Measure and Integration

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Contents

1 Measures

1.5 Borel measures on the real line

Exercise. Let $E \subset \mathbb{R}$ and assume that $m^*(E) < \infty$. Prove that E is measurable if and only if for every $\varepsilon > 0$, there exists a finite union of open intervals U such that $m^*(E \triangle U) < \varepsilon$.

Proof. ($\Longrightarrow$) See more general case below.

($\Longleftarrow$) Let $\varepsilon > 0$ be given and let $A \subset \mathbb{R}$. There then exists a finite union of open intervals U such that $m^*(E \triangle U) < \varepsilon$. Note that U is measurable, so

$$m^*(A \cap U) + m^*(A \cap U^c) = m^*(A)$$

However, $E \subset U \cup (E \cap U^c)$ so

$$m^*(A \cap E) \leq m^*(A \cap U) + m^*(A \cap E \cap U^c) \leq m^*(A \cap U) + \varepsilon.$$

Similarly, $E^c \subset (E^c \cap U) \cup U^c$

$$m^*(A \cap E^c) = m^*(A \cap E^c \cap U) + m^*(A \cap U^c) \leq \varepsilon + m^*(A \cap U^c).$$

This implies that

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary this implies that E is measurable. $\square$

Exercise (Folland 1.26)**.** If $E \in \mathfrak{M}_\mu$ and $\mu(E) < \infty$, then for every $\varepsilon > 0$ there is a finite union of open itervals U such that $\mu(E \triangle U) < \varepsilon$.

Proof. Recall that

$$\mu(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j)) : E \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}.$$

It follows that there exists open intervals $\{I_j\}_{j=1}^{\infty}$ such that $E \subset \bigcup_{j=1}^{\infty} I_j$ and $\mu(\bigcup_{j=1}^{\infty} I_j) \leq \mu(E) + \varepsilon$. By monotone continuity, there exists N such that

$$\mu\left(\bigcup_{j=1}^{N} I_j\right) \geq \mu\left(\bigcup_{j=1}^{\infty} I_j\right) - \varepsilon.$$

Let $U = \bigcup_{j=1}^{N} I_j$. This is a finite union of open intervals. We then have $\mu(U \cap E^c) \leq \mu(\bigcup_{j=1}^{\infty} I_j \setminus E) \leq \varepsilon$. Meanwhile, $\mu(E \cap U^c) \leq \mu(\bigcup_{j=N+1}^{\infty} I_j) \leq \varepsilon$. This impies that

$$\mu(E \triangle U) = \mu(E \cap U^c) + \mu(E^c \cap U) \leq 2\varepsilon.$$

Since ε is arbitrary this completes the proof. $\square$

3 Signed measure and differentiation

3.1 Signed measures

Exercise (Folland 3.2)**.** If ν a signed measure, E is ν-null if and only if $|\nu|(E) = 0$. Also if ν and μ are signed measures, then $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof. It is clear that $|\nu|(E) = 0$ implies that E is ν-null. Now suppose E is ν-null and let $X = P \cup N$ be the Hahn decomposition of ν. Suppose for contradiction that $|\nu|(E) > 0$, then it follows that $\nu^+(E) > 0$ and $\nu^-(E) > 0$. We then have $\nu^+(E \cap P) = \nu^+(E \cap P) + \nu^+(E \cap N) = \nu^+(E) > 0$, but $\nu^-(E \cap P) \leq \nu^-(P) = 0$. Therefore, $\nu(E \cap P) > 0$, a contradiction with E being ν-null. Therefore, $|\nu|(E) = 0$.

Suppose $\nu \perp \mu$, then there is $X = E \cup F$ such that E is ν-null and F is μ-null. It follows that $|\nu|(E) = 0$, so $|\nu| \perp \mu$. Therefore, $\nu \perp \mu$ implies $|\nu| \perp \mu$. It is clear that $|\nu| \perp \mu$ implies $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Now suppose $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Then we have $X = E^+ \cup F^+ = E^- \cup F^-$ where F^+ is ν^+ null, F^- is ν^- null, and $E^\pm$ is μ-null. Let $E = E^+ \cup E^-$ and $F = F^+ \cap F^- = E^c$. Then we can verify that E is μ-null and F is ν-null. Therefore, $\nu \perp \mu$ and the proof is complete. $\square$

Exercise (Folland 3.3)**.** Let ν be a signed measure on $(X, \mathfrak{M})$.

