

# Measure and Integration

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### 3 Signed measure and differentiation

#### 3.1 Signed measures

**Exercise** (Folland 3.2). If  $\nu$  a signed measure,  $E$  is  $\nu$ -null if and only if  $|\nu|(E) = 0$ . Also if  $\nu$  and  $\mu$  are signed measures, then  $\nu \perp \mu$  if and only if  $|\nu| \perp \mu$  if and only if  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ .

*Proof.* It is clear that  $|\nu|(E) = 0$  implies that  $E$  is  $\nu$ -null. Now suppose  $E$  is  $\nu$ -null and let  $X = P \cup N$  be the Hahn decomposition of  $\nu$ . Suppose for contradiction that  $|\nu|(E) > 0$ , then it follows that  $\nu^+(E) > 0$  and  $\nu^-(E) > 0$ . We then have  $\nu^+(E \cap P) = \nu^+(E \cap P) + \nu^+(E \cap N) = \nu^+(E) > 0$ , but  $\nu^-(E \cap P) \leq \nu^-(P) = 0$ . Therefore,  $\nu(E \cap P) > 0$ , a contradiction with  $E$  being  $\nu$ -null. Therefore,  $|\nu|(E) = 0$ .

Suppose  $\nu \perp \mu$ , then there is  $X = E \cup F$  such that  $E$  is  $\nu$ -null and  $F$  is  $\mu$ -null. It follows that  $|\nu|(E) = 0$ , so  $|\nu| \perp \mu$ . Therefore,  $\nu \perp \mu$  implies  $|\nu| \perp \mu$ . It is clear that  $|\nu| \perp \mu$  implies  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ . Now suppose  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ . Then we have  $X = E^+ \cup F^+ = E^- \cup F^-$  where  $F^+$  is  $\nu^+$  null,  $F^-$  is  $\nu^-$  null, and  $E^\pm$  is  $\mu$ -null. Let  $E = E^+ \cup E^-$  and  $F = F^+ \cap F^- = E^c$ . Then we can verify that  $E$  is  $\mu$ -null and  $F$  is  $\nu$ -null. Therefore,  $\nu \perp \mu$  and the proof is complete.  $\square$

**Exercise** (Folland 3.3). Let  $\nu$  be a signed measure on  $(X, \mathfrak{M})$ .

1.  $L^1(\nu) = L^1(|\nu|)$ .
2. If  $f \in L^1(\nu)$ , then  $|\int f d\nu| \leq \int |f| d|\nu|$ .
3. If  $E \in \mathfrak{M}$ ,  $|\nu|(E) = \sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \right\}$

*Proof.* 1. Since we have

$$\int |f| d\nu = \int |f| d\nu^+ - \int |f| d\nu^-, \quad \int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^-,$$

it follows immediately that  $L^1(\nu) = L^1(|\nu|)$ .

2. We have

$$\left| \int f d\nu \right| = \left| \int f d\nu^+ - \int f d\nu^- \right| \leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|.$$

3. By the previous item, we know

$$|\nu|(E) = \int_E 1 d|\nu| \geq \int_E |f| d|\nu| \geq \left| \int_E f d\nu \right|.$$

for any  $|f| \leq 1$ . Also,  $|\nu|(E) = \left| \int (\chi_P + \chi_N) d\nu \right|$ , where  $X = P \cup N$  is the Hahn decomposition so  $|\chi_P + \chi_N| = 1$ .

$\square$

**Exercise** (Folland 3.4). If  $\nu$  is a signed measure and  $\lambda, \mu$  are positive measures such that  $\nu = \lambda - \mu$ , then  $\lambda \geq \nu^+$  and  $\mu \geq \nu^-$ .

