Probability

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2 Laws of large numbers

2.2 Weak laws of large numbers

Theorem (L^2 weak law)**.** Let $X_1, X_2, \ldots$ be uncorrelated random variables with $EX_i = \mu$ and $EX_i^2 \leq C < \infty$. If $S_n = X_1 + \cdots + X_n$ then as $n \to \infty$, $S_n/n \to \mu$ in L^2 and in probability.

Lemma. If $Y \geq 0$ and $p > 0$ then

$$E(Y^p) = \int_0^\infty p y^{p-1} P(Y > y)\, dy.$$

2.3 Borel-Cantelli Lemmas

Definition. Let A_n be a sequence of subsets of Ω, define

$$\limsup A_n = \bigcap_{m=0}^{\infty} \bigcup_{n=m}^{\infty} A_n = \{\omega \in A_n \text{ infintely often}\},$$
$$\liminf A_n = \bigcup_{m=0}^{\infty} \bigcap_{n=m}^{\infty} A_n = \{\omega \text{ in all but finitely many } A_n\}.$$

Proposition. We have

$$\limsup_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_{\limsup A_n}, \quad \liminf_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_{\liminf A_n},$$

and

$$P(\limsup A_n) \geq \limsup P(A_n), \quad P(\liminf A_n) \leq \liminf P(A_n).$$

Theorem (Borel-Cantelli Lemma)**.** If $\sum_{n=1}^\infty P(A_n) < \infty$, then

$$P(A_n \text{ i.o.}) = 0.$$

Theorem. $X_n \to X$ in probability iff for every subsequence $X_{n(m)}$, there is a further subsequence $X_{n(m_k)}$ such that $X_{n(m_k)}$ converges almost surely to X.

Theorem (First strong law of large numbers)**.** Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = \mu$ and $EX_i^4 < \infty$. If $S_n = X_1 + \cdots + X_n$, then $S_n/n \to \mu$ a.s.

Theorem (The 2nd Borel-Cantelli Lemma)**.** If events A_n are independent, then $\sum_{n=1}^\infty P(A_n) = 1$ implies $P(A_n \text{ i.o.}) = 1$.

Theorem. If $X_1, X_2, \ldots$ are i.i.d. with $E|X_i| = \infty$, then $P(|X_n| \geq n \text{ i.o.}) = 1$. Therefore, if $S_n = X_1 + \cdots + X_n$ then

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} \text{ exists in } (-\infty, \infty)\right) = 0.$$

Theorem. If $A_1, A_2, \ldots$ are pairwise independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then

$$\frac{\sum_{m=1}^{n} \mathbf{1}_{A_m}}{\sum_{m=1}^{n} P(A_n)} \to 1 \text{ a.s.}$$

as $n \to \infty$.

2.4 Strong law of large numbers

Theorem (Strong law of large numbers)**.** Let $X_1, X_2, \ldots$ be pairwise independent identically distributed random variables with $E|X_i| < \infty$. Let $EX_i = \mu$ and $S_n = X_1 + \cdots + X_n$. Then $S_n/n \to \mu$ a.s. as $n \to \infty$.

Theorem. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i^+ = \infty$ and $EX_i^- < \infty$. If $S_n = X_1 + \cdots + X_n$, then $S_n/n \to \infty$ a.s.