1. $L^1(\nu) = L^1(|\nu|)$.
2. If $f \in L^1(\nu)$, then $\left|\int f \, d\nu\right| \leq \int |f| \, d|\nu|$.
3. If $E \in \mathfrak{M}$, $|\nu|(E) = \sup\left\{\left|\int_E f \, d\nu\right| : |f| \leq 1\right\}$

Proof.

1. Since we have

$$\int |f| \, d\nu = \int |f| \, d\nu^+ - \int |f| \, d\nu^-, \quad \int |f| \, d|\nu| = \int |f| \, d\nu^+ + \int |f| \, d\nu^-,$$

it follows immediately that $L^1(\nu) = L^1(|\nu|)$.

2. We have

$$\left|\int f \, d\nu\right| = \left|\int f d\nu^+ - \int f \, d\nu^-\right| \leq \int |f| \, d\nu^+ + \int f \, d\nu^- = \int |f| \, d|\nu| \, .$$

3. By the previous item, we know

$$|\nu|(E) = \int_E 1 \, d|\nu| \geq \int_E |f| \, d|\nu| \geq \left|\int f \, d\nu\right| .$$

for any $|f| \leq 1$. Also, $|\nu|(E) = \left|\int (\chi_P + \chi_N) \, d\nu\right|$, where $X = P \cup N$ is the Hahn decomposition so $|\chi_P + \chi_N| = 1$.

$\square$

Exercise (Folland 3.4)**.** If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Proof. Let $\nu = \nu^+ - \nu^-$ be the unique decomposition. We then have $\lambda - \nu^+ = \mu - \nu^-$. Pick a set $E \in \mathfrak{M}$ such that $\nu^-(E) = 0$. We then have

$$\lambda(E) - \nu^+(E) = \mu(E) \geq 0.$$

Therefore $\lambda(E) \geq \nu^+(E)$. On the other hand, for $E \in \mathfrak{M}$ such that $\nu^+(E) = 0$, we have $\lambda(E) \geq 0 = \nu^+(E)$. In light of the Hahn decomposition for ν and the additivity of measure, we can conclude that $\lambda \geq \nu^+$, and thus $\mu \geq \nu^-$. $\square$

Exercise (Folland 3.5)**.** If ν_1, ν_2 are both signed measures that omits the values $\pm\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Use Exercise 3.4)

Proof. Note that we have $\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)$. Write $\nu = \nu_1 + \nu_2$. By Exercise 3.4, we have $\nu_1^+ + \nu_2^+ \geq \nu^+$ and $\nu_1^- + \nu_2^- \geq \nu^-$. It follows that

$$|\nu_1 + \nu_2| = \nu^+ + \nu^- \leq \nu_1^+ + \nu_2^+ + \nu_1^- + \nu_2^- = |\nu_1| + |\nu_2|,$$

as desired. $\square$

Exercise (Folland 3.6)**.** Suppose $\nu(E) = \int_E f \, d\mu$ where μ is a positive measure and f is an extended μ-integrable function. Describe the Hahn decomposition for ν and express the positive, negative, and total variation of ν in terms of f and μ.

Proof. A Hahn decomposition of ν is $X = P \cup N$ where $P = \{f \geq 0\}$ and $N = P^c$. We also have

$$\nu^+(E) = \int_E f^+ \, d\mu, \quad \nu^-(E) = \int_E f^- \, d\mu, \quad |\nu|(E) = \int_E |f| \, d\mu.$$

$\square$

Exercise (Folland 3.7)**.** Suppose ν is a signed measure on $(X, \mathfrak{M})$ and $E \in \mathfrak{M}$.

1. $\nu^+(E) = \sup\{\nu(F) : F \subset E, F \in \mathfrak{M}\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \subset E, F \in \mathfrak{M}\}$.
2. $|\nu|(E) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \ldots, E_n \text{ disjoint, and } \bigcup_{j=1}^n E_j = E\right\}$.

Proof. 1. First let $F \subset E$, we have $\nu(F) = \nu^+(F) - \nu^-(F)$. It follows that $\nu(F) \leq \nu^+(F) \leq \nu^+(E)$. Let $X = P \cup N$ be the Hahn decomposition for ν, then we have $\nu^+(E) = \nu(E \cap P)$. Similarly for $\nu^-(E)$.

2. Let $E_1, \ldots, E_n$ be disjoint and $\bigcup_{j=1}^n E_j = E$. It follows that

$$|\nu|(E) = \sum_{j=1}^n |\nu|(E_j) \geq \sum_{j=1}^n |\nu(E_j)|.$$

Let $X = P \cup N$ be the Hahn decomposition for ν, we then have

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu^+(E \cap P) + \nu^-(E \cap N) = |\nu(E \cap P)| + |\nu(E \cap N)|.$$

This completes the proof.