*Proof.* Let  $\nu = \nu^+ - \nu^-$  be the unique decomposition. We then have  $\lambda - \nu^+ = \mu - \nu^-$ . Pick a set  $E \in \mathfrak{M}$  such that  $\nu^-(E) = 0$ . We then have

$$\lambda(E) - \nu^+(E) = \mu(E) \geq 0.$$

Therefore  $\lambda(E) \geq \nu^+(E)$ . On the other hand, for  $E \in \mathfrak{M}$  such that  $\nu^+(E) = 0$ , we have  $\lambda(E) \geq 0 = \nu^+(E)$ . In light of the Hahn decomposition for  $\nu$  and the additivity of measure, we can conclude that  $\lambda \geq \nu^+$ , and thus  $\mu \geq \nu^-$ .  $\square$

**Exercise** (Folland 3.5). If  $\nu_1, \nu_2$  are both signed measures that omits the values  $\pm\infty$ , then  $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$ . (Use Exercise 3.4)

*Proof.* Note that we have  $\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)$ . Write  $\nu = \nu_1 + \nu_2$ . By Exercise 3.4, we have  $\nu_1^+ + \nu_2^+ \geq \nu^+$  and  $\nu_1^- + \nu_2^- \geq \nu^-$ . It follows that

$$|\nu_1 + \nu_2| = \nu^+ + \nu^- \leq \nu_1^+ + \nu_2^+ + \nu_1^- + \nu_2^- = |\nu_1| + |\nu_2|,$$

as desired.  $\square$

**Exercise** (Folland 3.6). Suppose  $\nu(E) = \int_E f d\mu$  where  $\mu$  is a positive measure and  $f$  is an extended  $\mu$ -integrable function. Describe the Hahn decomposition for  $\nu$  and express the positive, negative, and total variation of  $\nu$  in terms of  $f$  and  $\mu$ .

*Proof.* A Hahn decomposition of  $\nu$  is  $X = P \cup N$  where  $P = \{f \geq 0\}$  and  $N = P^c$ . We also have

$$\nu^+(E) = \int_E f^+ d\mu, \quad \nu^-(E) = \int_E f^- d\mu, \quad |\nu|(E) = \int_E |f| d\mu.$$

$\square$

**Exercise** (Folland 3.7). Suppose  $\nu$  is a signed measure on  $(X, \mathfrak{M})$  and  $E \in \mathfrak{M}$ .

1.  $\nu^+(E) = \sup \{\nu(F) : F \subset E, F \in \mathfrak{M}\}$  and  $\nu^-(E) = -\inf \{\nu(F) : F \subset E, F \in \mathfrak{M}\}$ .
2.  $|\nu|(E) = \sup \left\{ \sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \dots, E_n \text{ disjoint, and } \bigcup_{j=1}^n E_j = E \right\}$ .

*Proof.* 1. First let  $F \subset E$ , we have  $\nu(F) = \nu^+(F) - \nu^-(F)$ . It follows that  $\nu(F) \leq \nu^+(F) \leq \nu^+(E)$ . Let  $X = P \cup N$  be the Hahn decomposition for  $\nu$ , then we have  $\nu^+(E) = \nu(E \cap P)$ . Similarly for  $\nu^-(E)$ .

2. Let  $E_1, \dots, E_n$  be disjoint and  $\bigcup_{j=1}^n E_j = E$ . It follows that

$$|\nu|(E) = \sum_{j=1}^n |\nu|(E_j) \geq \sum_{j=1}^n |\nu(E_j)|.$$

Let  $X = P \cup N$  be the Hahn decomposition for  $\nu$ , we then have

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu^+(E \cap P) + \nu^-(E \cap N) = |\nu(E \cap P)| + |\nu(E \cap N)|.$$

This completes the proof. □

### 3.2 The Lebesgue-Randon-Nikodym theorem

**Exercise** (Folland 3.8).  $\nu \ll \mu$  if and only if  $|\nu| \ll \mu$  if and only if  $\nu^+ \ll \mu$  and  $\nu^- \ll \mu$ .

