aligned}$$

Also,

$$\begin{aligned}
\left\langle u, \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\rangle &= \sum_{n=0}^{N} \overline{\langle u, e_n\rangle} \langle u, e_n\rangle \\
&= \sum_{n=0}^{N} |\langle u, e_n\rangle|^2 .
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &\leq \left\| u - \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\|^2 \\
&= \|u\|^2 + \left\|\sum_{n=0}^{N} \langle u, e_n\rangle e_n\right\|^2 - 2\operatorname{Re}\left\langle u, \sum_{n=0}^{N} \langle u, e_n\rangle e_n \right\rangle \\
&= \|u\|^2 - \sum_{n=0}^{N} |\langle u, e_n\rangle|^2 ,
\end{aligned}$$

as desired.

For the infinite case, just take the limit as $N\to\infty$. □

Definition (Maximal orthonormal subset)**.** An orthonormal subset $\{e_\lambda\}_\lambda$ of a pre-Hilbert space is **maximal** if $u\in H$ and $\langle u, e_\lambda\rangle = 0$ for all $\lambda\in\Lambda$ implies that $u=0$.

Theorem. Every non-trivial pre-Hilbert space has a maximal orthonormal subset.

This can be proved using Zorn's Lemma. We will prove something less strong but often equally useful by hand, without applying Zorn's Lemma.

Theorem. Every non-trivial separable pre-Hilbert space has a countable maximal orthonormal subset.

Proof. Use the Gram-Schimdt process. Let $\{v_j\}_{j=0}^{\infty}$ be a countable dense subset of H where $v_0\neq 0$. Claim that for any $n\in\mathbb{N}$, there exists $m(n)\leq n$ and an orthonormal subset $\{e_1,\ldots,e_{m(n)}\}$ such that

1. $\operatorname{span}\{e_1, \ldots, e_{m(n)}\} = \operatorname{span}\{v_1, \ldots, v_n\}$.
2. If $v_n \in \operatorname{span}\{v_1, \ldots, v_{n-1}\}$, we have
$$\{e_1, \ldots, e_{m(n)}\} = \{e_1, \ldots, e_{m(n-1)}\} \cup \emptyset.$$
Otherwise, we have
$$\{e_1, \ldots, e_{m(n)}\} = \{e_1, \ldots, e_{m(n-1)}\} \cup e_{m(n)}$$
for some $e_{m(n)} \in H$.

Prove this by induction. For the base case, let $e_1 = \frac{v_1}{\|v_1\|}$. For the inductive step, suppose the claim holds for $n = k$. If $v_{k+1} \in \operatorname{span}\{v_1, \ldots, v_k\}$, then
$$\operatorname{span}\{e_1, \ldots, e_{n(k)}\} = \operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{span}\{v_1, \ldots, v_{k+1}\}.$$
Now suppose $v_{k+1} \notin \operatorname{span}\{v_1, \ldots, v_k\}$. Define
$$w_{k+1} = v_{k+1} - \sum_{j=1}^{m(k)} \langle v_{k+1}, e_j \rangle e_j.$$
Note that $w_{k+1} \neq 0$ and define $e_{m(k+1)} = \frac{w_{k+1}}{\|w_{k+1}\|}$. Then, $\|e_{m(k+1)}\| = 1$ and for all $1 \leq l \leq m(k)$,
$$\begin{aligned}\langle e_{m(k+1)}, e_l \rangle &= \frac{1}{\|w_{k+1}\|} \left\langle v_{k+1} - \sum_{j=1}^{m(k)} \langle v_{k+1}, e_j \rangle, e_l \right\rangle \\ &= \frac{1}{\|w_{k+1}\|} \left(\langle v_{k+1}, e_l \rangle - \langle v_{k+1}, e_l \rangle \right) \\ &= 0.\end{aligned}$$
Therefore, $e_{m(k+1)}$ is the desired vector we want and we have completed the proof for the claim.

Now let
$$S = \bigcup_{n=0}^{\infty} \{e_1, \ldots, e_{m(n)}\}.$$
Then S is a countable orthonormal subset of H. Now we show S is maximal. Suppose $u \in H$ and $\langle u, e_l \rangle = 0$. Since $\{v_j\}_{j=0}^{\infty}$ is dense in H, there exists $\{v_{j(k)}\}_{k=0}^{\infty}$ such that $v_{j(k)} \to u$ as $k \to \infty$. By our claim, we know $v_{j(k)} \in \operatorname{span}\{e_1, \ldots, e_{m(j(k))}\}$. By Bessel's inequality,
$$\|v_{j(k)}\|^2 = \sum_{l=1}^{m(j(k))} |\langle v_{j(k)}, e_l \rangle|^2 = \sum_{l=1}^{m(j(k))} |\langle v_{j(k)} - u, e_l \rangle|^2 \leq \|v_{j(k)} - u\|^2,$$
where for the first equality we used the fact that $v_{j(k)} \in \operatorname{span}\{e_1, \ldots, e_{m(j(k))}\}$. Since $v_{j(k)} \to u$ as $k \to \infty$, the inequality implies that $\|v_{j(k)}\| \to 0$ as $k \to \infty$ and thus $\|u\| = 0$, showing that S is indeed a maximal orthonormal subset of H. $\square$

Corollary. ℓ^2 and L^2 have countable maximal orthonormal subset since they are both separable.

