

Measure and Integration

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3 Signed measure and differentiation

Exercise (Folland 3.2). If ν a signed measure, E is ν -null if and only if $|\nu|(E) = 0$. Also if ν and μ are signed measures, then $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof. It is clear that $|\nu|(E) = 0$ implies that E is ν -null. Now suppose E is ν -null and let $X = P \cup N$ be the Hahn decomposition of ν . Suppose for contradiction that $|\nu|(E) > 0$, then it follows that $\nu^+(E) > 0$ and $\nu^-(E) > 0$. We then have $\nu^+(E \cap P) = \nu^+(E \cap P) + \nu^+(E \cap N) = \nu^+(E) > 0$, but $\nu^-(E \cap P) \leq \nu^-(P) = 0$. Therefore, $\nu(E \cap P) > 0$, a contradiction with E being ν -null. Therefore, $|\nu|(E) = 0$.

Suppose $\nu \perp \mu$, then there is $X = E \cup F$ such that E is ν -null and F is μ -null. It follows that $|\nu|(E) = 0$, so $|\nu| \perp \mu$. Therefore, $\nu \perp \mu$ implies $|\nu| \perp \mu$. It is clear that $|\nu| \perp \mu$ implies $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Now suppose $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Then we have $X = E^+ \cup F^+ = E^- \cup F^-$ where F^+ is ν^+ null, F^- is ν^- null, and E^\pm is μ -null. Let $E = E^+ \cup E^-$ and $F = F^+ \cap F^- = E^c$. Then we can verify that E is μ -null and F is ν -null. Therefore, $\nu \perp \mu$ and the proof is complete. \square

Exercise (Folland 3.3). Let ν be a signed measure on (X, \mathfrak{M}) .

1. $L^1(\nu) = L^1(|\nu|)$.
2. If $f \in L^1(\nu)$, then $|\int f d\nu| \leq \int |f| d|\nu|$.
3. If $E \in \mathfrak{M}$, $|\nu|(E) = \sup \{|\int_E f d\nu| : |f| \leq 1\}$

Proof. 1. Since we have

$$\int |f| d\nu = \int |f| d\nu^+ - \int |f| d\nu^-, \quad \int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^-,$$

it follows immediately that $L^1(\nu) = L^1(|\nu|)$.

2. We have

$$\left| \int f d\nu \right| = \left| \int f d\nu^+ - \int f d\nu^- \right| \leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|.$$

3. By the previous item, we know

$$|\nu|(E) = \int_E 1 d|\nu| \geq \int_E |f| d|\nu| \geq \left| \int f d\nu \right|.$$

for any $|f| \leq 1$. Also, $|\nu|(E) = \int |\chi_P + \chi_N| d\nu$, where $X = P \cup N$ is the Hahn decomposition so $|\chi_P + \chi_N| = 1$.

\square

Exercise (Folland 3.4). If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Proof. *** TO-DO *** □

Exercise (Folland 3.7). Suppose ν is a signed measure on (X, \mathfrak{M}) and $E \in \mathfrak{M}$.

1. $\nu^+(E) = \sup \{\nu(F) : F \subset E, F \in \mathfrak{M}\}$ and $\nu^-(E) = -\inf \{\nu(F) : F \subset E, F \in \mathfrak{M}\}$.
2. $|\nu|(E) = \sup \left\{ \sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \dots, E_n \text{ disjoint, and } \bigcup_{j=1}^n E_j = E \right\}$.

