

# Probability

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Spring 2026

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# 1 Measure theory review

## 1.1 Measurable space and mapping

**Definition** ( $\sigma$ -field). A set of sets  $\Sigma \subset 2^\Omega$  is a  $\sigma$ -field if

- $\emptyset \in \Sigma$ ,
- If  $A \in \Sigma$ , then  $A^c \in \Sigma$ ,
- If  $\{A_i\}_{i=1}^\infty \subset \Sigma$ , then  $\bigcup_{i=1}^\infty A_i \in \Sigma$ .

The pair  $(\Omega, \Sigma)$  is called a measurable space.

**Definition** (atom). Let  $\Sigma$  be a  $\sigma$ -field. Say  $A \in \Sigma$  is an atom if for all  $B \in \Sigma$  either  $A \subset B$  or  $A \cap B = \emptyset$ .

**Proposition.** For all  $\omega \in \Omega$ , there exists atom  $A \in \Sigma$  containing  $\omega$  if  $\Omega$  is finite or countable.

*Proof.* Only prove this for  $\Omega$  finite. Define  $\tilde{A} = \bigcap \{B \in \Sigma : \omega \in B\}$ .  $\square$

**Corollary.** If  $\Omega$  is finite or countable, there exists a partition  $\Omega = \bigsqcup_i \Omega_i$ , where each  $\Omega_i$  is an atom of  $\Sigma$ . With this partition,  $\Sigma$  is just the power set with respect to  $\{\Omega_i\}_i$ .

**Definition.** If  $F \subset 2^\Omega$ , then the  $\sigma$ -field generated by  $F$  is the smallest  $\sigma$ -field containing all elements of  $F$ .

**Example.** Let  $\Omega = \{1, 2, 3, 4, 5\}$  and  $F = \{\{2, 3\}, \{3, 4\}\}$ . Construct  $\sigma$ -field  $\Sigma$  generated by  $F$ .  $\Sigma$  is all possible union of sets from the collection  $\{\{2\}, \{3\}, \{4\}, \{1, 5\}\}$ .

**Definition** (measurable mapping). Given two measurable spaces  $(\Omega, \Sigma)$  and  $(\tilde{\Omega}, \tilde{\Sigma})$ . Then  $f : \Omega \rightarrow \tilde{\Omega}$  is measurable if  $f^{-1}(B) \in \Sigma$  for all  $B \in \tilde{\Sigma}$ .

**Definition** (Borel  $\sigma$ -field). Let  $(T, \tau)$  be a topological space. Then the Borel  $\sigma$ -field  $\mathcal{B}(T, \tau)$  is defined as the smallest  $\sigma$ -field containing all open sets.

**Definition** (product measurable space). Given two measurable spaces  $(\Omega, \Sigma)$  and  $(\tilde{\Omega}, \tilde{\Sigma})$ . We can define the product measurable space as follows: let the ground set be  $\Omega \times \tilde{\Omega}$ , and let  $\Sigma \otimes \tilde{\Sigma}$  be the smallest  $\sigma$ -field containing all rectangles  $B \times \tilde{B}$  where  $B \in \Sigma$  and  $\tilde{B} \in \tilde{\Sigma}$ .

More generally, let  $\Lambda$  be an index set and  $(\Omega_\lambda, \Sigma_\lambda)_{\lambda \in \Lambda}$ . Define the product  $\sigma$ -field  $\bigotimes_{\lambda \in \Lambda} \Sigma_\lambda$  be the smallest  $\sigma$ -field containing all elements in the form of  $\prod_{\lambda \in \Lambda} B_\lambda$  where  $B_\lambda \in \Sigma_\lambda$  and  $B_\lambda = \Omega_\lambda$  for all but countably many indices.

**Proposition.** Let  $(\Omega_i, \Sigma_i)_{i=1}^n$  be measurable spaces and  $(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$  be the product space. Let  $(\Omega, \Sigma)$  be the domain and  $f = (f_1, \dots, f_n) : (\Omega, \Sigma) \rightarrow (\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$ . Suppose  $f$  is measurable, then every coordinate projection  $f_i : \Omega \rightarrow \Omega_i$  is measurable.

This is also true for arbitrary index set.

## 1.2 Measure space

**Definition** (measure). Let  $(\Omega, \Sigma)$  be a measurable space. Then  $\mu : \Sigma \rightarrow [0, \infty]$  is a measure if

- $\mu(\emptyset) = 0$ .
- If  $A_i \in \Sigma$  is pairwise disjoint then  $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ .

**Proposition** (continuity of measure). If  $A_1 \subset A_2 \subset \dots$  is a nested sequence of elements of  $\Sigma$  and  $\mu$  be any measure on  $(\Omega, \Sigma)$ . Then

$$\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \lim_{i \rightarrow \infty} \mu(A_i).$$

If  $A_1 \supset A_2 \supset \dots$  is a nested sequence of elements of  $\Sigma$  and  $\mu(A_n) < \infty$  for some  $n$ . Then

$$\mu \left( \bigcap_{i=1}^{\infty} A_i \right) = \lim_{i \rightarrow \infty} \mu(A_i).$$