Probability

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Contents

1 Measure theory review

1.1 Measurable space and mapping

Definition (σ-field)**.** A collection of subsets $\Sigma \subset 2^\Omega$ is a σ-field if

- $\emptyset \in \Sigma$.
- If $A \in \Sigma$, then $A^c \in \Sigma$.
- If $\{A_i\}_{i=1}^\infty \subset \Sigma$, then $\bigcup_{i=1}^\infty A_i \in \Sigma$.

The pair (Ω, Σ) is called a measurable space.

Definition (atom)**.** Let Σ be a σ-field. Say $A \in \Sigma$ is an atom if for all $B \in \Sigma$ either $A \subset B$ or $A \cap B = \emptyset$.

Proposition. For all $\omega \in \Omega$, there exists atom $A \in \Sigma$ containing ω if Ω is finite or countable.

Proof. Define $\widetilde{A} = \bigcap \{B \in \Sigma : \omega \in B\}$. We can check that $\widetilde{A} \in \Sigma$ and $\widetilde{A}$ is an atom containing ω. $\square$

Corollary. If Ω is finite or countable, there exists a partition $\Omega = \bigsqcup_i \Omega_i$, where each Ω_i is an atom of Σ. With this partition, Σ is just the power set with respect to $\{\Omega_i\}_i$.

Definition. If $F \subset 2^\Omega$, then the σ-field generated by F is the smallest σ-field containing all elements of F. Write this σ-field as $\sigma(F)$.

Example. Let $\Omega = \{1, 2, 3, 4, 5\}$ and $F = \{\{2, 3\}, \{3, 4\}\}$. Construct σ-field Σ generated by F. Σ is all possible union of sets from the collection $\{\{2\}, \{3\}, \{4\}, \{1, 5\}\}$.

Definition (measurable mapping)**.** Given two measurable spaces (Ω, Σ) and $(\widetilde{\Omega}, \widetilde{\Sigma})$. Then $f : \Omega \to \widetilde{\Omega}$ is measurable if $f^{-1}(B) \in \Sigma$ for all $B \in \widetilde{\Sigma}$.

Definition (Borel σ-field)**.** Let (T, τ) be a topological space. Then the Borel σ-field $\mathcal{B}(T, \tau)$ is defined as the smallest σ-field containing all open sets.

Definition (product measurable space)**.** Given two measurable spaces (Ω, Σ) and $(\widetilde{\Omega}, \widetilde{\Sigma})$. We can define the product measurable space as follows: let the ground set be $\Omega \times \widetilde{\Omega}$, and let $\Sigma \otimes \widetilde{\Sigma}$ be the smallest σ-field containing all rectangles $B \times \widetilde{B}$ where $B \in \Sigma$ and $\widetilde{B} \in \widetilde{\Sigma}$.

More generally, let Λ be an index set and $(\Omega_\lambda, \Sigma_\lambda)_{\lambda \in \Lambda}$. Define the product σ-field $\bigotimes_{\lambda \in \Lambda} \Sigma_\lambda$ be the smallest σ-field containing all elements in the form of $\prod_{\lambda \in \Lambda} B_\lambda$ where $B_\lambda \in \Sigma_\lambda$ and $B_\lambda = \Omega_\lambda$ for all but countably many indices.

Proposition. Let $(\Omega_i, \Sigma_i)_{i=1}^n$ be measurable spaces and $(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$ be the product space. Let (Ω, Σ) be the domain and $f = (f_1, \ldots, f_n) : (\Omega, \Sigma) \to (\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$. Suppose f is measurable, then every coordinate projection $f_i : \Omega \to \Omega_i$ is measurable.

This is also true for arbitrary index set.

Proposition. If $f : (\Omega, \Sigma) \to (\Omega_f, \Sigma_f)$ and $g : (\Omega, \Sigma) \to (\Omega_g, \Sigma_g)$, then the concatenation (f, g) is measurable w.r.t. the product space $(\Omega_f \times \Omega_g, \Sigma_f \otimes \Sigma_g)$.

Proof. Let $A \times B$ be such that $A \in \Sigma_f$ and $B \in \Sigma_g$. Then the preimage

$$(f,g)^{-1}(A \times B) = f^{-1}(A) \cap g^{-1}(B) \in \Sigma.$$

By definition, the product σ-field is generated by rectangles, so the proof is complete. $\square$

1.2 Measure space

Definition (measure)**.** Let (Ω, Σ) be a measurable space. Then $\mu : \Sigma \to [0, \infty]$ is a measure if

- $\mu(\emptyset) = 0$.
- If $A_i \in \Sigma$ is pairwise disjoint then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Proposition (continuity of measure)**.** If $A_1 \subset A_2 \subset \dots$ is a nested sequence of elements of Σ and μ be any measure on (Ω, Σ). Then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i).$$

If $A_1 \supset A_2 \supset \dots$ is a nested sequence of elements of Σ and $\mu(A_n) < \infty$ for some n. Then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i).$$

Definition. Let (Ω, Σ, μ) be a measure space.

Say μ is σ-finite if there is a representation $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ where $\Omega_i \in \Sigma$ and $\mu(\Omega_i) < \infty$.

