

# Introduction to Functional Analysis

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# 1 Banach space theory

## 1.1 Quotient spaces, Baire category and uniform boundedness

**Theorem.** Let  $\|\cdot\|$  be a **seminorm** on a vector space  $V$ . If we define  $E = \{v \in V : \|v\| = 0\}$ , then  $E$  is a subspace of  $V$ , and the function on  $V/E$  defined by

$$\|v + E\| = \|v\|$$

for any  $v + E \in V/E$  defines a **norm**.

**Theorem** (Baire Category Theorem). Let  $M$  be a complete metric space, and let  $\{C_n\}_{n=1}^\infty$  be a collection of closed subsets of  $M$  such that  $M = \bigcup_{n=1}^\infty C_n$ . Then at least one of the  $C_n$  contains an open ball  $B(x, r) = \{y \in M : d(x, y) < r\}$ .

**Theorem** (Uniform Boundedness Theorem). Let  $B$  be Banach space and  $V$  a normed vector space. Let  $\{T_n\}_{n=1}^\infty$  be a sequence in  $\mathcal{B}(B, V)$ . Then if for all  $b \in B$  we have  $\sup_n \|T_n b\| < \infty$  (that is, this sequence is pointwise bounded), then  $\sup_n \|T_n\| < \infty$  (the operator norms are bounded).

*Proof.* For each  $k \in \mathbb{N}$ , define

$$C_k = \left\{ b \in B : \|b\| \leq 1, \sup_{n \in \mathbb{N}} \|T_n b\| \leq k \right\}.$$

This set is closed for each  $k \in \mathbb{N}$ , but by assumption, we have

$$\{b \in B : \|b\| \leq 1\} = \bigcup_{k=1}^\infty C_k.$$

The left hand side is a closed subset of  $B$ , and is thus a complete metric space. By Baire Category Theorem, there exists  $k \in \mathbb{N}$  such that  $C_k$  contains an open ball  $B(b_0, \delta_0)$ . Then, if  $b \in B(0, \delta_0)$ , we have  $b_0 + b \in B(b_0, \delta_0)$  and thus

$$\sup_{n \in \mathbb{N}} \|T_n(b_0 + b)\| \leq k.$$

It follows that

$$\sup_{n \in \mathbb{N}} \|T_n b\| \leq \sup_{n \in \mathbb{N}} \|T_n(b_0 + b)\| + \sup_{n \in \mathbb{N}} \|T_n b_0\| \leq 2k.$$

Suppose  $\|b\| = 1$ , then  $\frac{\delta_0}{2}b \in B(0, \delta_0)$  and thus for all  $n \in \mathbb{N}$ , we have

$$\left\| T_n \left( \frac{\delta_0}{2} b \right) \right\| \leq 2k.$$

Therefore,

$$\sup_{n \in \mathbb{N}} \|T_n\| \leq \frac{4k}{\delta_0}.$$

□

## 2 Hilbert space theory

### 2.1 Basic Hilbert space theory

**Definition** (Pre-Hilbert space). A **pre-Hilbert** space  $H$  is a vector space over  $\mathbb{C}$  with a **Hermitian inner product**, which is a map  $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{C}$  satisfying the following properties.

1. For all  $\lambda_1, \lambda_2 \in \mathbb{C}$  and  $v_1, v_2, w \in H$ , we have

$$\langle \lambda_1 v_1 + \lambda_2 v_2, \lambda_1 v_1 + \lambda_2 v_2 \rangle = \lambda_1 \langle v_1, v_1 \rangle + \lambda_2 \langle v_2, v_2 \rangle.$$

2. For all  $v, w \in H$ , we have  $\langle v, v \rangle = \overline{\langle w, w \rangle}$ .
3. For all  $v \in H$ , we have  $\langle v, v \rangle \geq 0$ , with equality if and only if  $v = 0$ .

**Definition.** Let  $H$  be a pre-Hilbert space. For all  $v \in H$ , we define

$$\|v\| = \langle v, v \rangle^{\frac{1}{2}}.$$

**Theorem** (Cauchy-Schwarz inequality). Let  $H$  be a pre-Hilbert space. For all  $u, v \in H$ , we have

$$|\langle u, u \rangle| \leq \|u\| \|v\|.$$

*Proof.* Define  $f(t) = \|u + tv\|^2$ . Notice that

$$\begin{aligned} f(t) &= \langle u + tv, u + tv \rangle \\ &= \langle u, u \rangle + t^2 \langle v, v \rangle + t \langle u, v \rangle + t \langle v, u \rangle \\ &= \|u\|^2 + t^2 \|v\|^2 + 2t \operatorname{Re}(\langle u, v \rangle). \end{aligned}$$

This implies that

$$0 \leq f(t_{\min}) = \|u\|^2 - \frac{\operatorname{Re}(\langle u, v \rangle)^2}{\|v\|^2}.$$

It follows that

$$|\operatorname{Re}(\langle u, v \rangle)| \leq \|u\| \|v\|.$$

This is almost what we want. To finish up, first note that if  $\langle u, v \rangle = 0$  then there is nothing to prove, so suppose  $\langle u, v \rangle \neq 0$ , and define

$$\lambda = \frac{\overline{\langle u, v \rangle}}{|\langle u, v \rangle|}.$$

Note that we have  $|\lambda| = 1$  and we have the chain of equalities of real numbers:

$$|\langle u, v \rangle| = \lambda \langle u, v \rangle = \langle \lambda u, \lambda v \rangle = \operatorname{Re} \langle \lambda u, \lambda v \rangle \leq \|\lambda u\| \|\lambda v\|.$$

However,  $\|\lambda u\| = \|u\|$ , so the proof is complete. □

**Theorem.** If  $H$  is a pre-Hilbert space, then  $\|\cdot\|$  is a norm on  $H$ .

*Proof.* Note that

$$\|v\| = 0 \iff \langle v, v \rangle = 0 \iff v = 0.$$

Now if  $\lambda \in \mathbb{C}$  and  $v \in H$ , then

$$\langle \lambda v, \lambda v \rangle = \lambda \bar{\lambda} \langle v, v \rangle = |\lambda|^2 \|v\|^2.$$

Therefore,  $\|\lambda v\| = |\lambda| \|v\|$ .

Finally, let  $u, v \in H$ , then

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \|u\|^2 + \|v\|^2 + 2 \operatorname{Re} \langle u, v \rangle \\ &\leq \|u\|^2 + \|v\|^2 + 2 |\langle u, v \rangle| \\ &\leq \|u\|^2 + \|v\|^2 + 2 \|u\| \|v\| \\ &= (\|u\| + \|v\|)^2. \end{aligned}$$

This completes the proof.  $\square$

**Theorem.** If  $u_n \rightarrow u$  and  $v_n \rightarrow v$  in a pre-Hilbert space  $H$ , then  $\langle u_n, v_n \rangle \rightarrow \langle u, v \rangle$ .

*Proof.* If  $u_n \rightarrow u$  and  $v_n \rightarrow v$ , then  $\|u_n - u\| \rightarrow 0$  and  $\|v_n - v\| \rightarrow 0$ . It follows that

$$\begin{aligned} |\langle u_n, v_n \rangle - \langle u, v \rangle| &= |\langle u_n - u, v_n - v \rangle + \langle u, v_n - v \rangle| \\ &\leq \|u_n - u\| \|v_n - v\| + \|u\| \|v_n - v\| \\ &\leq \|u_n - u\| \sup_{k \in \mathbb{N}} \|v_k\| + \|u\| \|v - v_n\| \\ &\rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . This completes the proof.  $\square$

**Definition** (Hilbert space). A **Hilbert space** is a pre-Hilbert space that is complete with respect to the norm  $\|\cdot\| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$ .

**Example.** Some examples of Hilbert spaces:

- $\mathbb{C}^n = \{z = (z_1, \dots, z_n) : z_j \in \mathbb{C}\}$  with  $\langle z, z \rangle = \sum_j z_j \bar{z}_j$  is a Hilbert space.
- $\ell^2 = \left\{a = \{a_k\}_{k=1}^\infty : a_k \in \mathbb{C}, \sum_{k=1}^\infty |a_k|^2 < \infty\right\}$  with  $\langle a, a \rangle = \sum_{k=1}^\infty a_k \bar{a}_k$  is a Hilbert space.
- If  $E \subset \mathbb{R}$  is measurable, then  $L^2(E) = \left\{f : E \rightarrow \mathbb{C}, \int_E |f|^2 < \infty\right\}$  with  $\langle f, f \rangle = \int_E f \bar{f}$  is a Hilbert space.

We will show that each separable Hilbert space is isometrically isomorphic to either  $\mathbb{C}^n$  or  $\ell^2$ .

