Probability

Notes taken by Runqiu Ye
Carnegie Mellon University

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1 Measure theory review

1.1 Measurable space and mapping

Definition (atom)**.** Let Σ be a σ-field. Say $A \in \Sigma$ is an atom if for all $B \in \Sigma$ either $A \subset B$ or $A \cap B = \emptyset$.

Proposition. For all $\omega \in \Omega$, there exists atom $A \in \Sigma$ containing ω if Ω is finite or countable.

Proof. Only prove this for Ω finite. Define $\widetilde{A} = \bigcap \{B \in \Sigma : \omega \in B\}$. □

Corollary. If Ω is finite or countable, there exists a partition $\Omega = \bigsqcup_i \Omega_i$, where each Ω_i is an atom of Σ. With this partition, Σ is just the power set with respect to $\{\Omega_i\}_i$.

Definition. If $F \subset 2^{\Omega}$, then the σ-field generated by F is the smallest σ-field containing all elements of F.

Example. Let $\Omega = \{1,2,3,4,5\}$ and $F = \{\{2,3\},\{3,4\}\}$. Construct σ-field Σ generated by F. Σ is all possible union of sets from the collection $\{\{2\},\{3\},\{4\},\{1,5\}\}$.

Definition (measurable mapping)**.** Given two measurable spaces (Ω, Σ) and $(\widetilde{\Omega}, \widetilde{\Sigma})$. Then $f : \Omega \to \widetilde{\Omega}$ is measurable if $f^{-1}(B) \in \Sigma$ for all $B \in \widetilde{\Sigma}$.

Definition (Borel σ-field)**.** Let (T, τ) be a topological space. Then the Borel σ-field $\mathcal{B}(T, \tau)$ is defined as the smallest σ-field containing all open sets.

Definition (product measurable space)**.** Given two measurable spaces (Ω, Σ) and $(\widetilde{\Omega}, \widetilde{\Sigma})$. We can define the product measurable space as follows: let the ground set be $\Omega \times \widetilde{\Omega}$, and let $\Sigma \otimes \widetilde{\Sigma}$ be the smallest σ-field containing all rectangles $B \times \widetilde{B}$ where $B \in \Sigma$ and $\widetilde{B} \in \widetilde{\Sigma}$.

More generally, let Λ be an index set and $(\Omega_\lambda, \Sigma_\lambda)_{\lambda \in \Lambda}$. Define the product σ-field $\bigotimes_{\lambda \in \Lambda} \Sigma_\lambda$ be the smallest σ-field containing all elements in the form of $\prod_{\lambda \in \Lambda} B_\lambda$ where $B_\lambda \in \Sigma_\lambda$ and $B_\lambda = \Omega_\lambda$ for all but countably many indices.

Proposition. Let $(\Omega_i, \Sigma_i)_{i=1}^n$ be measurable spaces and $(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$ be the product space. Let (Ω, Σ) be the domain and $f = (f_1, \ldots, f_n) : (\Omega, \Sigma) \to (\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$. Suppose f is measurable, then every coordinate projection $f_i : \Omega \to \Omega_i$ is measurable.

This is also true for arbitrary index set.

1.2 Measure space

Definition (measure)**.** Let (Ω, Σ) be a measurable space. Then $\mu : \Sigma \to [0, \infty]$ is a measure if

- $\mu(\emptyset) = 0$.
- If $A_i \in \Sigma$ is pairwise disjoint then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Proposition (continuity of measure)**.** If $A_1 \subset A_2 \subset \dots$ is a nested sequence of elements of Σ and μ be any measure on (Ω, Σ). Then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

If $A_1 \supset A_2 \supset \dots$ is a nested sequence of elements of Σ and $\mu(A_n) < \infty$ for some n. Then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$