Probability

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2 Laws of large numbers

2.2 Weak laws of large numbers

Theorem (L^2 weak law)**.** Let $X_1, X_2, \ldots$ be uncorrelated random variables with $EX_i = \mu$ and $EX_i^2 \leq C < \infty$. If $S_n = X_1 + \cdots + X_n$ then as $n \to \infty$, $S_n/n \to \mu$ in L^2 and in probability.

Lemma. If $Y \geq 0$ and $p > 0$ then

$$E(Y^p) = \int_0^\infty p y^{p-1} P(Y > y)\, dy.$$

Theorem (Weak law of triangular arrays)**.** For each n let $X_{n,k}$, $1 \leq k \leq n$ be independent. Let $b_n > 0$ with $b_n \to \infty$, and let $\overline{X}_{n,k} = X_{n,k}\, \mathbf{1}_{(|X_{n,k}| < b_n)}$. Suppose that as $n \to \infty$,

1. $\sum_{k=1}^n P(|X_{n,k}| > b_n) \to 0$, and
2. $b_n^{-2} \sum_{k=1}^n E\overline{X}_{n,k}^2 \to 0$.

If we let $S_n = X_{n,1} + \cdots + X_{n,n}$ and put $a = \sum_{k=1}^n E\overline{X}_{n,k}$, then

$$\frac{S_n - a_n}{b_n} \to 0 \text{ in probability.}$$

Theorem (Weak law of large numbers)**.** Let $X_1, X_2, \ldots$ be i.i.d. with

$$xP(|X_i| > x) \to 0$$

as $x \to \infty$. Let $S_n = X_1 + \cdots + X_n$ and let $\mu_n = E(X_1\, \mathbf{1}_{(|X_1| \leq n)})$. Then $S_n/n - \mu_n \to 0$ in probability.

Theorem. Let $X_1, X_2, \ldots$ be i.i.d. with $E|X_i| < \infty$. Let $S = X_1 + \cdots + X_n$ and let $\mu = EX_1$. Then $S_n/n \to \mu$ in probability.

2.3 Borel-Cantelli Lemmas

Definition. Let A_n be a sequence of subsets of Ω, define

$$\limsup A_n = \bigcap_{m=0}^\infty \bigcup_{n=m}^\infty A_n = \{\omega \in A_n \text{ infintely often}\},$$
$$\liminf A_n = \bigcup_{m=0}^\infty \bigcap_{n=m}^\infty A_n = \{\omega \text{ in all but finitely many } A_n\}.$$

Proposition. We have

$$\limsup_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_{\limsup A_n}, \quad \liminf_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_{\liminf A_n},$$

and

$$P(\limsup A_n) \geq \limsup P(A_n), \quad P(\liminf A_n) \leq \liminf P(A_n).$$

Theorem (Borel-Cantelli Lemma)**.** If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then

$$P(A_n \text{ i.o.}) = 0.$$

Theorem. $X_n \to X$ in probability iff for every subsequence $X_{n(m)}$, there is a further subsequence $X_{n(m_k)}$ such that $X_{n(m_k)}$ converges almost surely to X.

Theorem (First strong law of large numbers)**.** Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = \mu$ and $EX_i^4 < \infty$. If $S_n = X_1 + \cdots + X_n$, then $S_n/n \to \mu$ a.s.

Theorem (Second Borel-Cantelli Lemma)**.** If events A_n are independent, then $\sum_{n=1}^{\infty} P(A_n) = 1$ implies $P(A_n \text{ i.o.}) = 1$.

Theorem. If $X_1, X_2, \ldots$ are i.i.d. with $E|X_i| = \infty$, then $P(|X_n| \geq n \text{ i.o.}) = 1$. Therefore, if $S_n = X_1 + \cdots + X_n$ then

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} \text{ exists in } (-\infty, \infty)\right) = 0.$$

Theorem. If $A_1, A_2, \ldots$ are pairwise independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then

$$\frac{\sum_{m=1}^{n} \mathbf{1}_{A_m}}{\sum_{m=1}^{n} P(A_n)} \to 1 \text{ a.s.}$$

as $n \to \infty$.

2.4 Strong law of large numbers

Theorem (Strong law of large numbers)**.** Let $X_1, X_2, \ldots$ be pairwise independent identically distributed random variables with $E|X_i| < \infty$. Let $EX_i = \mu$ and $S_n = X_1 + \cdots + X_n$. Then $S_n/n \to \mu$ a.s. as $n \to \infty$.

Theorem. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i^+ = \infty$ and $EX_i^- < \infty$. If $S_n = X_1 + \cdots + X_n$, then $S_n/n \to \infty$ a.s.