Now we have seen that  $\ell^2$  and  $L^2$  spaces are Hilbert spaces. This is expected since the definition of the inner product in these spaces uses the fact that they are  $\ell^2$  or  $L^2$ . A natural question then is whether other  $\ell^p$  or  $L^p$  spaces are also Hilbert spaces with respect to some inner product? It turns out there is a simple way to decide whether a norm come from an inner-product, and thus whether a Banach space is a Hilbert space.

**Theorem** (Parallelogram Law). If  $H$  is a pre-Hilbert space, then for all  $u, v \in H$ , we have

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

In addition, if  $H$  is a normed vector space satisfying this equality, then  $H$  is a pre-Hilbert space.

Using the previous theorem, we can verify that  $\ell^p$  and  $L^p$  with  $p \neq 2$  are **not** Hilbert spaces.

**Definition** (Orthogonal). If  $H$  is a pre-Hilbert space,  $u, v \in H$  are **orthogonal** if  $\langle u, v \rangle = 0$ . We denote this as  $u \perp v$ .

**Definition** (Orthonormal sets). If  $H$  is a pre-Hilbert space, a subset  $\{e_\lambda\}_{\lambda \in \Lambda} \subset H$  is **orthonormal** if for all  $\lambda \in \Lambda$ , we have  $\|e_\lambda\| = 1$  and  $\lambda_1 \neq \lambda_2$  implies  $e_{\lambda_1} \perp e_{\lambda_2}$ .

**Remark.** we will mainly be interested in the case where we have a countable orthonormal set.

**Example.** The set  $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n \in \mathbb{Z}}$  as elements in  $L^2([-\pi, \pi])$  is an orthonormal subset of  $L^2([-\pi, \pi])$ . Indeed, for any  $m, n \in \mathbb{Z}$ , we have

$$\int_{-\pi}^{\pi} e^{imx} \overline{e^{inx}} = \int_{-\pi}^{\pi} e^{i(m-n)x} = \begin{cases} 2\pi & (m = n), \\ 0 & (m \neq n). \end{cases}$$

Therefore,  $\left\langle \frac{e^{inx}}{\sqrt{2\pi}}, \frac{e^{inx}}{\sqrt{2\pi}} \right\rangle = \delta_{mn}$ , and  $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n \in \mathbb{Z}}$  is an orthonormal subset of  $L^2([-\pi, \pi])$ .

**Theorem** (Bessel). If  $\{e_n\}_{n=1}^\infty$  is countable orthonormal subset of a pre-Hilbert space  $H$ , then for all  $u \in H$ , we have

$$\sum_{n=1}^{\infty} |\langle u, e_n \rangle|^2 \leq \|u\|^2.$$

*Proof.* We first do the finite case. Suppose  $\{e_n\}_{n=1}^N$  is an orthonormal subset of  $H$ . Then,

$$\begin{aligned} \left\| \sum_{n=1}^N \langle u, e_n \rangle e_n \right\|^2 &= \left\langle \sum_{n=1}^N \langle u, e_n \rangle e_n, \sum_{n=1}^N \langle u, e_n \rangle e_n \right\rangle \\ &= \sum_{n=1}^N \sum_{m=1}^N \langle u, e_n \rangle \overline{\langle u, e_m \rangle} \langle e_n, e_m \rangle \\ &= \sum_{n=1}^N |\langle u, e_n \rangle|^2. \end{aligned}$$

Also,

$$\begin{aligned}\langle u, u \rangle &= \sum_{n=0}^N \overline{\langle u, u \rangle} \langle u, u \rangle \\ &= \sum_{n=0}^N |\langle u, u \rangle|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}0 &\leq \left\| u - \sum_{n=0}^N \langle u, u \rangle e_n \right\|^2 \\ &= \|u\|^2 + \left\| \sum_{n=0}^N \langle u, u \rangle e_n \right\|^2 - 2 \operatorname{Re} \langle u, u \rangle \\ &= \|u\|^2 - \sum_{n=0}^N |\langle u, u \rangle|^2,\end{aligned}$$

as desired.

For the infinite case, just take the limit as  $N \rightarrow \infty$ .  $\square$

**Definition** (Maximal orthonormal subset). An orthonormal subset  $\{e_\lambda\}_\lambda$  of a pre-Hilbert space is **maximal** if  $u \in H$  and  $\langle u, u \rangle = 0$  for all  $\lambda \in \Lambda$  implies that  $u = 0$ .

**Theorem.** Every non-trivial pre-Hilbert space has a maximal orthonormal subset.

This can be proved using Zorn's Lemma. We will prove something less strong but often equally useful by hand, without applying Zorn's Lemma.

**Theorem.** Every non-trivial separable pre-Hilbert space has a countable maximal orthonormal subset.

*Proof.* Use the Gram-Schmidt process. Let  $\{v_j\}_{j=1}^\infty$  be a countable dense subset of  $H$  where  $v_0 \neq 0$ . Claim that for any  $n \in \mathbb{N}$ , there exists  $m(n) \leq n$  and an orthonormal subset  $\{e_1, \dots, e_{m(n)}\}$  such that

1.  $\operatorname{span} \{e_1, \dots, e_{m(n)}\} = \operatorname{span} \{v_1, \dots, v_n\}$ .
2. If  $v_n \in \operatorname{span} \{v_1, \dots, v_{n-1}\}$ , we have

$$\{e_1, \dots, e_{m(n)}\} = \{e_1, \dots, e_{m(n-1)}\} \cup \emptyset.$$

Otherwise, we have

$$\{e_1, \dots, e_{m(n)}\} = \{e_1, \dots, e_{m(n-1)}\} \cup e_{m(n)}$$

for some  $e_{m(n)} \in H$ .

Prove this by induction. For the base case, let  $e_1 = \frac{v_1}{\|v_1\|}$ . For the inductive step, suppose the claim holds for  $n = k$ . If  $v_{k+1} \in \text{span}\{v_1, \dots, v_k\}$ , then

$$\text{span}\{e_1, \dots, e_{n(k)}\} = \text{span}\{v_1, \dots, v_k\} = \text{span}\{v_1, \dots, v_{k+1}\}.$$

Now suppose  $v_{k+1} \notin \text{span}\{v_1, \dots, v_k\}$ . Define

$$w_{k+1} = v_{k+1} - \sum_{j=1}^{m(k)} \langle v_{k+1}, v_{k+1} \rangle e_j.$$

Note that  $w_{k+1} \neq 0$  and define  $e_{m(k+1)} = \frac{w_{k+1}}{\|w_{k+1}\|}$ . Then,  $\|e_{m(k+1)}\| = 1$  and for all  $1 \leq l \leq m(k)$ ,

$$\begin{aligned} \langle e_{m(k+1)}, e_{m(k+1)} \rangle &= \frac{1}{\|w_{k+1}\|} \left\langle v_{k+1} - \sum_{j=1}^{m(k)} \langle v_{k+1}, v_{k+1} \rangle e_j, v_{k+1} - \sum_{j=1}^{m(k)} \langle v_{k+1}, v_{k+1} \rangle e_j \right\rangle \\ &= \frac{1}{\|w_{k+1}\|} (\langle v_{k+1}, v_{k+1} \rangle - \langle v_{k+1}, v_{k+1} \rangle) \\ &= 0. \end{aligned}$$

Therefore,  $e_{m(k+1)}$  is the desired vector we want and we have completed the proof for the claim.

Now let

$$S = \bigcup_{n=1}^{\infty} \{e_1, \dots, e_{m(n)}\}.$$

Then  $S$  is a countable orthonormal subset of  $H$ . Now we show  $S$  is maximal. Suppose  $u \in H$  and  $\langle u, u \rangle = 0$ . Since  $\{v_j\}_{j=1}^{\infty}$  is dense in  $H$ , there exists  $\{v_{j(k)}\}_{k=1}^{\infty}$  such that  $v_{j(k)} \rightarrow u$  as  $k \rightarrow \infty$ . By our claim, we know  $v_{j(k)} \in \text{span}\{e_1, \dots, e_{m(j(k))}\}$ . By Bessel's inequality,

$$\|v_{j(k)}\|^2 = \sum_{l=1}^{m(j(k))} |\langle v_{j(k)}, v_{j(k)} \rangle|^2 = \sum_{l=1}^{m(j(k))} |\langle v_{j(k)} - u, v_{j(k)} - u \rangle|^2 \leq \|v_{j(k)} - u\|^2,$$

where for the first equality we used the fact that  $v_{j(k)} \in \text{span}\{e_1, \dots, e_{m(j(k))}\}$ . Since  $v_{j(k)} \rightarrow u$  as  $k \rightarrow \infty$ , the inequality implies that  $\|v_{j(k)}\| \rightarrow 0$  as  $k \rightarrow \infty$  and thus  $\|u\| = 0$ , showing that  $S$  is indeed a maximal orthonormal subset of  $H$ .  $\square$

**Corollary.**  $\ell^2$  and  $L^2$  have countable maximal orthonormal subset since they are both separable.

## 2.2 Orthonormal bases and Fourier Series

**Definition** (Orthonormal basis). Let  $H$  be a Hilbert space. An **orthonormal basis** of  $H$  is a countable maximal orthonormal subset  $\{e_n\}_n$  of  $H$ .