*Proof.* It is clear that  $|\nu| \ll \mu$  implies  $\nu^+ \ll \mu$  and  $\nu^- \ll \mu$  implies  $\nu \ll \mu$ . It then remains to show that  $\nu \ll \mu$  implies  $|\nu| \ll \mu$ . Let  $X = P \cup N$  be the Hahn decomposition for  $\nu$  and  $E \in \mathfrak{M}$  be such that  $\mu(E) = 0$ . It follows that  $\mu(E \cap P) = 0$  and  $\mu(E \cap N) = 0$ . Then  $\nu(E \cap P) = \nu^+(E \cap P) = 0$  and  $\nu(E \cap N) = \nu^-(E \cap N) = 0$ . Therefore,

$$\nu(E) = \nu^+(E \cap P) + \nu^-(E \cap N) = 0,$$

as desired. □

**Exercise** (Folland 3.9). Suppose  $\{\nu_j\}$  is a sequence of positive measures. If  $\nu_j \perp \mu$  for all  $j$ , then  $\sum_{j=1}^{\infty} \nu_j \perp \mu$ , and if  $\nu_j \ll \mu$  for all  $j$ , then  $\sum_{j=1}^{\infty} \nu_j \perp \mu$ .

*Proof.* Suppose  $\nu_j \perp \mu$  for all  $j$ . For each  $j \in \mathbb{N}$ , let  $X = E_j \cup F_j$  where  $\nu_j$  is null on  $E_j$  and  $\mu$  is null on  $F_j$ . Let  $E = \bigcap_{j=1}^{\infty} E_j$  and  $F = \bigcup_{j=1}^{\infty} F_j = E^c$ . Then it is easy to verify that  $\sum_{j=1}^{\infty} \nu_j$  is null on  $E$  and  $\mu$  is null on  $F$ .

Suppose  $\nu_j \ll \mu$  for all  $j$ . Then for  $E \in \mathfrak{M}$  such that  $\mu(E) = 0$ , we have  $\nu_j(E) = 0$  for all  $j$ . Therefore,  $\sum_{j=1}^{\infty} \nu_j(E) = 0$  and thus  $\sum_{j=1}^{\infty} \nu_j \perp \mu$ . □

### 3.3 Complex measures

**Exercise** (Folland 3.18). Let  $\nu$  be a complex measure on  $(X, \mathfrak{M})$ . Prove that  $L^1(\nu) = L^1(|\nu|)$  and if  $f \in L^1(\nu)$ , then  $|\int f d\nu| \leq \int |f| d|\nu|$ .

*Proof.* For  $L^1(\nu) \subset L^1(|\nu|)$ , consider  $f \in L^1(\nu)$ . Note that  $\nu = \nu_r + i\nu_i$  and it is easy to verify that  $|i\nu_i| = |\nu_i|$ . Therefore by Proposition 3.14, we have  $|\nu| \leq |\nu_r| + |\nu_i|$ . It follows that

$$\begin{aligned} \int |f| d|\nu| &\leq \int |f| d|\nu_r| + \int |f| d|\nu_i| \\ &= \int |f| d\nu_r^+ + \int |f| d\nu_r^- + \int |f| d\nu_i^+ + \int |f| d\nu_i^-. \end{aligned}$$

Since  $f \in L^1(\nu)$ , all four terms are finite and thus  $f \in L^1(|\nu|)$ .

For  $L^1(|\nu|) \subset L^1(\nu)$ , consider  $f \in L^1(|\nu|)$ . Then we have

$$\int |f| d\nu = \int |f| \frac{d\nu}{d|\nu|} d|\nu| \leq \int |f| \left| \frac{d\nu}{d|\nu|} \right| d|\nu| = \int |f| d|\nu|,$$

where we have used the fact that  $d\nu/d|\nu|$  has absolute value 1  $|\nu|$ -a.e. This shows that  $f \in L^1(\nu)$ .

Moreover, we have that  $|\int f d\nu| \leq \int |f| d\nu$ . Therefore,

$$\left| \int f d\nu \right| \leq \int |f| d|\nu|,$$

as desired.  $\square$

**Exercise** (Folland 3.19). If  $\nu, \mu$  are complex measures and  $\lambda$  is a positive measure, then  $\nu \perp \mu$  if and only if  $|\nu| \perp |\mu|$ , and  $\nu \ll \lambda$  if and only if  $|\nu| \ll \lambda$ .

*Proof.* The “if” direction is clear for both propositions.

