Measure and Integration

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3 Signed measure and differentiation

Exercise (Folland 3.18)**.** Let ν be a complex measure on $(X, \mathfrak{M})$. Prove that $L^1(\nu) = L^1(|\nu|)$ and if $f \in L^1(\nu)$, then $\left|\int f\, d\nu\right| \leq \int |f|\; d\,|\nu|$.

Proof. For $L^1(\nu) \subset L^1(|\nu|)$, consider $f \in L^1(\nu)$. Note that $\nu = \nu_r + i\nu_i$ and it is easy to verify that $|i\nu_i| = |\nu_i|$. Therefore by Proposition 3.14, we have $|\nu| \leq |\nu_r| + |\nu_i|$. It follows that

$$\begin{aligned}\int |f|\; d\,|\nu| &\leq \int |f|\; d\,|\nu_r| + \int |f|\; d\,|\nu_i| \\ &= \int |f|\; d\nu_r^+ + \int |f|\; d\nu_r^- + \int |f|\; d\nu_i^+ + \int |f|\; d\nu_i^-.\end{aligned}$$

Since $f \in L^1(\nu)$, all four terms are finite and thus $f \in L^1(|\nu|)$.

For $L^1(|\nu|) \subset L^1(\nu)$, consider $f \in L^1(|\nu|)$. Then we have

$$\int |f|\; d\nu = \int |f| \frac{d\nu}{d\,|\nu|}\, d\,|\nu| \leq \int |f| \left|\frac{d\nu}{d\,|\nu|}\right| d\,|\nu| = \int |f|\; d\,|\nu|\,,$$

where we have used the fact that $d\nu/d\,|\nu|$ has absolute value 1 $|\nu|$-a.e. This shows that $f \in L^1(\nu)$.

Moreover, we have that $\left|\int f\, d\nu\right| \leq \int |f|\; d\nu$. Therefore,

$$\left|\int f\, d\nu\right| \leq \int |f|\; d\,|\nu|\,,$$

as desired. □

Exercise (Folland 3.19)**.** If ν, μ are complex measures and λ is a positive measure, then $\nu \perp \mu$ if and only if $|\nu| \perp |\mu|$, and $\nu \ll \lambda$ if and only if $|\nu| \ll \lambda$.

Proof. ***** TO-DO ***** □

Exercise (Folland 3.20)**.** If ν is a complex measure on $(X, \mathfrak{M})$ and $\nu(X) = |\nu|\,(X)$, then $\nu = |\nu|$.

Proof. By Lebesgue-Randon-Nikodym theorem, we have $d\nu = f\, d\mu$ for some function f and positive measure μ. It follows that $d\,|\nu| = |f|\; d\mu$ and

$$\int f\, d\mu = \int |f|\; d\mu.$$

This shows that $f = |f|$ μ-a.e., and thus $\nu = |\nu|$. □