**Theorem.** If  $\{e_n\}_{n=1}^{\infty}$  is an orthonormal basis in Hilbert space  $H$ , then for all  $u \in H$ , we have

$$\sum_{n=1}^{\infty} \langle u, u \rangle e_n = u.$$

This is the Fourier-Bessel series.

This tells us we can write each element in  $H$  as a infinite linear combination of the orthonormal basis.

*Proof.* We first prove the sequence of partial sums  $\{\sum_{n=0}^m \langle u, u \rangle e_n\}_m$  is Cauchy. Let  $\varepsilon > 0$ . By Bessel's inequality, we have

$$\sum_{n=1}^{\infty} |\langle u, u \rangle|^2 \leq \|u\|^2 < \infty.$$

Therefore, there exists  $M \in \mathbb{N}$  such that  $N \geq M$  implies  $\sum_{n=N+1}^{\infty} |\langle u, u \rangle|^2 < \varepsilon^2$ . Then for all  $m > l \geq M$ , we have

$$\left\| \sum_{n=0}^m \langle u, u \rangle e_n - \sum_{n=0}^l \langle u, u \rangle e_n \right\|^2 \leq \sum_{n=l+1}^m |\langle u, u \rangle|^2 \leq \sum_{n=l+1}^{\infty} |\langle u, u \rangle|^2 < \varepsilon^2.$$

Therefore, the sequence of partial sum is Cauchy. Since  $H$  is complete, there exists  $\bar{u} \in H$  such that  $\bar{u} = \sum_{n=1}^{\infty} \langle u, u \rangle e_n$ . It remains to show that  $\bar{u} = u$ . By continuity of inner-product, for all  $l \in \mathbb{N}$ , we have

$$\begin{aligned} \langle u - \bar{u}, u - \bar{u} \rangle &= \lim_{m \rightarrow \infty} \left\langle u - \sum_{n=0}^m \langle u, u \rangle e_n, u - \sum_{n=0}^m \langle u, u \rangle e_n \right\rangle \\ &= \lim_{m \rightarrow \infty} \left[ \langle u, u \rangle - \sum_{n=0}^m \langle u, u \rangle \langle e_n, e_n \rangle \right] \\ &= 0. \end{aligned}$$

Since  $\{e_n\}_{n=1}^{\infty}$  is maximal, this implies that  $u - \bar{u} = 0$  and the proof is complete.  $\square$

**Theorem.** Let  $H$  be a Hilbert space. If  $H$  has an orthonormal basis, then  $H$  is separable.

*Proof.* Suppose  $\{e_n\}_{n=1}^{\infty}$  is an orthonormal basis for  $H$ . Then

$$S = \bigcup_{m \in \mathbb{N}} \left\{ \sum_{n=0}^m q_n e_n : q_n \in \mathbb{Q} + i\mathbb{Q} \right\}$$

is a countable set. Also, by the previous theorem,  $S$  is dense in  $H$ .  $\square$

**Remark.** Let  $H$  be a Hilbert space.  $H$  is separable if and only if  $H$  has an orthonormal basis.

**Theorem** (Parseval's identity). If  $H$  is a Hilbert space and  $\{e_n\}_{n=1}^\infty$  is a countable orthonormal basis, then for all  $u \in H$ , we have

$$\sum_n |\langle u, e_n \rangle|^2 = \|u\|^2$$

*Proof.* We have  $u = \sum_n \langle u, e_n \rangle e_n$ . This implies that

$$\begin{aligned} \|u\|^2 &= \lim_{m \rightarrow \infty} \left\langle \sum_{n=0}^m \langle u, e_n \rangle e_n, \sum_{n=0}^m \langle u, e_n \rangle e_n \right\rangle \\ &= \lim_{m \rightarrow \infty} \sum_{n=0}^m \sum_{l=0}^m \langle u, e_n \rangle \overline{\langle u, e_l \rangle} \langle e_n, e_l \rangle \\ &= \lim_{m \rightarrow \infty} \sum_{n=0}^m |\langle u, e_n \rangle|^2 \\ &= \sum_{n=1}^\infty |\langle u, e_n \rangle|^2. \end{aligned}$$

□

**Theorem.** If  $H$  is an infinite dimensional separable Hilbert space, then  $H$  is isometrically isomorphic to  $\ell^2$ . That is, there exists bijective bounded linear map  $T : H \rightarrow \ell^2$  such that for all  $u, v \in H$ , we have

$$\|Tu\|_{\ell^2} = \|u\|_H \quad \text{and} \quad \langle Tu, Tv \rangle_{\ell^2} = \langle u, v \rangle_H.$$

*Proof.* Since  $H$  is separable, there exists an orthonormal basis  $\{e_n\}_{n=1}^\infty$ . For all  $u \in H$ , the previous theorem gives

$$\|u\| = \left( \sum_{n=1}^\infty |\langle u, e_n \rangle|^2 \right)^{\frac{1}{2}}.$$

Define  $T : H \rightarrow \ell^2$  by

$$Tu = \{\langle u, e_n \rangle\}_{n=1}^\infty \in \ell^2.$$

It is easy to check that  $T$  is the desired isometric isomorphism. □

Next we use the theories we learned in a more concrete setting — the Fourier series.

**Theorem.** The subset  $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}}$  is an orthonormal subset of  $L^2([-\pi, \pi])$ .

**Definition.** Let  $f \in L^2([-\pi, \pi])$ . Then the  $n$ -th **Fourier coefficient** of  $f$  is

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt.$$

The  $N$ -th Fourier sum of  $f$  is

$$S_N f(x) = \sum_{|n| \leq N} \widehat{f}(n) e^{inx} = \sum_{|n| \leq N} \langle f, f \rangle \frac{e^{inx}}{\sqrt{2\pi}}.$$

The **Fourier series** of  $f$  is the formal series  $\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{-inx}$ .

The natural question now is whether we have for all  $f \in L^2([-\pi, \pi])$ ,

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}.$$

That is, whether we have the following convergence in  $L^2$ .

$$\lim_{N \rightarrow \infty} \|f - S_N f\|_2 = 0.$$

This question is then equivalent to whether  $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}}$  is maximal in  $L^2([-\pi, \pi])$ . That is, whether  $\widehat{f}(n) = 0$  for all  $n \in \mathbb{N}$  implies  $f = 0$ .

The answer to the question is yes, but it is going to take some work. We first do some simple calculation.

**Theorem.** For all  $f \in L^2([-\pi, \pi])$  and for all  $N \in \mathbb{N}$ , we have

$$S_N f(x) = \int_{-\pi}^{\pi} D_N(x-t) f(t) dt,$$

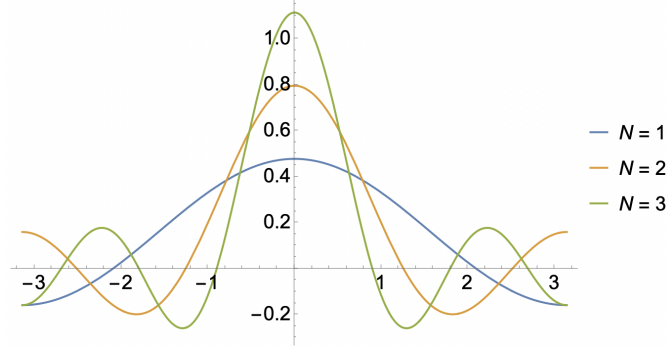
where

$$D_N(x) = \begin{cases} \frac{2N+1}{2\pi} & (x = 0) \\ \frac{\sin(N+\frac{1}{2})x}{2\pi \sin \frac{x}{2}} & (x \neq 0) \end{cases}$$

it the **Dirichlet kernel**. Figure 1 shows a plot of  $D_N(x)$  on  $[-\pi, \pi]$  for  $N = 1, 2, 3$ . Note that  $D_N$  is a smooth function.

*Proof.* If  $f \in L^2([-\pi, \pi])$ , we have

$$\begin{aligned} S_N f(x) &= \sum_{|n| \leq N} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \right) e^{inx} \\ &= \int_{-\pi}^{\pi} f(t) \left( \frac{1}{2\pi} \sum_{|n| \leq N} e^{in(x-t)} \right) dt. \end{aligned}$$

Figure 1: Plot of Dirichlet kernel  $D_N(x)$  on  $[-\pi, \pi]$  for  $N = 1, 2, 3$ .