□

3.2 The Lebesgue-Randon-Nikodym theorem

Exercise (Folland 3.8)**.** $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Proof. It is clear that $|\nu| \ll \mu$ implies $\nu^+ \ll \mu$ and $\nu^- \ll \mu$ implies $\nu \ll \mu$. It then remains to show that $\nu \ll \mu$ implies $|\nu| \ll \mu$. Let $X = P \cup N$ be the Hahn decomposition for ν and $E \in \mathfrak{M}$ be such that $\mu(E) = 0$. It follows that $\mu(E \cap P) = 0$ and $\mu(E \cap N) = 0$. Then $\nu(E \cap P) = \nu^+(E \cap P) = 0$ and $\nu(E \cap N) = \nu^-(E \cap N) = 0$. Therefore,

$$|\nu|(E) = \nu^+(E \cap P) + \nu^-(E \cap N) = 0,$$

as desired. □

Exercise (Folland 3.9)**.** Suppose $\{\nu_j\}$ is a sequence of positive measures. If $\nu_j \perp \mu$ for all j, then $\sum_{j=1}^\infty \nu_j \perp \mu$, and if $\nu_j \ll \mu$ for all j, then $\sum_{j=1}^\infty \nu_j \perp \mu$.

Proof. Suppose $\nu_j \perp \mu$ for all j. For each $j \in \mathbb{N}$, let $X = E_j \cup F_j$ where ν_j is null on E_j and μ is null on F_j. Let $E = \bigcap_{j=1}^\infty E_j$ and $F = \bigcup_{j=1}^\infty F_j = E^c$. Then it is easy to verify that $\sum_{j=1}^\infty \nu_j$ is null on E and μ is null on F.

Suppose $\nu_j \ll \mu$ for all j. Then for $E \in \mathfrak{M}$ such that $\mu(E) = 0$, we have $\nu_j(E) = 0$ for all j. Therefore, $\sum_{j=1}^\infty \nu_j(E) = 0$ and thus $\sum_{j=1}^\infty \nu_j \perp \mu$. □

Exercise (Follan 3.10)**.** Theorem 3.5 might fail if ν is not finite. (Consider $d\nu(x) = dx/x$ and $d\mu(x) = dx$ on $(0,1)$, or ν counting measure and $\mu(E) = \sum_{n \in E} 2^{-n}$ on $\mathbb{N}$).

Proof. ***** TO-DO ***** □

3.3 Complex measures

Exercise (Folland 3.18)**.** Let ν be a complex measure on $(X, \mathfrak{M})$. Prove that $L^1(\nu) = L^1(|\nu|)$ and if $f \in L^1(\nu)$, then $\left|\int f\, d\nu\right| \leq \int |f|\, d|\nu|$.

Proof. For $L^1(\nu) \subset L^1(|\nu|)$, consider $f \in L^1(\nu)$. Note that $\nu = \nu_r + i\nu_i$ and it is easy to verify that $|i\nu_i| = |\nu_i|$. Therefore by Proposition 3.14, we have $|\nu| \leq |\nu_r| + |\nu_i|$. It follows that

$$\begin{aligned}\int |f|\, d|\nu| &\leq \int |f|\, d|\nu_r| + \int |f|\, d|\nu_i| \\ &= \int |f|\, d\nu_r^+ + \int |f|\, d\nu_r^- + \int |f|\, d\nu_i^+ + \int |f|\, d\nu_i^-.\end{aligned}$$

Since $f \in L^1(\nu)$, all four terms are finite and thus $f \in L^1(|\nu|)$.

For $L^1(|\nu|) \subset L^1(\nu)$, consider $f \in L^1(|\nu|)$. Then we have

$$\int |f|\, d\nu_{ri} \leq \left| \int |f| \frac{d\nu}{d|\nu|}\, d|\nu| \right| \leq \int |f| \left| \frac{d\nu}{d|\nu|} \right| d|\nu| = \int |f|\, d|\nu|,$$

where we have used the fact that $d\nu/d|\nu|$ has absolute value 1 $|\nu|$-a.e. This shows that $f \in L^1(\nu)$.

Moreover, write $d\nu = g\, d\mu$ so $d|\nu| = |g|\, d\mu$. Then we have

$$\left| \int f\, d\nu \right| = \left| \int fg\, d\mu \right| \leq \int |f|\, |g|\, d\mu = \int |f|\, d|\nu|,$$

as desired. $\square$

Exercise (Folland 3.19)**.** If ν, μ are complex measures and λ is a positive measure, then $\nu \perp \mu$ if and only if $|\nu| \perp |\mu|$, and $\nu \ll \lambda$ if and only if $|\nu| \ll \lambda$.