Suppose  $\nu \perp \mu$ . Then  $\nu_r \perp \mu_r$ ,  $\nu_i \perp \mu_r$ ,  $\nu_r \perp \mu_i$ , and  $\nu_i \perp \mu_i$ . It follows that  $|\nu_r| \perp \mu_r$  and  $|\nu_i| \perp \mu_r$ . Since  $|\nu| \leq |\nu_r| + |\nu_i|$ , we have  $|\nu| \perp \mu_r$ . Similarly  $|\nu| \perp \mu_i$ . Following the same reasoning, we obtain  $|\nu| \perp |\mu|$ , as desired.

Suppose  $\nu \ll \lambda$ . Since  $\nu = \nu_r + i\nu_i$ , we have  $\nu_r \ll \lambda$  and  $\nu_i \ll \lambda$ . Recall from Exercise 3.8 that this implies  $|\nu_r| \ll \lambda$  and  $|\nu_i| \ll \lambda$ . Moreover,  $|\nu| \leq |\nu_r| + |\nu_i|$ , so  $|\nu| \ll \lambda$ .  $\square$

**Exercise** (Folland 3.20). If  $\nu$  is a complex measure on  $(X, \mathfrak{M})$  and  $\nu(X) = |\nu|(X)$ , then  $\nu = |\nu|$ .

*Proof.* By Lebesgue-Randon-Nikodym theorem, we have  $d\nu = f d\mu$  for some function  $f$  and positive measure  $\mu$ . It follows that  $d|\nu| = |f| d\mu$  and

$$\int f d\mu = \int |f| d\mu.$$

Now let  $E \in \mathfrak{M}$ . We then have

$$\int_E f d\mu + \int_{E^c} f d\mu = \int_E |f| d\mu + \int_{E^c} |f| d\mu.$$

It follows that

$$0 \leq \int_E |f| - f_r d\mu = \int_{E^c} f_r - |f| d\mu \leq 0.$$

Since  $E \in \mathfrak{M}$  is arbitrary,  $f_r = |f|$  and  $f_i = 0$  a.e. It follows that  $d\nu = f d\mu = |f| d\mu = d|\nu|$ .  $\square$

**Exercise** (Folland 3.21). Let  $\nu$  be a complex measure on  $(X, \mathfrak{M})$ . If  $E \in \mathfrak{M}$ , define

$$\begin{aligned}\mu_1(E) &= \sup \left\{ \sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \dots, E_n \text{ disjoint}, E = \bigcup_{j=1}^n E_j \right\}, \\ \mu_2(E) &= \sup \left\{ \sum_{j=1}^{\infty} |\nu(E_j)| : E_1, E_2, \dots \text{ disjoint}, E = \bigcup_{j=1}^{\infty} E_j \right\}, \\ \mu_3(E) &= \sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \right\}.\end{aligned}$$

Then  $\mu_1 = \mu_2 = \mu_3 = |\nu|$ . (First show that  $\mu_1 \leq \mu_2 \leq \mu_3$ . To see  $\mu_3 = |\nu|$ , let  $f = \overline{d\nu/d|\nu|}$  and apply Proposition 3.13. To see  $\mu_3 \leq \mu_1$ , approximate  $f$  by simple functions.)

*Proof.* It is clear that  $\mu_1 \leq \mu_2$  by letting  $E_j = \emptyset$  for  $j > n$ . To show  $\mu_2 \leq \mu_3$ , consider disjoint collection of sets  $\{E_j\}_{j=1}^{\infty}$  such that  $E = \bigcup_{j=1}^{\infty} E_j$ . Let

$$f = \sum_{j=1}^{\infty} \frac{\overline{\nu(E_j)}}{|\nu(E_j)|} \chi_{E_j}.$$

Since  $\{E_j\}_{j=1}^{\infty}$  are disjoint, we have  $|f| \leq 1$ . Moreover, 1 is integrable on  $X$ , so dominated convergence theorem implies

$$\int_E f d\nu = \int_E \sum_{j=1}^{\infty} \frac{\overline{\nu(E_j)}}{|\nu(E_j)|} \chi_{E_j} d\nu = \sum_{j=1}^{\infty} \int_E \frac{\overline{\nu(E_j)}}{|\nu(E_j)|} \chi_{E_j} d\nu = \sum_{j=1}^{\infty} |\nu(E_j)|.$$

Therefore,  $\mu_2 \leq \mu_3$ .