Let  $D_N(x) = \frac{1}{2\pi} \sum_{|n| \leq N} e^{inx}$ . Then for  $x \neq 0$ , we have

$$\begin{aligned}
 D_N(x) &= \frac{1}{2\pi} \sum_{n=-N}^N e^{-inx} \\
 &= \frac{1}{2\pi} e^{-iNx} \sum_{n=0}^{2N} (e^{ix})^n \\
 &= \frac{1}{2\pi} e^{-iNx} \frac{1 - e^{i(2N+1)x}}{1 - e^{ix}} \\
 &= \frac{1}{2\pi} \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}} \\
 &= \frac{1}{2\pi} \frac{2i \sin(N + \frac{1}{2})x}{2i \sin \frac{x}{2}} \\
 &= \frac{1}{2\pi} \frac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}},
 \end{aligned}$$

as desired. For  $x = 0$ , we also clearly have  $D_N(0) = \frac{(2N+1)}{2\pi}$ . The proof is thus complete.  $\square$

**Definition.** If  $f \in L^2([-\pi, \pi])$ , we define the  $N$ -th **Cesaro-Fourier mean** of  $f$  by

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{k=0}^N S_k f(x).$$

The idea behind defining the Cesaro mean is that if the original sequence converges, the Cesaro mean also converge to the same limit. However, Cesaro have even better property — the Cesaro mean can converge even if the original sequence does not converge. Therefore, it has better

convergence properties and hopefully we can show it converge to  $f$  in  $L^2$  more easily. The goal now is then to show

$$\|\sigma_N f - f\|_2 \rightarrow 0 \text{ as } N \rightarrow \infty.$$

This would tell us if all Fourier coefficients are zero, then the Cesaro means are zero, and the limit above would tell us  $f$  is zero.

### 2.3 Fejer's theorem and convergence of Fourier series

In this section, we will show that if  $f \in L^2([-\pi, \pi])$ , then  $\|\sigma_N f - f\|_2 \rightarrow 0$  as  $N \rightarrow \infty$ .

First we will rewrite the Cesaro Fourier mean, just like what we did for the partial Fourier sum using the Dirichlet kernel.

**Theorem.** For any  $f \in L^2([-\pi, \pi])$ , we have

$$\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(x-t) f(t) dt,$$

where

$$K_N(x) = \begin{cases} \frac{N+1}{2\pi} & (x = 0) \\ \frac{1}{2\pi(N+1)} \frac{\sin^2 \frac{N+1}{2} x}{\sin^2 \frac{x}{2}} & (x \neq 0) \end{cases}$$

is the Fejer kernel.

Moreover, we have

1.  $K_N(x) \geq 0$ ,  $K_N(x) = K_N(-x)$ , and  $K_N(x)$  is  $2\pi$  periodic.
2.  $\int_{-\pi}^{\pi} K_N(t) dt = 1$ .
3. If  $\delta \in (0, \pi]$ , then for all  $\delta \leq |x| \leq \pi$ , we have

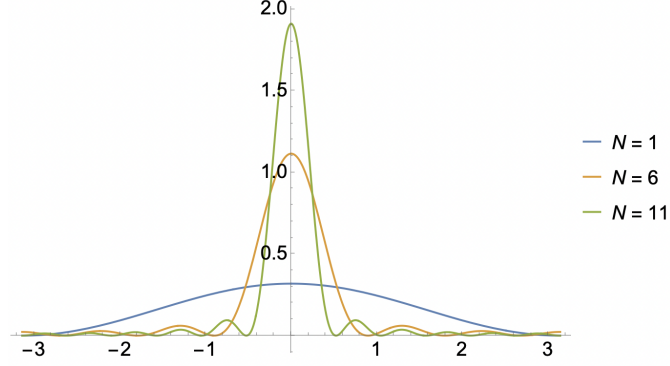
$$|K_N(x)| \leq \frac{1}{2\pi(N+1) \sin^2 \frac{\delta}{2}}.$$

A plot for  $K_N(x)$  is shown in Figure 2.

Note that  $K_N(x)$  is concentrated at 0 when  $N$  is very large. In this case, we have

$$\begin{aligned} \sigma_N f(x) &= \int_{-\pi}^{\pi} K_N(x-t) f(t) dt \\ &\approx f(x) \int_{-\pi}^{\pi} K_N(t) dt \\ &= f(x). \end{aligned}$$

This provides a rough intuition behind the Fejer kernel. The fact that  $K_N$  is non-negative makes a huge difference compared to the Dirichlet kernel, since it gives much better properties.

Figure 2: Plot of Dirichlet kernel  $D_N(x)$  on  $[-\pi, \pi]$  for  $N = 1, 6, 11$ .

*Proof.* Recall that

$$S_k f(x) = \int_{-\pi}^{\pi} D_k(x-t) f(t) dt,$$

where

$$D_k(t) = \begin{cases} \frac{2N+1}{2\pi} & (t = 0), \\ \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})t}{\sin \frac{t}{2}} & (t \neq 0). \end{cases}$$

It follows that

$$\begin{aligned} \sigma_N f(x) &= \frac{1}{N+1} \sum_{k=0}^N S_k f(x) \\ &= \int_{-\pi}^{\pi} \frac{1}{N+1} \sum_{k=0}^N D_k(x-t) f(t) dt. \end{aligned}$$

Then for  $x \neq 0$ , we have

$$\begin{aligned} K_N(x) &= \frac{1}{N+1} \sum_{k=0}^N D_k(x) \\ &= \frac{1}{2\pi(N+1)} \frac{1}{2\sin^2 \frac{x}{2}} \sum_{k=0}^N 2\sin \frac{x}{2} \sin \left(k + \frac{1}{2}\right) x \\ &= \frac{1}{2\pi(N+1)} \frac{1}{2\sin^2 \frac{x}{2}} \sum_{k=0}^N [\cos kx - \cos(k+1)x] \\ &= \frac{1}{2\pi(N+1)} \frac{1}{\sin^2 \frac{x}{2}} \frac{1 - \cos(N+1)x}{2} \\ &= \frac{1}{2\pi(N+1)} \frac{\sin^2 \frac{N+1}{2} x}{\sin^2 \frac{x}{2}}. \end{aligned}$$

It follows immediately that  $K_N(x) \geq 0$ ,  $K_N(x)$  is even and  $2\pi$  periodic.

For property 2, note that for all  $k$ ,

$$\int_{-\pi}^{\pi} D_k(t) dt = \int_{-\pi}^{\pi} \sum_{n=-k}^k e^{int} dt = 1.$$

Then,

$$\int_{-\pi}^{\pi} K_N(t) dt = \frac{1}{N+1} \sum_{k=0}^N \int_{-\pi}^{\pi} D_k(t) dt = 1,$$

as desired.

For property 3, let  $\delta \in (0, \pi]$ . Note that  $\sin^2 \frac{x}{2}$  is even and increasing on  $[0, \pi]$ . It follows that  $\delta \leq |x| \leq \pi$  implies  $\sin^2 \frac{x}{2} \geq \sin^2 \frac{\delta}{2}$ . Therefore,

$$K_N(x) \leq \frac{1}{2\pi(N+1)} \frac{\sin^2 \frac{N+1}{2}x}{\sin^2 \frac{\delta}{2}} \leq \frac{1}{2\pi(N+1) \sin^2 \frac{\delta}{2}}.$$

□

Since the continuous functions that vanishes at both end points is dense in  $L^2([-\pi, \pi])$ , it make sense to first prove the theorem for continuous functions. We have the following theorem by Fejer.

**Theorem (Fejer).** If  $f \in C([-\pi, \pi])$  is  $2\pi$ -periodic,  $f(\pi) = f(-\pi)$ , then  $\sigma_N f \rightarrow f$  uniformly on  $[-\pi, \pi]$ .

*Proof.* First we extend  $f$  by periodicity to all of  $\mathbb{R}$ . Then  $f \in C(\mathbb{R})$ ,  $2\pi$ -periodic. This implies that  $f$  is uniformly continuous and bounded.