2.2 Orthonormal bases and Fourier Series

Definition (Orthonormal basis)**.** Let H be a Hilbert space. An **orthonormal basis** of H is a countable maximal orthonormal subset $\{e_n\}_n$ of H.

Theorem. If $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis in Hilbert space H, then for all $u \in H$, we have
$$\sum_{n=0}^{\infty} \langle u, e_n \rangle e_n = u.$$
This is the Fourier-Bessel series.

This tells us we can write each element in H as a infinite linear combination of the orthonormal basis.

Proof. We first prove the sequence of partial sums $\{\sum_{n=0}^{m} \langle u, e_n \rangle e_n\}_m$ is Cauchy. Let $\varepsilon > 0$. By Bessel's inequality, we have

$$\sum_{n=0}^{\infty} |\langle u, e_n \rangle|^2 \leq \|u\|^2 < \infty.$$

Therefore, there exsits $M \in \mathbb{N}$ such that $N \geq M$ implies $\sum_{n=N+1}^{\infty} |\langle u, e_n \rangle|^2 < \varepsilon^2$. Then for all $m > l \geq M$, we have

$$\left\| \sum_{n=0}^{m} \langle u, e_n \rangle e_n - \sum_{n=0}^{l} \langle u, e_n \rangle e_n \right\|^2 \leq \sum_{n=l+1}^{m} |\langle u, e_n \rangle|^2 \leq \sum_{n=l+1}^{\infty} |\langle u, e_n \rangle|^2 < \varepsilon^2.$$

Therefore, the sequence of partial sum is Cauchy. Since H is complete, there exists $\overline{u} \in H$ such that $\overline{u} = \sum_{n=0}^{\infty} \langle u, e_n \rangle e_n$. It remains to show that $\overline{u} = u$. By continuity of inner-product, for all $l \in \mathbb{N}$, we have

$$\begin{aligned}
\langle u - \overline{u}, e_l \rangle &= \lim_{m \to \infty} \left\langle u - \sum_{n=0}^{m} \langle u, e_n \rangle e_n, e_l \right\rangle \\
&= \lim_{m \to \infty} \left[\langle u, e_l \rangle - \sum_{n=0}^{m} \langle u, e_n \rangle \langle e_n, e_l \rangle \right] \\
&= 0.
\end{aligned}$$

Since $\{e_n\}_{n=0}^{\infty}$ is maximal, this implies that $u - \overline{u} = 0$ and the proof is complete. □

Theorem. Let H be a Hilbert space. If H has an orthonormal basis, then H is separable.

Proof. Suppose $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for H. Then

$$S = \bigcup_{m \in \mathbb{N}} \left\{ \sum_{n=0}^{m} q_n e_n : q_n \in \mathbb{Q} + i\mathbb{Q} \right\}$$

is a countable set. Also, by the previous theorem, S is dense in H. □

Remark. Let H be a Hilbert space. H is separable if and only if H has an orthonormal basis.

Theorem (Parseval's identity)**.** If H is a Hilbert space and $\{e_n\}_{n=0}^{\infty}$ is a countable orthonormal basis, then for all $u \in H$, we have

$$\sum_{n} |\langle u, e_n \rangle|^2 = \|u\|^2$$

Proof. We have $u = \sum_n \langle u, e_n \rangle e_n$. This implies that

$$\begin{aligned}
\|u\|^2 &= \lim_{m \to \infty} \left\langle \sum_{n=0}^{m} \langle u, e_n \rangle e_n, \sum_{l=0}^{m} \langle u, e_l \rangle e_l \right\rangle \\
&= \lim_{m \to \infty} \sum_{n=0}^{m} \sum_{l=0}^{m} \langle u, e_n \rangle \overline{\langle u, e_l \rangle} \langle e_n, e_l \rangle \\
&= \lim_{m \to \infty} \sum_{n=0}^{m} |\langle u, e_n \rangle|^2 \\
&= \sum_{n=0}^{\infty} |\langle u, e_n \rangle|^2 .
\end{aligned}$$