Say μ is a probability measure if $\mu(\Omega) = 1$.

Definition (completion of measure space)**.** Let (Ω, Σ, μ) be a measure space. Let

$$\widetilde{\Sigma} = \left\{A \cup B : A \in \Sigma, B \subset \Omega, \text{there exists } C \in \Sigma \text{ with } \mu(C) = 0 \text{ and } B \subset C\right\}.$$

We can check $\widetilde{\Sigma}$ is a σ-field. If $\widetilde{\mu}$ is a measure on $(\Omega, \widetilde{\Sigma})$ which agrees with μ on Σ, then $(\Omega, \widetilde{\Sigma}, \widetilde{\mu})$ is called a completion of (Ω, Σ, μ).

1.3 π-λ theorem

Definition (π-system)**.** Let Ω be a set and $\mathcal{P}$ be a collection of subsets of Ω. Then $\mathcal{P}$ is a π-system if it is closed with respect to taking finite intersections. That is, $A, B \in \mathcal{P}$ implies $A \cap B \in \mathcal{P}$.

Example. On the real line $\mathbb{R}$, both $\mathcal{P}_1 = \{(a, b) : a < b\}$ and $\mathcal{P}_2 = \{(-\infty, a] : a \in \mathbb{R}\}$ are π-systems.

Definition (λ-system)**.** Let Ω be a set and $\mathcal{L}$ be a collection of subsets of Ω. Say $\mathcal{L}$ is a λ-system if

- $\emptyset \in \mathcal{L}$.
- $A \in \mathcal{L}$ implies $A^c \in \mathcal{L}$.

– for all countable collection of disjoint elements $A_i \in \mathcal{L}$, we have $\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$.

Theorem (π-λ theorem)**.** Let Ω be a set, $\mathcal{P}$ be a π-system and $\mathcal{L}$ be a λ-system. Also suppose $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

Proof. Let $\ell(\mathcal{P})$ be the smallest λ-system on Ω containing $\mathcal{P}$. The goal is to show that $\ell(\mathcal{P})$ is a σ-field. We need to show that if $A_i \in \ell(\mathcal{P})$ for $1 \leq i < \infty$, then $\bigcup_{i=1}^{\infty} A_i \in \ell(\mathcal{P})$. Note that

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left(A_i \setminus \bigcup_{j=1}^{i-1} A_i \right),$$

so it suffices to show that $A, B \in \ell(\mathcal{P})$ implies $A \cap B \in \ell(\mathcal{P})$.

For $A \in \ell(\mathcal{P})$ we define

$$W_A = \{B \subset \Omega : A \cap B \in \ell(\mathcal{P})\}.$$

It can be directly verified that W_A is a λ-system.

Take $A \in \mathcal{P}$, then for any $B \in \mathcal{P}$ we have $A \cap B \in \mathcal{P} \subset \ell(\mathcal{P})$. Hence, $\mathcal{P} \subset W_A$ and thus $\ell(\mathcal{P}) \subset W_A$ for all $A \in \mathcal{P}$, as $\ell(\mathcal{P})$ is the smallest λ-system on Ω containing $\mathcal{P}$. Now take $A \in \ell(\mathcal{P})$, we have $A \in W_B$ for all $B \in \mathcal{P}$. It follows that $A \cap B \in \ell(\mathcal{P})$ and thus $B \in W_A$. Hence similarly $\ell(\mathcal{P}) \subset W_A$ for all $A \in \ell(\mathcal{P})$.

Now for any pair $B, C \in \ell(\mathcal{P})$, we have $C \in W_B$ and thus $B \cap C \in \ell(\mathcal{P})$. This completes the proof. □

1.4 Extension theorems

Definition (semi-field)**.** A collection of subsets $S \subset 2^{\Omega}$ is a semi-field if

– $\emptyset \in S$ and $\Omega \in S$.

– $A, B \in S$ implies $A \cap B \in S$.

– If $A \in S$, then A^c is a finite disjoint union of sets in S.

Theorem (Caratheorody's extension theorem)**.** Let S be a semi-field and let μ be a non-negative function on S satisfying:

– $\mu(\emptyset) = 0$.

– If $A_1, \ldots, A_n$ are disjoint and $\bigcup_{i=1}^{n} A_i \in S$, then $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$.

– If $A_1, A_2, \ldots$ are such that $\bigcup_{i=1}^{\infty} A_i \in S$, then $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

Then μ admits a unique extension $\overline{\mu}$ which is a measure on $\overline{S}$, the field (algebra) generated by S. Moreover, if $\overline{\mu}$ is σ-finite then $\overline{\mu}$ admits a unique extension $\widetilde{\mu}$ to $\sigma(S)$.

Definition. Let T be any set. Write

$$\mathbb{R}^T = \left\{ (\omega_t)_{t \in T} : \omega_t \in \mathbb{R} \right\}.$$

Also write $\mathcal{R}^T$ as the σ-field generated by rectangles of the form $\prod_{t\in T} I_t$, where for each $t \in T$, I_t is either a semi-open interval of the form $(a, b]$ with $a < b$ or $I_t = \mathbb{R}$, and $I_t = \mathbb{R}$ for all but finitely many $t \in T$.