Let  $\varepsilon > 0$ . Since  $f$  is uniformly continuous, there exists  $\delta > 0$  such that  $|y - z| < \delta$  implies that  $|f(y) - f(z)| < \frac{\varepsilon}{2}$ . Choose  $M \in \mathbb{N}$  such that

$$\frac{2 \|f\|_{\infty}}{(N+1) \sin^2 \frac{\delta}{2}} < \frac{\varepsilon}{2}.$$

for all  $N \geq M$ . Also, since  $f$  and  $K_N$  are both  $2\pi$ -periodic, we have

$$\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(x-t) f(t) dt = \int_{x-\pi}^{x+\pi} K_N(\tau) f(x-\tau) d\tau = \int_{-\pi}^{\pi} K_N(\tau) f(x-\tau) d\tau.$$

Then for all  $N \geq M$  and for all  $x \in [-\pi, \pi]$ , we have

$$\begin{aligned}
|\sigma_N f(x) - f(x)| &= \left| \int_{-\pi}^{\pi} K_N(t) f(x-t) dt - \int_{-\pi}^{\pi} K_N(t) f(x) dt \right| \\
&\leq \int_{-\pi}^{\pi} K_N(t) |f(x-t) - f(x)| dt \\
&\leq \int_{|t| < \delta} K_N(t) |f(x-t) - f(x)| dt + \int_{\delta \leq |t| \leq \pi} K_N(t) |f(x-t) - f(x)| dt \\
&\leq \frac{\varepsilon}{2} \int_{|t| < \delta} K_N(t) dt + 2 \|f\|_{\infty} \int_{\delta \leq |t| \leq \pi} \frac{1}{2\pi(N+1) \sin^2 \frac{\delta}{2}} dt \\
&\leq \frac{\varepsilon}{2} + \frac{2 \|f\|_{\infty}}{(N+1) \sin^2 \frac{\delta}{2}} \\
&\leq \varepsilon.
\end{aligned}$$

□

**Remark.** The same proof can be modified if instead of  $K_N(x) \geq 0$ , we have

$$\sup_{N \in \mathbb{N}} \int_{-\pi}^{\pi} |K_N(x)| dx < \infty.$$

Note that

$$\int_{-\pi}^{\pi} |D_N(x)| dx \sim \log N,$$

so we cannot reproduct the proof using Dirichlet kernel.

We only need some last bit of information to conclude the answer of our main question.

**Theorem.** For all  $f \in L^2([-\pi, \pi])$ , we have  $\|\sigma_N f\|_2 \leq \|f\|_2$ .

*Proof.* Suppose first the  $f \in C([-\pi, \pi])$  and  $2\pi$ -periodic. Then  $\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(t) f(x-t) dt$ . It follows that

$$\begin{aligned}
\int_{-\pi}^{\pi} |\sigma_N f(x)|^2 dx &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x-s) \overline{f(x-t)} K_N(s) K_N(t) ds dt dx \\
&= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_N(s) K_N(t) \left[ \int_{-\pi}^{\pi} f(x-s) \overline{f(x-t)} dx \right] ds dt \\
&\leq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_N(s) K_N(t) \|f(\cdot - s)\|_2 \|f(\cdot - t)\|_2 ds dt \\
&\leq \|f\|_2^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_N(s) K_N(t) ds dt \\
&= \|f\|_2^2,
\end{aligned}$$



where we used Cauchy-Schwarz inequality. This implies that  $\|\sigma_N f\|_2 \leq \|f\|_2$ .

Now for the general case, by density there exists sequence  $\{f_n\}_{n=1}^\infty$  of  $2\pi$ -periodic continuous function that  $\|f_n - f\|_2 \rightarrow 0$ . Then,  $\|\sigma_N f_n - \sigma_N f\| \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,

$$\|\sigma_N f\|_2 = \lim_{n \rightarrow \infty} \|\sigma_N f_n\|_2 \leq \lim_{n \rightarrow \infty} \|f_n\|_2 = \|f\|_2.$$

□

**Theorem.** For all  $f \in L^2([-\pi, \pi])$ , we have  $\|\sigma_N f - f\|_2 \rightarrow 0$  as  $N \rightarrow \infty$ . In particular, as a immediate corollary, if  $\hat{f}(n) = 0$  for all  $n \in \mathbb{Z}$ , then  $f = 0$ .

*Proof.* Let  $f \in L^2([-\pi, \pi])$  and  $\varepsilon > 0$ . Again by density there exists  $g \in C([-\pi, \pi])$   $2\pi$ -periodic such that  $\|f - g\|_2 \leq \frac{\varepsilon}{3}$ . Since  $\sigma_N g \rightarrow g$  uniformly on  $[-\pi, \pi]$ , there exists  $M \in \mathbb{N}$  such that for all  $N \geq M$  and all  $x \in [-\pi, \pi]$ , we have

$$|\sigma_N g(x) - g(x)| < \frac{\varepsilon}{3\sqrt{2\pi}}.$$

Then for all  $N \geq M$ ,

$$\begin{aligned} \|\sigma_N f - f\|_2 &\leq \|\sigma_N(f - g)\|_2 + \|\sigma_N g - g\| + \|g - f\|_2 \\ &\leq 2\|f - g\|_2 + \left( \int_{-\pi}^{\pi} |\sigma_N g(x) - g(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \varepsilon. \end{aligned}$$

□

**Remark.** We have shown that for all  $f \in L^2([-\pi, \pi])$ ,  $\|S_N f - f\|_2 \rightarrow 0$ . This does not say  $S_N f$  converge to  $f$  almost everywhere. However, by a theorem by Carleson, for all  $f \in L^2([-\pi, \pi])$ , we actually do have  $S_N f \rightarrow f$  almost everywhere. Also, for all  $1 < p < \infty$ ,  $\|S_N f - f\|_p \rightarrow 0$ . This is not true for  $p = 1$  or  $p = \infty$ .

## 2.4 Minimizers, orthogonal complements, and Riesz representation theorem

### Length minimizers

**Theorem.** Suppose  $H$  a Hilbert space and  $C \subset H$  is a subset such that

1.  $C \neq \emptyset$ .
2.  $C$  is closed.
3.  $C$  is convex. That is, if  $v_1, v_2 \in C$  and  $t \in [0, 1]$ , then  $tv_1 + (1 - t)v_2 \in C$ .

Then, there exists a unique  $v \in C$  such that  $\|v\| = \inf_{u \in C} \|u\|$ .

*Proof.* Let  $d = \inf_{u \in C} \|u\|$ , which we know exists. Then there exists sequence  $\{u_n\}_{n=1}^\infty \subset C$  such that  $\|u_n\| \rightarrow d$ .

Claim that  $\{u_n\}_{n=1}^\infty$  is Cauchy. Let  $\varepsilon > 0$  be given. There exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies

$$2\|u_n\|^2 < 2d^2 + \frac{\varepsilon^2}{2}.$$

It follows that for all  $n, m \geq N$ , we have

$$\|u_n - u_m\|^2 = 2\|u_n\|^2 + 2\|u_m\|^2 - 4\left\|\frac{u_n + u_m}{2}\right\|^2,$$

by the Parallelogram law. Note that  $\frac{u_n + u_m}{2} \in C$ . Therefore,

$$\|u_n - u_m\|^2 \leq 2d^2 + \frac{\varepsilon^2}{2} + 2d^2 + \frac{\varepsilon^2}{2} - 4d^2 = \varepsilon^2.$$

This shows that  $\{u_n\}_{n=1}^\infty$  is Cauchy.

Since  $H$  is Hilbert space, there exists  $v \in H$  such that  $u_n \rightarrow v$ . Since  $C$  is closed,  $v \in C$ . It is also clear that  $\|v\| = d$ . To show this element is unique, suppose  $v, \bar{v} \in C$  and  $\|v\| = \|\bar{v}\| = d$ . Then,

$$\|v - \bar{v}\|^2 = 2\|v\|^2 + 2\|\bar{v}\|^2 - 4\left\|\frac{v + \bar{v}}{2}\right\|^2 \leq 4d^2 - 4d^2 = 0.$$

This implies that  $v = \bar{v}$  and the proof is complete.  $\square$

### Orthocomplements

**Theorem.** If  $H$  is a Hilbert space,  $W \subset H$  is a subspace, then

$$W^\perp = \{u \in H : \langle u, w \rangle = 0 \text{ for all } w \in W\}$$

is a closed linear subspace of  $H$ .

Moreover, if  $W$  is closed, then

$$H = W \oplus W^\perp.$$

That is, for all  $u \in H$ , there exists unique  $w \in W$  and  $w^\perp \in W^\perp$  such that  $u = w + w^\perp$ .

*Proof.* It is easy to show that  $W^\perp$  is a subspace of  $H$ , and  $W \cap W^\perp = \{0\}$ . To show  $W^\perp$  is closed, let  $\{u_n\}_{n=1}^\infty$  be a sequence in  $W^\perp$  and  $u \in H$  such that  $u_n \rightarrow u$ . We need to show that  $u \in W^\perp$ . Let  $w \in W$ , then

$$\langle u, w \rangle = \lim_{n \rightarrow \infty} \langle u_n, w \rangle = 0.$$

Therefore,  $u \in W^\perp$  and  $W^\perp$  is a closed linear subspace of  $H$ .