Proof. The "if" direction is clear for both propositions.

Suppose $\nu \perp \mu$. Then $\nu_r \perp \mu_r$, $\nu_i \perp \mu_r$, $\nu_r \perp \mu_i$, and $\nu_i \perp \mu_i$. It follows from Exercise 3.8 that $|\nu_r| \perp \mu_r$ and $|\nu_i| \perp \mu_r$. Since $|\nu| \leq |\nu_r| + |\nu_i|$, we have $|\nu| \perp \mu_r$. Similarly $|\nu| \perp \mu_i$. Following the same reasoning, we obtain $|\nu| \perp |\mu|$, as desired.

Suppose $\nu \ll \lambda$. Since $\nu = \nu_r + i\nu_i$, we have $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$. Recall from Exercise 3.8 that this implies $|\nu_r| \ll \lambda$ and $|\nu_i| \ll \lambda$. Moreover, $|\nu| \leq |\nu_r| + |\nu_i|$, so $|\nu| \ll \lambda$. $\square$

Exercise (Folland 3.20)**.** If ν is a complex measure on $(X, \mathfrak{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

Proof. By Lebesgue-Randon-Nikodym theorem, we have $d\nu = f\, d\mu$ for some function f and positive measure μ. It follows that $d|\nu| = |f|\, d\mu$ and

$$\int f\, d\mu = \int |f|\, d\mu.$$

Now let $E \in \mathfrak{M}$. We then have

$$\int_E f\,d\mu + \int_{E^c} f\,d\mu = \int_E |f|\,d\mu + \int_{E^c} |f|\,d\mu.$$

It follows that

$$0 \leq \int_E |f| - f_r\,d\mu = \int_{E^c} f_r - |f|\,d\mu \leq 0.$$

Since $E \in \mathfrak{M}$ is arbitrary, $f_r = |f|$ and $f_i = 0$ a.e. It follows that $d\nu = f\,d\mu = |f|\,d\mu = d\,|\nu|$. $\square$

Exercise (Folland 3.21)**.** Let ν be a complex measure on $(X, \mathfrak{M})$. If $E \in \mathfrak{M}$, define

$$\mu_1(E) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N},\, E_1, \ldots, E_n \text{ disjoint, } E = \bigcup_{j=1}^n E_j\right\},$$

$$\mu_2(E) = \sup\left\{\sum_{j=1}^\infty |\nu(E_j)| : E_1, E_2, \ldots \text{ disjoint, } E = \bigcup_{j=1}^\infty E_j\right\},$$

$$\mu_3(E) = \sup\left\{\left|\int_E f\,d\nu\right| : |f| \leq 1\right\}.$$

Then $\mu_1 = \mu_2 = \mu_3 = |\nu|$. (First show that $\mu_1 \leq \mu_2 \leq \mu_3$. To see $\mu_3 = |\nu|$, let $f = \overline{d\nu/d\,|\nu|}$ and apply Proposition 3.13. To see $\mu_3 \leq \mu_1$, approximate f by simple functions.)

Proof. It is clear that $\mu_1 \leq \mu_2$ by letting $E_j = \emptyset$ for $j > n$. To show $\mu_2 \leq \mu_3$, consider disjoint collection of sets $\{E_j\}_{j=1}^\infty$ such that $E = \bigcup_{j=1}^\infty E_j$. Let

$$f = \sum_{j=1}^\infty \frac{\overline{\nu(E_j)}}{|\nu(E_j)|}\chi_{E_j}.$$

Since $\{E_j\}_{j=1}^\infty$ are disjoint, we have $|f| \leq 1$. Moreover, 1 is integrable on X, so dominated convergence theorem implies

$$\int_E f\,d\nu = \int_E \sum_{j=1}^\infty \frac{\overline{\nu(E_j)}}{|\nu(E_j)|}\chi_{E_j}\,d\nu = \sum_{j=1}^\infty \int_E \frac{\overline{\nu(E_j)}}{|\nu(E_j)|}\chi_{E_j}\,d\nu = \sum_{j=1}^\infty |\nu(E_j)|.$$

Therefore, $\mu_2 \leq \mu_3$.

