Measure and Integration

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Contents

3 Signed measure and differentiation

Exercise (Folland 3.2)**.** If ν a signed measure, E is ν-null if and only if $|\nu|(E) = 0$. Also if ν and μ are signed measures, then $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof. It is clear that $|\nu|(E) = 0$ implies that E is ν-null. Now suppose E is ν-null and let $X = P \cup N$ be the Hahn decomposition of ν. Suppose for contradiction that $|\nu|(E) > 0$, then it follows that $\nu^+(E) > 0$ and $\nu^-(E) > 0$. We then have $\nu^+(E \cap P) = \nu^+(E \cap P) + \nu^+(E \cap N) = \nu^+(E) > 0$, but $\nu^-(E \cap P) \leq \nu^-(P) = 0$. Therefore, $\nu(E \cap P) > 0$, a contradiction with E being ν-null. Therefore, $|\nu|(E) = 0$.

Suppose $\nu \perp \mu$, then there is $X = E \cup F$ such that E is ν-null and F is μ-null. It follows that $|\nu|(E) = 0$, so $|\nu| \perp \mu$. Therefore, $\nu \perp \mu$ implies $|\nu| \perp \mu$. It is clear that $|\nu| \perp \mu$ implies $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Now suppose $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Then we have $X = E^+ \cup F^+ = E^- \cup F^-$ where F^+ is ν^+ null, F^- is ν^- null, and $E^\pm$ is μ-null. Let $E = E^+ \cup E^-$ and $F = F^+ \cap F^- = E^c$. Then we can verify that E is μ-null and F is ν-null. Therefore, $\nu \perp \mu$ and the proof is complete. □

Exercise (Folland 3.3)**.** Let ν be a signed measure on $(X, \mathfrak{M})$.

1. $L^1(\nu) = L^1(|\nu|)$.
2. If $f \in L^1(\nu)$, then $\left|\int f \, d\nu\right| \leq \int |f| \, d|\nu|$.
3. If $E \in \mathfrak{M}$, $|\nu|(E) = \sup\left\{\left|\int_E f \, d\nu\right| : |f| \leq 1\right\}$

Proof.

1. Since we have
$$\int |f| \, d\nu = \int |f| \, d\nu^+ - \int |f| \, d\nu^-, \quad \int |f| \, d|\nu| = \int |f| \, d\nu^+ + \int |f| \, d\nu^-,$$
it follows immediately that $L^1(\nu) = L^1(|\nu|)$.
2. We have
$$\left|\int f \, d\nu\right| = \left|\int f d\nu^+ - \int f \, d\nu^-\right| \leq \int |f| \, d\nu^+ + \int f \, d\nu^- = \int |f| \, d|\nu|.$$
3. By the previous item, we know
$$|\nu|(E) = \int_E 1 \, d|\nu| \geq \int_E |f| \, d|\nu| \geq \left|\int f \, d\nu\right|.$$
for any $|f| \leq 1$. Also, $|\nu|(E) = \int |\chi_P + \chi_N| \, d\nu$, where $X = P \cup N$ is the Hahn decomposition so $|\chi_P + \chi_N| = 1$.

□

Exercise (Folland 3.4)**.** If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Proof. Let $\nu = \nu^+ - \nu^-$ be the unique decomposition. We then have $\lambda - \nu^+ = \mu - \nu^-$. Pick a set $E \in \mathfrak{M}$ such that $\nu^-(E) = 0$. We then have

$$\lambda(E) - \nu^+(E) = \mu(E) \geq 0.$$

Therefore $\lambda(E) \geq \nu^+(E)$. On the other hand, for $E \in \mathfrak{M}$ such that $\nu^+(E) = 0$, we have $\lambda(E) \geq 0 = \nu^+(E)$. In light of the Hahn decomposition for ν and the additivity of measure, we can conclude that $\lambda \geq \nu^+$, and thus $\mu \geq \nu^-$. $\square$

Exercise (Folland 3.7)**.** Suppose ν is a signed measure on $(X, \mathfrak{M})$ and $E \in \mathfrak{M}$.

1. $\nu^+(E) = \sup\{\nu(F) : F \subset E, F \in \mathfrak{M}\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \subset E, F \in \mathfrak{M}\}$.
2. $|\nu|(E) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \ldots, E_n \text{ disjoint, and } \bigcup_{j=1}^n E_j = E\right\}$.

Proof. 1. First let $F \subset E$, we have $\nu(F) = \nu^+(F) - \nu^-(F)$. It follows that $\nu(F) \leq \nu^+(F) \leq \nu^+(E)$. Let $X = P \cup N$ be the Hahn decomposition for ν, then we have $\nu^+(E) = \nu(E \cap P)$. Similarly for $\nu^-(E)$.

2. Let $E_1, \ldots, E_n$ be disjoint and $\bigcup_{j=1}^n E_j = E$. It follows that

$$|\nu|(E) = \sum_{j=1}^n |\nu|(E_j) \geq \sum_{j=1}^n |\nu(E_j)|.$$

Let $X = P \cup N$ be the Hahn decomposition for ν, we then have

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu^+(E \cap P) + \nu^-(E \cap N) = |\nu(E \cap P)| + |\nu(E \cap N)|.$$

This completes the proof.