Now suppose  $W$  is closed. If  $W = H$ , then  $W^\perp = \{0\}$  and  $H = W \oplus W^\perp$ . Now assume that  $W \neq H$ . Let  $u \in H \setminus W$  and define

$$C = u + W = \{u + w : w \in W\}.$$

Note that  $u \in C$  so  $C \neq \emptyset$ . Also,  $C$  is convex, since if  $u + w_1 \in C$ ,  $u + w_2 \in C$ , and  $t \in [0, 1]$ , then

$$t(u + w_1) + (1 - t)(u + w_2) = u + (tw_1 + (1 - t)w_2) \in u + W.$$

Claim that  $C$  is also closed. Suppose  $\{u + w_n\}_{n=1}^\infty \subset C$  is such that  $u + w_n \rightarrow v$  for some  $v \in H$ . We want to show that  $v \in C$ . This implies that  $w_n \rightarrow v - u$  and since  $W$  is closed,  $v - u \in W$ . It follows that  $v = u + (v - u)$  so  $v \in C$ .

Since  $C$  is nonempty, closed, and convex, there exists unique element  $v \in C$  such that

$$\|v\| = \inf_{w \in W} \|u + w\|.$$

Note that  $v \in C$  so  $u - v \in W$ . Also,  $u = (u - v) + v$ . Claim that  $v \in W^\perp$ . Let  $w \in W$  and

$$f(t) = \|v + tw\|^2 = \|v\|^2 + t^2 \|w\|^2 + 2t \operatorname{Re} \langle v, w \rangle.$$

Then  $f(t)$  has a minimum at  $t = 0$ , which implies  $f'(0) = \langle v, w \rangle = 0$ . Repeat the previous argument with  $iw$  in place of  $w$  to obtain  $\operatorname{Re} \langle v, w \rangle = \operatorname{Im} \langle v, w \rangle = 0$ . This shows that  $w \in W^\perp$  and thus  $H = W + W^\perp$ .

To show the decomposition is unique, suppose  $u = w_1 + w_1^\perp = w_2 + w_2^\perp$ . This implies that

$$w_2 - w_1 = w_1^\perp - w_2^\perp \in W \cap W^\perp.$$

However,  $W \cap W^\perp = \{0\}$ , so  $w_1 = w_2$  and  $w_1^\perp = w_2^\perp$ . □

**Theorem.** If  $W \subset H$  is a subspace, then

$$\overline{W} = (W^\perp)^\perp,$$

where  $\overline{W}$  is the closure of  $W$ .

*Proof.* Homework. □

**Definition** (Projection). A bounded linear operator  $P : H \rightarrow H$  is a **projection** if  $P^2 = P$ .

**Theorem.** Let  $H$  be a Hilbert space,  $W \subset H$  be a closed subspace. Then by the previous theorem we have  $H = W \oplus W^\perp$ . Define  $\Pi_W : H \rightarrow H$  in the following way: for  $v = w + w^\perp$ , define

$$\Pi_W(v) = w.$$

Then  $\Pi_W$  is a projection.

*Proof.* It is easy to verify that  $\Pi_W$  is linear and  $\Pi_W^2 = \Pi_W$ . Claim  $\Pi_W$  is bounded. Suppose  $v = w + w^\perp$ . It follows that

$$\|v\|^2 = \|w + w^\perp\|^2 = \|w\|^2 + \|w^\perp\|^2 \geq \|w\|^2.$$

This shows that  $\|\Pi_W(v)\| \leq \|v\|$  so  $\Pi_W$  is a bounded linear operator.  $\square$

### Riesz representation theorem

**Theorem** (Riesz representation theorem). If  $H$  is a Hilbert space, then for all  $f \in H'$ , there exists a unique  $v \in H$  such that

$$f(u) = \langle u, v \rangle \text{ for all } u \in H.$$

*Proof.* For uniqueness, suppose  $f(u) = \langle u, v \rangle = \langle u, \tilde{v} \rangle$ . This implies that  $\langle u, v - \tilde{v} \rangle = 0$  for all  $u \in H$ . Setting  $u = v - \tilde{v}$  gives  $v = \tilde{v}$ .

Now we show existence. If  $f = 0$ , let  $v = 0$ . Suppose now  $f \neq 0$ . Then there exists  $u_1 \in H$  such that  $f(u_1) \neq 0$ . Now let  $u_0 = \frac{u_1}{f(u_1)}$ , then  $f(u_0) = 1$ . Let

$$C = \{u \in H : f(u) = 1\} = f^{-1}(\{1\}).$$

Then  $C$  is a nonempty and closed subset of  $H$ . Claim that  $C$  is also convex. If  $u_1, u_2 \in C$  and  $t \in [0, 1]$ , then

$$f(tu_1 + (1-t)u_2) = tf(u_1) + (1-t)f(u_2) = 1.$$

Therefore,  $C$  is also convex. This implies that there exists  $v_0 \in C$  such that

$$v_0 = \inf_{u \in C} \|u\|.$$

Note that  $v_0 \neq 0$  and let  $v = \frac{v_0}{\|v_0\|^2}$ . Claim this is the desired vector. Let  $N = f^{-1}(\{0\})$ . Then,  $C = v_0 + N$  and  $\|v_0\| = \inf_{w \in N} \|v_0 + w\|$ . By a similar argument as a previous theorem, we have  $v_0 \in N^\perp$ . Let  $u \in H$ , then

$$f(u - f(u)v_0) = f(u) - f(u)f(v_0) = 0.$$

Therefore,  $u - f(u)v_0 \in N$ . Since  $v_0 \in N^\perp$ , we have  $\langle u - f(u)v_0, v_0 \rangle = 0$ . This implies that

$$\begin{aligned} \langle u, v_0 \rangle &= \frac{1}{\|v_0\|^2} \langle u, v_0 \rangle \\ &= \frac{1}{\|v_0\|^2} (\langle u - f(u)v_0, v_0 \rangle + f(u) \langle v_0, v_0 \rangle) \\ &= f(u), \end{aligned}$$

completing the proof.  $\square$

## 2.5 Adjoint of a bounded linear operator on a Hilbert space

**Theorem.** Let  $H$  be a Hilbert space and  $A : H \rightarrow H$  be a bounded linear operator. Then there exists a unique bounded linear operator  $A^* : H \rightarrow H$  (the **adjoint** of  $A$ ) such that for all  $u, v \in H$ ,

$$\langle Au, Au \rangle = \langle u, u \rangle.$$

Moreover,  $\|A^*\| = \|A\|$ .

*Proof.* The uniqueness of  $A^*$  follows from a similar argument of uniqueness for Riesz representation theorem.

Let  $v \in H$ . Define  $f_v : H \rightarrow \mathbb{C}$  by

$$f_v(u) = \langle Au, Au \rangle.$$

Then for any  $u_1, u_2 \in H$  and  $\lambda_1, \lambda_2 \in \mathbb{C}$ , we have

$$\begin{aligned} f_v(\lambda_1 u_1 + \lambda_2 u_2) &= \langle A(\lambda_1 u_1 + \lambda_2 u_2), A(\lambda_1 u_1 + \lambda_2 u_2) \rangle \\ &= \lambda_1 \langle Au_1, Au_1 \rangle + \lambda_2 \langle Au_2, Au_2 \rangle \\ &= \lambda_1 f_v(u_1) + \lambda_2 f_v(u_2). \end{aligned}$$

Therefore,  $f_v$  is a linear map. Next, suppose  $\|u\| = 1$ , then

$$|f_v(u)| = |\langle Au, Au \rangle| \leq \|Au\| \|v\| \leq \|A\| \|v\|.$$

This shows that  $f_v \in H'$ . Then, by Riesz representation theorem, there is a unique  $A^*v \in H$  such that for all  $u \in H$ ,

$$f_v(u) = \langle Au, Au \rangle = \langle u, u \rangle.$$

Claim that  $v \mapsto A^*v$  is linear. Let  $v_1, v_2 \in H$  and  $\lambda_1, \lambda_2 \in \mathbb{C}$ , then for all  $u \in H$ , we have

$$\begin{aligned} \langle u, u \rangle &= \langle Au, Au \rangle \\ &= \overline{\lambda_1} \langle Au, Au \rangle + \overline{\lambda_2} \langle Au, Au \rangle \\ &= \overline{\lambda_1} \langle u, u \rangle + \overline{\lambda_2} \langle u, u \rangle \\ &= \langle u, u \rangle. \end{aligned}$$

This implies that

$$A^*(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 A^*v_1 + \lambda_2 A^*v_2.$$

Therefore,  $A^*$  is a linear map. Next we show  $A^*$  is bounded and  $\|A^*\| = \|A\|$ . Suppose  $\|v\| = 1$ . If  $A^*v = 0$ , then clearly  $\|A^*v\| \leq \|A\|$ . Suppose now  $A^*v \neq 0$ , then

$$\|A^*v\|^2 = \langle A^*v, A^*v \rangle = \langle AA^*v, AA^*v \rangle \leq \|AA^*v\| \|v\| \leq \|A\| \|A^*v\|.$$

Therefore,  $\|A^*\| \leq \|A\|$  and  $A^*$  is a bounded linear operator.