□

Theorem. If H is an infinte dimensional separable Hilbert space, then H is isometrically isomorphic to ℓ^2. That is, there exists bijective boudned linear map $T : H \to \ell^2$ such that for all $u, v \in H$, we have

$$\|Tu\|_{\ell^2} = \|u\|_H \text{ and } \langle Tu, Tv \rangle_{\ell^2} = \langle u, v \rangle_H .$$

Proof. Since H is separable, there exists an orthonormal basis $\{e_n\}_{n=0}^{\infty}$. For all $u \in H$, the previous theorem gives

$$\|u\| = \left(\sum_{n=0}^{\infty} |\langle u, e_n\rangle|^2\right)^{\frac{1}{2}}.$$

Define $T : H \to \ell^2$ by

$$Tu = \{\langle u, e_n\rangle\}_{n=0}^{\infty} \in \ell^2.$$

It is easy to check that T is the desired isometric isomorphism. □

Next we use the theories we learned in a more concrete setting — the Fourier series.

Theorem. The subset $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ is an orthonormal subset of $L^2([-\pi, \pi])$.

Definition. Let $f \in L^2([-\pi, \pi])$. Then the **n-th Fourier coefficient** of f is

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\, dt.$$

The **N-th Fourier sum** of f is

$$S_n f(x) = \sum_{|n|\le N} \widehat{f}(n)e^{inx} = \sum_{|n|\le N} \left\langle f, \frac{e^{int}}{\sqrt{2\pi}}\right\rangle \frac{e^{inx}}{\sqrt{2\pi}}.$$

The **Fourier series** of f is the formal series $\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{-inx}$.

The natural question now is whether we have for all $f \in L^2([-\pi, \pi])$,

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{inx}.$$

That is, whether we have the following convergence in L^2.

$$\lim_{N\to\infty} \|f - S_N f\|_2 = 0.$$

This question is then equivalent to whether $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ is maximal in $L^2([-\pi, \pi])$. That is, whether $\widehat{f}(n) = 0$ for all $n \in \mathbb{N}$ implies $f = 0$.

The answer to the question is yes, but it is going to take some work. We first do some simple calculation.

Theorem. For all $f \in L^2([-\pi, \pi])$ and for all $N \in \mathbb{N}$, we have

$$S_N f(x) = \int_{-\pi}^{\pi} D_N(x - t)f(t)\, dt,$$

where

$$D_N(x) = \begin{cases} \frac{2N+1}{2\pi} & (x = 0) \\ \frac{\sin\left(N+\frac{1}{2}\right)x}{2\pi \sin\frac{x}{2}} & (x \neq 0) \end{cases}$$

it the **Dirichlet kernel**. Note that D_N is a smooth function.

Proof. If $f \in L^2([-\pi, \pi])$, we have

$$\begin{aligned} S_N f(x) &= \sum_{|n|\le N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\, dt\right) e^{inx} \\ &= \int_{-\pi}^{\pi} f(t)\left(\frac{1}{2\pi}\sum_{|n|\le N} e^{in(x-t)}\right) dt. \end{aligned}$$

Let $D_N(x) = \frac{1}{2\pi}\sum_{|n|\leq N} e^{in(x-t)}$. Then for $x \neq 0$, we have

$$\begin{aligned}
D_N(x) &= \frac{1}{2\pi}\sum_{n=-N}^{N} e^{-inx} \\
&= \frac{1}{2\pi}e^{-iNx}\sum_{n=0}^{2N}\left(e^{ix}\right)^n \\
&= \frac{1}{2\pi}e^{-iNx}\frac{1-e^{i(2N+1)x}}{1-e^{ix}} \\
&= \frac{1}{2\pi}\frac{e^{i(N+\frac{1}{2})x}-e^{-i(N+\frac{1}{2})x}}{e^{\frac{ix}{2}}-e^{-\frac{ix}{2}}} \\
&= \frac{1}{2\pi}\frac{2i\sin(N+\frac{1}{2})x}{2i\sin\frac{x}{2}} \\
&= \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}},
\end{aligned}$$

as desired. For $x = 0$, we also clearly have $D_N(0) = \frac{(2N+1)}{2\pi}$. The proof is thus complete. □

Definition. If $f \in L^2([-\pi,\pi])$, we define the N**-th Cesaro-Fourier mean** of f by

$$\sigma_N f(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_N f(x).$$

The idea behind defining the Cesaro mean is that if the original sequence converges, the Cesaro mean also converge to the same limit. However, Cesaro have even better property — the Cesaro mean can converge even if the original sequence does not converge. Therefore, it has better convergence properties and hopefully we can show it converge to f in L^2 more easily. The goal now is then to show

$$\|\sigma_N f - f\|_2 \to 0 \text{ as } N \to \infty.$$

This would tell us if all Fourier coefficients are zero, then the Cesaro means are zero, and the limit above would tell us f is zero.

2.3 Fejer's theorem and Convergence of Fourier series