$\square$

Exercise (Folland 3.8)**.** $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Proof. It is clear that $|\nu| \ll \mu$ implies $\nu^+ \ll \mu$ and $\nu^- \ll \mu$ implies $\nu \ll \mu$. It then remains to show that $\nu \ll \mu$ implies $|\nu| \ll \mu$. Let $X = P \cup N$ be the Hahn decomposition for ν and $E \in \mathfrak{M}$ be such that $\mu(E) = 0$. It follows that $\mu(E \cap P) = 0$ and $\mu(E \cap N) = 0$. Then $\nu(E \cap P) = \nu^+(E \cap P) = 0$ and $\nu(E \cap N) = \nu^-(E \cap N) = 0$. Therefore,

$$\nu(E) = \nu^+(E \cap P) + \nu^-(E \cap N) = 0,$$

as desired. $\square$

Exercise (Folland 3.18)**.** Let ν be a complex measure on $(X, \mathfrak{M})$. Prove that $L^1(\nu) = L^1(|\nu|)$ and if $f \in L^1(\nu)$, then $\left|\int f \, d\nu\right| \leq \int |f| \, d|\nu|$.

Proof. For $L^1(\nu) \subset L^1(|\nu|)$, consider $f \in L^1(\nu)$. Note that $\nu = \nu_r + i\nu_i$ and it is easy to verify that $|i\nu_i| = |\nu_i|$. Therefore by Proposition 3.14, we have $|\nu| \leq |\nu_r| + |\nu_i|$. It follows that

$$\begin{aligned}\int |f| \, d|\nu| &\leq \int |f| \, d|\nu_r| + \int |f| \, d|\nu_i| \\ &= \int |f| \, d\nu_r^+ + \int |f| \, d\nu_r^- + \int |f| \, d\nu_i^+ + \int |f| \, d\nu_i^- .\end{aligned}$$

Since $f \in L^1(\nu)$, all four terms are finite and thus $f \in L^1(|\nu|)$.

For $L^1(|\nu|) \subset L^1(\nu)$, consider $f \in L^1(|\nu|)$. Then we have

$$\int |f| \, d\nu = \int |f| \frac{d\nu}{d|\nu|} \, d|\nu| \leq \int |f| \left| \frac{d\nu}{d|\nu|} \right| \, d|\nu| = \int |f| \, d|\nu| ,$$

where we have used the fact that $d\nu / d|\nu|$ has absolute value 1 $|\nu|$-a.e. This shows that $f \in L^1(\nu)$.

Moreover, we have that $\left| \int f \, d\nu \right| \leq \int |f| \, d\nu$. Therefore,

$$\left| \int f \, d\nu \right| \leq \int |f| \, d|\nu| ,$$

as desired. □

Exercise (Folland 3.19)**.** If ν, μ are complex measures and λ is a positive measure, then $\nu \perp \mu$ if and only if $|\nu| \perp |\mu|$, and $\nu \ll \lambda$ if and only if $|\nu| \ll \lambda$.

Proof. The "if" direction is clear for both propositions.

Suppose $\nu \perp \mu$. Then $\nu_r \perp \mu_r$, $\nu_i \perp \mu_r$, $\nu_r \perp \mu_i$, and $\nu_i \perp \mu_i$. It follows that $|\nu_r| \perp \mu_r$ and $|\nu_i| \perp \mu_r$. Since $|\nu| \leq |\nu_r| + |\nu_i|$, we have $|\nu| \perp \mu_r$. Similarly $|\nu| \perp \mu_i$. Following the same reasoning, we obtain $|\nu| \perp |\mu|$, as desired.

Suppose $\nu \ll \lambda$. Since $\nu = \nu_r + i\nu_i$, we have $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$. Recall from Exercise 3.8 that this implies $|\nu_r| \ll \lambda$ and $|\nu_i| \ll \lambda$. Moreover, $|\nu| \leq |\nu_r| + |\nu_i|$, so $|\nu| \ll \lambda$. □

Exercise (Folland 3.20)**.** If ν is a complex measure on $(X, \mathfrak{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

Proof. By Lebesgue-Randon-Nikodym theorem, we have $d\nu = f \, d\mu$ for some function f and positive measure μ. It follows that $d|\nu| = |f| \, d\mu$ and

$$\int f \, d\mu = \int |f| \, d\mu.$$

Now let $E \in \mathfrak{M}$. We then have

$$\int_E f \, d\mu + \int_{E^c} f \, d\mu = \int_E |f| \, d\mu + \int_{E^c} |f| \, d\mu.$$

It follows that

$$0 \leq \int_E |f| - \mathrm{Re}(f)\, d\mu = \int_{E^c} \mathrm{Re}(f) - |f|\, d\mu \leq 0.$$

Since $E \in \mathfrak{M}$ is arbitrary, $\mathrm{Re}(f) = |f|$ and $\mathrm{Im}(f) = 0$ a.e. It follows that $d\nu = f\, d\mu = |f|\, d\mu = d\,|\nu|$. $\square$