Note that for all  $u, v \in H$ ,

$$\langle A^*u, A^*u \rangle = \overline{\langle v, v \rangle} = \overline{\langle Av, Av \rangle} = \langle u, u \rangle.$$

This implies that  $A = (A^*)^*$ . Therefore,  $\|A\| = \|(A^*)^*\| \leq \|A^*\|$  and thus  $\|A^*\| = \|A\|$ .  $\square$

**Example.** Suppose  $u = (u_1, \dots, u_n) \in \mathbb{C}^n$  and  $(Au)_i = \sum_j A_{ij}u_j$ . Then,

$$\langle Au, Au \rangle = \sum_i (Au)_i \overline{(Au)_i} = \sum_{i,j} A_{ij}u_j \overline{(Au)_i} = \sum_j u_j \overline{\sum_i A_{ij}u_j} = \sum_j u_j \overline{A^*v_j}.$$

This implies that  $(A^*)_{ij} = \overline{A_{ji}}$ .

**Example.** Suppose  $\{A_{ij}\}_{i,j=0}^\infty$  is a double sequence in  $\mathbb{C}^n$  such that

$$\sum_{i,j} |A_{ij}|^2 = \lim_{N \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^N |A_{ij}|^2 < \infty.$$

Define  $A : \ell^2 \rightarrow \ell^2$  by

$$Aa = \sum_{j=1}^\infty A_{ij}a_j,$$

where  $a = \{a_j\}_{j=1}^\infty \in \ell^2$ . Then,  $A \in \mathcal{B}(\ell^2, \ell^2)$  and  $(A^*)_{ij} = \overline{A_{ji}}$ .

**Example.** Suppose  $K \in C([0, 1] \times [0, 1])$  and define  $A : L^2([0, 1]) \rightarrow L^2([0, 1])$  by

$$Af(x) = \int_0^1 K(x, y)f(y) dy.$$

Then we can verify that

$$A^*g(x) = \int_0^1 \overline{K(y, x)}g(y) dy.$$

**Theorem.** Suppose  $H$  is a Hilbert space and  $A : H \rightarrow H$  is a bounded linear operator. Then,

$$(\text{range } A)^\perp = \ker(A^\perp).$$

*Proof.*  $v \in \ker(A^*)$  if and only if  $\langle u, v \rangle = 0$  for all  $u \in H$ . This is equivalent to  $\langle Au, Au \rangle = 0$  for all  $u \in H$ , and this is equivalent to  $v \in (\text{range } A)^\perp$ .  $\square$

**Corollary.** Suppose  $\text{range } A$  is closed, then  $A$  is surjective if and only if  $A^*$  is injective.

With these useful tools we developed, we are soon going to discuss the solvability of linear equations, that is, equations in the form of  $Au = v$ . We take for granted that a bounded linear operator takes bounded sets to bounded sets in finite-dimensional spaces, and so we can find a convergent subsequence using Heine-Borel. So the point is that there is some compactness hidden in here in  $\mathbb{R}^n$  and  $\mathbb{C}^n$ , so we need to study some facts about how compactness and Hilbert spaces before we can talk about solvability of equations.

**Definition (Compact).** If  $X$  is a metric space, we say  $K \subset X$  is **compact** if every sequence in  $K$  has a subsequence converging to an element in  $K$ .

**Example.** Suppose  $H$  is an infinite dimensional separable Hilbert space, then  $B[0, 1]$  is not compact.

*Proof.* Let  $\{e_n\}_{n=1}^\infty$  be an orthonormal subset of  $H$ . Then for all  $n \neq k$ , we have

$$\|e_n - e_k\|^2 = \|e_n\|^2 + \|e_k\|^2 + 2 \operatorname{Re} \langle e_n, e_k \rangle = 2.$$

Therefore,  $\{e_n\}_{n=1}^\infty$  cannot have a convergent subsequence.  $\square$

Recall from Arzela-Ascoli, the extra condition for a set to be compact, other than closed and bounded, is **equicontinuity**. This motivates the following definition.

**Definition** (Equi-small tails). Let  $H$  be a Hilbert space. A subset  $K \subset H$  has **equi-small tails** with respect to a countable orthonormal subset  $\{e_k\}_{k=1}^\infty$  if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $v \in K$ ,

$$\sum_{k>N} |\langle v, e_k \rangle|^2 < \varepsilon^2.$$

**Example.** A finite set has equi-small tails with respect to any countable orthonormal subset  $\{e_n\}_{n=1}^\infty$ .

**Theorem.** Let  $H$  be a Hilbert space and  $\{v_n\}_{n=1}^\infty$  a sequence with  $v_n \rightarrow v$ . Let  $\{e_k\}_{k=1}^\infty$  be a countable orthonormal subset. Then,

1.  $K = \{v_n : n \in \mathbb{N}\} \cup \{v\}$  is compact.
2.  $K$  has equi-small tails with respect to  $\{e_k\}_{k=1}^\infty$ .

*Proof.* 1. Exercise.

2. Since  $v_n \rightarrow v$ , there exists  $M \in \mathbb{N}$  such that for all  $n \geq M$ ,  $\|v_n - v\| < \frac{\varepsilon}{2}$ . Also, choose  $N \in \mathbb{N}$  so large that

$$\sum_{k>N} |\langle v, e_k \rangle|^2 + \max_{1 \leq n \leq M-1} \sum_{k>N} |\langle v_n, e_k \rangle|^2 < \frac{\varepsilon^2}{4}.$$

Then,

$$\sum_{k>N} |\langle v, e_k \rangle|^2 < \frac{\varepsilon^2}{4} < \varepsilon^2.$$

and for all  $1 \leq n \leq M-1$ ,

$$\sum_{k>N} |\langle v_n, e_k \rangle|^2 < \frac{\varepsilon^2}{4} < \varepsilon^2.$$

If  $n \geq M$ , by Bessel's inequality we have

$$\begin{aligned}
 \left[ \sum_{k>N} |\langle v_n, v_n \rangle|^2 \right]^{\frac{1}{2}} &= \left[ \sum_{k>N} |\langle v_n - v, v_n - v \rangle + \langle v, v \rangle|^2 \right]^{\frac{1}{2}} \\
 &\leq \left[ \sum_{k>N} |\langle v_n - v, v_n - v \rangle|^2 \right]^{\frac{1}{2}} + \left[ \sum_{k>N} |\langle v, v \rangle|^2 \right]^{\frac{1}{2}} \\
 &\leq \|v_n - v\| + \frac{\varepsilon}{2} \\
 &< \varepsilon.
 \end{aligned}$$

This shows that  $K$  has equi-small tails with respect to  $\{e_k\}_{k=1}^\infty$ .

□

## 2.6 Compact subsets of a Hilbert space and finite-rank operators

**Theorem.** Let  $H$  be a separable Hilbert space, and  $\{e_k\}_{k=1}^\infty$  be an orthonormal basis of  $H$ . Then  $K \subset H$  is compact if and only if  $K$  is closed, bounded, and has equi-small tails with respect to  $\{e_k\}_{k=1}^\infty$ .

*Proof.* ( $\implies$ ) Suppose  $K$  is compact, then  $K$  is closed and bounded. Suppose for contradiction that  $K$  does not have equi-small tails with respect to  $\{e_k\}_{k=1}^\infty$ . Then there exists  $\varepsilon_0 > 0$  such that for all  $N \in \mathbb{N}$ , there exists  $u_N \in K$  such that

$$\sum_{k>N} |\langle u_N, u_N \rangle|^2 \geq \varepsilon_0^2.$$

Since  $K$  is compact, there exists subsequence  $\{v_n\}_{n=1}^\infty$  of  $\{u_N\}_N$  such that  $v_n \rightarrow v$  for some  $v \in K$ . Then for all  $n \in \mathbb{N}$ , we have

$$\sum_{k>n} |\langle v_n, v_n \rangle|^2 \geq \varepsilon_0^2.$$

Then  $\{v_n : n \in \mathbb{N}\} \cup \{v\}$  does not have equi-small tails. This is a contradiction to the previous theorem. Therefore,  $K$  has equi-small tails with respect to  $\{e_k\}_{k=1}^\infty$ .