Now we show $\mu_3 = |\nu|$. It is clear that

$$\left|\int_E f\,d\nu\right| \leq \int_E |f|\,d\,|\nu| \leq |\nu|\,(E),$$

so $\mu_3 \leq |\nu|$. On the other hand, let $f = \overline{d\nu/d\,|\nu|}$. Then, $|f| = 1$ a.e. and

$$\left|\int_E \overline{\frac{d\nu}{d\,|\nu|}}\,d\nu\right| = \left|\int_E \overline{\frac{d\nu}{d\,|\nu|}}\frac{d\nu}{d\,|\nu|}\,d\,|\nu|\right| = \int_E 1\,d\,|\nu| = |\nu|\,(E).$$

Therefore, $|\nu| \leq \mu_3$ and thus $|\nu| = \mu_3$.

Finally, to show $\mu_3 \leq \mu_1$, let φ be a simple function such that $|\varphi| \leq 1$. Let

$$\varphi = \sum_{j=1}^{n} v_j \chi_{E_j}$$

be the canonical representation of φ. We then have $|v_j| \leq 1$ for each j and $E_1, \ldots, E_n$ are disjoint. WLOG assume $E = \bigcup_{j=1}^{n} E_j$. It follows that

$$\left| \int_E \varphi \, d\nu \right| = \left| \sum_{j=1}^{n} v_j \nu(E_j) \right| \leq \sum_{j=1}^{n} |\nu(E_j)| \leq \mu_1(E).$$

We know any function f with $|f| \leq 1$ can be approximated by simple functions with absolute value less than or equal to 1, so $\mu_3 \leq \mu_1$. The proof is then complete. ☐

3.4 Differentiation on Euclidean spaces

Exercise (Folland 3.22)**.** If $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, then there exists $C, R > 0$ such that $Hf(x) \geq C\,|x|^{-n}$ for $|x| > R$. Hence $m(\{x : Hf(x) > \alpha\}) > C'/\alpha$ when α is small, so the estimate in the maximal theorem is essentially sharp.

Proof. Recall that

$$Hf(x) = \sup_{r>0} A_r\,|f|\,(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| \, dy.$$

Taking $r = 2\,|x|$ we have

$$Hf(x) \geq \frac{1}{2^n\,|x|^n\, m(B(0,1))} \int_{B(x,2|x|)} |f(y)| \, dy.$$

However, we have

$$\lim_{t\to\infty} \int_{B(0,t)} |f(y)| \, dy = \int_{\mathbb{R}^n} |f(y)| \, dy.$$

Write $M = \int |f(y)| \, dy$. Since $f \in L^1(\mathbb{R}^n)$ and $f \neq 0$, we have $0 < M < \infty$. There then exists an $R > 0$ such that

$$\int_{B(0,R)} |f(y)| \, dy \geq \frac{M}{2}.$$

Now for $|x| > R$, we have $B(0,R) \subset B(x, 2\,|x|)$, and thus

$$Hf(x) \geq \frac{M}{2^{n+1} m(B(0,1))} \,|x|^{-n}\,.$$

When $\alpha < \frac{1}{2}CR^{-n}$, we have $C\,|x|^{-n} > \alpha$ for $R < |x| < (C/\alpha)^{1/n}$ and

$$\frac{C}{\alpha} - R^n > \frac{C}{2\alpha}.$$

It follows that

$$m(\{x : Hf(x) > \alpha\}) \geq m(B(0,1))\left(\frac{C}{\alpha} - R^n\right) > \frac{Cm(B(0,1))}{2\alpha}.$$

The proof is then complete. □

Exercise (Folland 3.23)**.** A useful variant of the Hardy-Littlewood maximal function is

$$H^*f(x) = \sup\left\{\frac{1}{m(B)}\int_B |f(y)|\,dy : B \text{ is a ball and } x \in B\right\}.$$

Show that $Hf \leq H^*f \leq 2^n Hf$.

Proof. Recall that

$$Hf(x) = \sup_{r>0}\frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)|\,dy.$$

It is then clear that $Hf \leq H^*f$. Now let B be a ball with radius r and $x \in B$. It follows that $B \subset B(x,2r)$. Note also $m(B(x,2r)) = 2^n m(B)$, so

$$\frac{1}{m(B)}\int_B |f(y)|\,dy \leq \frac{2^n}{m(B(x,2r))}\int_{B(x,2r)} |f(y)|\,dy \leq 2^n Hf.$$

Since B is arbitrary, this implies that $H^*f \leq 2^n Hf$. □

Exercise (Follan 3.25)**.** If E is a Borel set in $\mathbb{R}^n$, the *density* $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r\to 0}\frac{m(E \cap B(x,r))}{m(B(x,r))},$$

whenever the limit exists.

1. Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.
2. Find examples of E such that $D_E(x)$ is a given number of $\alpha \in (0,1)$ or such that $D_E(x)$ does not exist.

Proof. 1.

□

3.5 Functions of bounded variation

*** TO-DO ***