Theorem (Kolmogorov's extension theorem)**.** For each finite non-empty subset $J \subset T$, let μ_J be a Borel probability measure in $\mathbb{R}^J$, and assume that the measures $(\mu_J)_{J\subset T,|J|<\infty}$ are compatible, in the sense that whenever $J_1 \subset J_2 \subset T$ with $0 \le |J_1| \le |J_2| < \infty$, $I_j \subset \mathbb{R}$ with $j \in J_1$ are Borel subsets of $\mathbb{R}$, and

$$\widetilde{I}_j = \begin{cases} I_j & (j \in J_1) \\ \mathbb{R} & (j \in J_2 \setminus J_1), \end{cases}$$

one has

$$\mu_{J_2}\left(\prod_{j\in J_2} \widetilde{I}_j\right) = \mu_{J_1}\left(\prod_{j\in J_1} I_j\right).$$

Then there exists a unique probability measure μ on $(\mathbb{R}^T, \mathcal{R}^T)$ consistent with $(\mu_J)_{J\subset T,|J|<\infty}$. That is, one has

$$\mu\left(\prod_{t\in T} I_t\right) = \mu_J\left(\prod_{j\in J} I_j\right)$$

whenever $J \subset T$ with $|J| < \infty$ and $I_t = \mathbb{R}$ for all $t \notin J$.

1.5 Lebesgue Integration

Here we provide a proof for dominated convergence theorem that uses the truncation technique, which will be a useful technique later in the course.

Theorem (dominated convergence theorem)**.** Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions on (Ω, Σ, μ) and $g \ge 0$ be another measurable function. Suppose

1. $\int g\, d\mu < \infty$.
2. $|f_n|(\omega) \le g(\omega)$ for all $\omega \in \Omega$ and $n \ge 1$.
3. $f_n \to f$ pointwise.

Then

$$\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu.$$

Proof. **Claim 1.** If h is a function on (Ω, Σ, μ) with $h \ge 0$ and $\int h\, d\mu < \infty$. Let $\{A_n\}_{n=1}^\infty$ be any sequence of elements of Σ with $\mu(A_n) \to 0$. Then

$$\int_{A_n} h\, d\mu \to 0.$$

Proof of claim. WLOG assume $\mu(A_n) \le 2^{-n}$ for all n. Define $h_n = h\, \mathbb{1}_{\bigcup_{i=n}^\infty A_i}$. We then have

1. The sequence $\{h_n\}_{n=1}^{\infty}$ is monotone.
2. h_n converges to 0 almost everywhere.

Monotone convergence theorem then implies $\lim_{n\to\infty} \int h_n \, d\mu = 0$. Meanwhile,

$$0 \le \int_{A_n} h \, d\mu \le \int h_n \, d\mu,$$

and the proof is complete.

Claim 2. Suppose $h \ge 0$ and $\int h \, d\mu < \infty$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of strictly positive numbers converging to zero. Define

$$B_n = \{\omega \in \Omega : h(\omega) \le \varepsilon_n\} \in \Sigma.$$

Then

$$\int_{B_n} h \, d\mu \to 0.$$

Proof of this claim is left as an exercise.

Now we prove the theorem. Fix $\varepsilon > 0$. By the previous two claims, there exists $M > 0$ and $\delta > 0$ such that

$$\int_{\{g \ge M\}} g \, d\mu < \varepsilon, \quad \int_{\{g \le \delta\}} g \, d\mu < \varepsilon.$$

Let $U = \{\omega : \delta < g(\omega) < M\}$. Since g is integrable, $\mu(U) < \infty$. For $\omega \in U$, let $n_\varepsilon(\omega)$ be the smallest index such that $n \ge n_\varepsilon(\omega)$ implies $|f_n(\omega) - f(\omega)| \le \varepsilon \mu(U)^{-1}$. It follows that there exists N such that

$$\mu\left(\{\omega \in U : n_\varepsilon(\omega) > N\}\right) \le \frac{\varepsilon}{M}.$$

Then, for $n \ge N$, we have

$$\left| \int_U (f_n - f) \, d\mu \right| \le \int_{n_\varepsilon(\omega) \le N} |f_n - f| \, d\mu + \int_{n_\varepsilon(\omega) > N} |f_n - f| \, d\mu \le 3\varepsilon.$$

Now for $n \ge N$, we have

$$\left| \int (f_n - f) \, d\mu \right| \le 3\varepsilon + \int_{U^c} |f - f_n| \, d\mu \le 3\varepsilon + 2 \int_{U^c} g \, d\mu \le 7\varepsilon.$$

□

Theorem (Markov-Chebyshev inequality)**.** Suppose we have probability measure space $(\Omega, \Sigma, \mathbb{P})$ and $f \ge 0$. Suppose also $\int f \, d\mathbb{P} < \infty$. Then, for all $t > 0$, we have

$$\mathbb{P}\left(\{\omega : f(\omega) > t\}\right) \le t^{-1} \int f \, d\mathbb{P}.$$