Now we show  $\mu_3 = |\nu|$ . It is clear that

$$\left| \int_E f d\nu \right| \leq \int_E |f| d|\nu| \leq |\nu|(E),$$

so  $\mu_3 \leq |\nu|$ . On the other hand, let  $f = \overline{d\nu/d|\nu|}$ . Then,  $|f| = 1$  a.e. and

$$\left| \int_E \frac{\overline{d\nu}}{d|\nu|} d\nu \right| = \left| \int_E \frac{\overline{d\nu}}{d|\nu|} \frac{d\nu}{d|\nu|} d|\nu| \right| = \int_E 1 d|\nu| = |\nu|(E).$$

Therefore,  $|\nu| \leq \mu_3$  and thus  $|\nu| = \mu_3$ .

Finally, to show  $\mu_3 \leq \mu_1$ , let  $\varphi$  be a simple function such that  $|\varphi| \leq 1$ . Let

$$\varphi = \sum_{j=1}^n v_j \chi_{E_j}$$

be the canonical representation of  $\varphi$ . We then have  $|v_j| \leq 1$  for each  $j$  and  $E_1, \dots, E_n$  are disjoint. WLOG assume  $E = \bigcup_{j=1}^n E_j$ . It follows that

$$\left| \int_E \varphi d\nu \right| = \left| \sum_{j=1}^n v_j \nu(E_j) \right| \leq \sum_{j=1}^n |\nu(E_j)| \leq \mu_1(E).$$

We know any function  $f$  with  $|f| \leq 1$  can be approximated by simple functions with absolute value less than or equal to 1, so  $\mu_3 \leq \mu_1$ . The proof is then complete.  $\square$

### 3.4 Differentiation on Euclidean spaces

**Exercise** (Folland 3.22). If  $f \in L^1(\mathbb{R}^n)$ ,  $f \neq 0$ , then there exists  $C, R > 0$  such that  $Hf(x) \geq C|x|^{-n}$  for  $|x| > R$ . Hence  $m(\{x : Hf(x) > \alpha\}) > C'/\alpha$  when  $\alpha$  is small, so the estimate in the maximal theorem is essentially sharp.

*Proof.* Recall that

$$Hf(x) = \sup_{r>0} A_r |f|(x) = \sup_{r>0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y)| dy.$$

Taking  $r = 2|x|$  we have

$$Hf(x) \geq \frac{1}{2^n |x|^n m(B(0, 1))} \int_{B(x, 2|x|)} |f(y)| dy.$$

However, we have

$$\lim_{t \rightarrow \infty} \int_{B(0, t)} |f(y)| dy = \int_{\mathbb{R}^n} |f(y)| dy.$$

Write  $M = \int |f(y)| dy$ . Since  $f \in L^1(\mathbb{R}^n)$  and  $f \neq 0$ , we have  $0 < M < \infty$ . There then exists an  $R > 0$  such that

$$\int_{B(0, R)} |f(y)| dy \geq \frac{M}{2}.$$

Now for  $|x| > R$ , we have  $B(0, R) \subset B(x, 2|x|)$ , and thus

$$Hf(x) \geq \frac{M}{2^{n+1} m(B(0, 1))} |x|^{-n}.$$

When  $\alpha < \frac{1}{2}CR^{-n}$ , we have  $C|x|^{-n} > \alpha$  for  $R < |x| < (C/\alpha)^{1/n}$  and

$$\frac{C}{\alpha} - R^n > \frac{C}{2\alpha}.$$

It follows that

$$m(\{x : Hf(x) > \alpha\}) \geq m(B(0, 1)) \left( \frac{C}{\alpha} - R^n \right) > \frac{Cm(B(0, 1))}{2\alpha}.$$

The proof is then complete.  $\square$