( $\impliedby$ ) Suppose  $K$  is closed, bounded, and has equi-small tails. Let  $\{u_n\}_{n=1}^\infty$  be a sequence in  $K$ . Since  $K$  is closed, it suffices to show that  $\{u_n\}_{n=1}^\infty$  has a convergent subsequence. Since  $K$  is bounded, there exists  $C \geq 0$  such that  $\|u_n\| \leq C$ . Then, for all  $k, n \in \mathbb{N}$ ,

$$|\langle v_n, v_n \rangle| \leq \|u_n\| \|e_k\| \leq C.$$

That is, for any  $k \in \mathbb{N}$ ,  $\{\langle u_n, u_n \rangle\}_{n=1}^\infty$  is a bounded sequence in  $\mathbb{C}$ . This implies that there exists a subsequence  $\{\langle u_{n_1(j)}, u_{n_1(j)} \rangle\}_{j=1}^\infty$  of  $\{\langle u_n, u_n \rangle\}_{n=1}^\infty$  that converges in  $\mathbb{C}$ . Now  $\{\langle u_{n_1(j)}, u_{n_1(j)} \rangle\}_{j=1}^\infty$



is a bounded sequence, there exists a subsequence  $\{\langle u_{n_2(j)}, u_{n_2(j)} \rangle\}_{j=1}^\infty$  that converges. Note that  $\lim_{j \rightarrow \infty} \langle u_{n_2(j)}, u_{n_2(j)} \rangle$  and  $\lim_{j \rightarrow \infty} \langle u_{n_2(j)}, u_{n_2(j)} \rangle$  both exist. Inductively, for any  $l \in \mathbb{N}$ , there exists subsequence  $\{n_l(j)\}_{j=1}^\infty$  of  $\{n_{l-1}(j)\}_{j=1}^\infty$  such that for all  $1 \leq k \leq l$ , the limit  $\lim_{j \rightarrow \infty} \langle u_{n_l(j)}, u_{n_l(j)} \rangle$  exists.

For all  $l \in \mathbb{N}$ , let  $v_l = u_{n_l(l)}$ , then  $\{v_l\}_{l=1}^\infty$  is a subsequence of  $\{u_n\}_{n=1}^\infty$  such that for any  $k \in \mathbb{N}$ ,  $\{\langle v_l, v_l \rangle\}_{l=1}^\infty$  converges. Claim that  $\{v_l\}_{l=1}^\infty$  is Cauchy. This will complete the proof since the closed subset  $K$  of Hilbert space  $H$  is complete. Since  $K$  has equi-small tails, there exists  $N \in \mathbb{N}$  such that for all  $l \in \mathbb{N}$ ,

$$\sum_{k > N} |\langle v_l, v_l \rangle| < \frac{\varepsilon^2}{16}.$$

Since the  $N$  sequences  $\{\langle v_l, v_l \rangle\}_{l=1}^\infty, \dots, \{\langle v_l, v_l \rangle\}_{l=1}^\infty$  all converges, there exists  $M \in \mathbb{N}$  such that for all  $l, m \geq M$ , we have

$$\sum_{k=0}^N |\langle v_l, v_l \rangle - \langle v_m, v_m \rangle|^2 < \frac{\varepsilon^2}{4}.$$

Then for all  $l, m \geq M$ , we have

$$\begin{aligned} \|v_l - v_m\| &= \left[ \sum_{k=0}^N |\langle v_l - v_m, v_l - v_m \rangle|^2 + \sum_{k > N} |\langle v_l - v_m, v_l - v_m \rangle|^2 \right]^{\frac{1}{2}} \\ &\leq \left[ \sum_{k=0}^N |\langle v_l - v_m, v_l - v_m \rangle|^2 \right]^{\frac{1}{2}} + \left[ \sum_{k > N} |\langle v_l - v_m, v_l - v_m \rangle|^2 \right]^{\frac{1}{2}} \\ &< \frac{\varepsilon}{2} + \left[ \sum_{k > N} |\langle v_l, v_l \rangle|^2 \right]^{\frac{1}{2}} + \left[ \sum_{k > N} |\langle v_m, v_m \rangle|^2 \right]^{\frac{1}{2}} \\ &< \varepsilon. \end{aligned}$$

□

**Example.** The Hilbert cube

$$K = \{ \{a_k\}_{k=1}^\infty \in \ell^2 : |a_k| \leq 2^{-k} \}$$

is compact.

**Theorem.** A subset  $K$  of a Hilbert space  $H$  is compact if and only if  $K$  is closed, bounded, and for all  $\varepsilon > 0$ , there exists a finite dimensional subspace  $W \subset H$  such that for all  $u \in K$ ,

$$\inf_{w \in W} \|u - w\| < \varepsilon.$$

### Finite rank operators

**Definition** (Finite rank operator). A bounded linear operator  $T \in \mathcal{B}(H)$  is a **finite rank operator** if  $\text{range}(T)$ , which is a subspace of  $H$ , is finite dimensional. Denote the set of all finite rank operator on  $H$  with  $\mathcal{R}(H)$ .

**Theorem.**  $\mathcal{R}(H)$  is a subspace of  $\mathcal{B}(H)$ .

**Theorem.**  $T \in \mathcal{R}(H)$  if and only if there exists an orthonormal set  $\{e_k\}_{k=0}^L$  and constants  $\{c_{ij}\}_{i,j=0}^L \subset \mathbb{C}$  such that

$$Tu = \sum_{i,j=0}^L c_{ij} \langle u, e_j \rangle e_i.$$

*Proof.* ( $\Leftarrow$ ). Clear.

( $\Rightarrow$ ). Since  $\text{range}(T)$  is finite dimensional, there exists orthonormal basis  $\{\bar{e}_k\}_{k=0}^N$  such that

$$\begin{aligned} Tu &= \sum_{k=0}^N \langle Tu, \bar{e}_k \rangle \bar{e}_k \\ &= \sum_{k=0}^N \langle u, u \rangle \bar{e}_k \\ &= \sum_{k=0}^N \langle u, u \rangle \bar{e}_k. \end{aligned}$$

Let  $\{e_0, \dots, e_L\}$  be the orthonormal subset of  $H$  obtained by applying the Gram-Schmidt process on  $\{\bar{e}_0, \dots, \bar{e}_N, v_0, \dots, v_N\}$ . Then, there exists  $a_{ki}, b_{kj}$  such that

$$\bar{e}_k = \sum_{i=0}^L a_{ki} e_i, \quad v_k = \sum_{j=0}^L b_{kj} e_j.$$

Then,

$$Tu = \sum_{k=0}^N \langle u, u \rangle \bar{e}_k = \sum_{i,j=0}^L \left( \sum_{k=0}^N a_{ki} \bar{b}_{kj} \right) \langle u, u \rangle e_i.$$

Setting  $c_{ij} = \sum_{k=0}^N a_{ki} \bar{b}_{kj}$  completes the proof.  $\square$

**Theorem.** The following holds:

1.  $T \in \mathcal{R}(H)$  implies  $T^* \in \mathcal{R}(H)$ .
2.  $T \in \mathcal{R}(H)$  and  $A, B \in \mathcal{B}(H)$  implies  $ATB \in \mathcal{R}(H)$ .

*Proof.* 1. Write

$$Tu = \sum_{i,j=0}^L c_{ij} \langle w, w \rangle e_i.$$

Then, for any  $u, v \in H$ , we have

$$\begin{aligned} \langle u, u \rangle &= \langle Tu, Tu \rangle \\ &= \left\langle \sum_{i,j} c_{ij} \langle u, u \rangle e_i, \sum_{i,j} c_{ij} \langle u, u \rangle e_i \right\rangle \\ &= \sum_{i,j} c_{ij} \langle u, u \rangle \langle e_i, e_i \rangle \\ &= \langle u, u \rangle \\ &= \langle u, u \rangle. \end{aligned}$$

This implies that

$$T^*v = \sum_{i,j=0}^L \overline{c_{ij}} \langle v, v \rangle e_j,$$

so  $T^* \in \mathcal{R}(H)$ .

2. Exercise.

□

Now a good question to investigate is whether  $\mathcal{R}(H)$  is a closed subspace in  $\mathcal{B}(H)$ ? That is, if  $T_n \in \mathcal{R}(H)$  and  $\|T_n - T\| \rightarrow 0$ , then is  $T \in \mathcal{R}(H)$ ? The answer turns out to be no.

**Example.** Take  $T_n : \ell^2 \rightarrow \ell^2$  defined by

$$T_n a = \left\{ \frac{a_1}{1}, \dots, \frac{a_n}{n}, 0, 0, \dots \right\}$$

for all  $a = \{a_k\}_{k=1}^\infty \in \ell^2$ . Then  $T_n \in \mathcal{R}(H)$ . Let  $T : \ell^2 \rightarrow \ell^2$  defined by

$$Ta = \left\{ \frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \dots \right\}.$$

We can verify that  $\|T - T_n\| \leq \frac{1}{n+1}$  but  $T$  is not finite rank.

As  $\mathcal{R}(H)$  is not closed, a question is what is the closure  $\overline{\mathcal{R}(H)}$  in  $\mathcal{B}(H)$ . The answer to this is the set of **compact operators**.

**Definition** (Compact operator).  $K \in \mathcal{B}(H)$  is a **compact operator** if  $\overline{K(B[0,1])}$  is compact.

**Theorem.**  $K \in \mathcal{B}(H)$  is a compact operator if and only if  $K \in \overline{\mathcal{R}(H)}$ .