Proof. *** TO-DO *** □

Exercise (Folland 3.8). $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Proof. It is clear that $|\nu| \ll \mu$ implies $\nu^+ \ll \mu$ and $\nu^- \ll \mu$ implies $\nu \ll \mu$. It then remains to show that $\nu \ll \mu$ implies $|\nu| \ll \mu$. Let $X = P \cup N$ be the Hahn decomposition for ν and $E \in \mathfrak{M}$ be such that $\mu(E) = 0$. It follows that $\mu(E \cap P) = 0$ and $\mu(E \cap N) = 0$. Then $\nu(E \cap P) = \nu^+(E \cap P) = 0$ and $\nu(E \cap N) = \nu^-(E \cap N) = 0$. Therefore,

$$\nu(E) = \nu^+(E \cap P) + \nu^-(E \cap N) = 0,$$

as desired. □

Exercise (Folland 3.18). Let ν be a complex measure on (X, \mathfrak{M}) . Prove that $L^1(\nu) = L^1(|\nu|)$ and if $f \in L^1(\nu)$, then $|\int f d\nu| \leq \int |f| d|\nu|$.

Proof. For $L^1(\nu) \subset L^1(|\nu|)$, consider $f \in L^1(\nu)$. Note that $\nu = \nu_r + i\nu_i$ and it is easy to verify that $|\nu_i| = |\nu_i|$. Therefore by Proposition 3.14, we have $|\nu| \leq |\nu_r| + |\nu_i|$. It follows that

$$\begin{aligned} \int |f| d|\nu| &\leq \int |f| d|\nu_r| + \int |f| d|\nu_i| \\ &= \int |f| d\nu_r^+ + \int |f| d\nu_r^- + \int |f| d\nu_i^+ + \int |f| d\nu_i^-. \end{aligned}$$

Since $f \in L^1(\nu)$, all four terms are finite and thus $f \in L^1(|\nu|)$.

For $L^1(|\nu|) \subset L^1(\nu)$, consider $f \in L^1(|\nu|)$. Then we have

$$\int |f| d\nu = \int |f| \frac{d\nu}{d|\nu|} d|\nu| \leq \int |f| \left| \frac{d\nu}{d|\nu|} \right| d|\nu| = \int |f| d|\nu|,$$

where we have used the fact that $d\nu/d|\nu|$ has absolute value 1 $|\nu|$ -a.e. This shows that $f \in L^1(\nu)$.

Moreover, we have that $|\int f d\nu| \leq \int |f| d|\nu|$. Therefore,

$$\left| \int f d\nu \right| \leq \int |f| d|\nu|,$$

as desired. □

Exercise (Folland 3.19). If ν, μ are complex measures and λ is a positive measure, then $\nu \perp \mu$ if and only if $|\nu| \perp |\mu|$, and $\nu \ll \lambda$ if and only if $|\nu| \ll \lambda$.

Proof. The “if” direction is clear for both propositions.

Suppose $\nu \perp \mu$. Then $\nu_r \perp \mu_r$, $\nu_i \perp \mu_r$, $\nu_r \perp \mu_i$, and $\nu_i \perp \mu_i$. It follows that $|\nu_r| \perp \mu_r$ and $|\nu_i| \perp \mu_r$. Since $|\nu| \leq |\nu_r| + |\nu_i|$, we have $|\nu| \perp \mu_r$. Similarly $|\nu| \perp \mu_i$. Following the same reasoning, we obtain $|\nu| \perp |\mu|$, as desired.

Suppose $\nu \ll \lambda$. Since $\nu = \nu_r + i\nu_i$, we have $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$. Recall from Exercise 3.8 that this implies $|\nu_r| \ll \lambda$ and $|\nu_i| \ll \lambda$. Moreover, $|\nu| \leq |\nu_r| + |\nu_i|$, so $|\nu| \ll \lambda$. \square

Exercise (Folland 3.20). If ν is a complex measure on (X, \mathfrak{M}) and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

Proof. By Lebesgue-Randon-Nikodym theorem, we have $d\nu = f d\mu$ for some function f and positive measure μ . It follows that $d|\nu| = |f| d\mu$ and

$$\int f d\mu = \int |f| d\mu.$$

Now let $E \in \mathfrak{M}$. We then have

$$\int_E f d\mu + \int_{E^c} f d\mu = \int_E |f| d\mu + \int_{E^c} |f| d\mu.$$

It follows that

$$0 \leq \int_E |f| - \operatorname{Re}(f) d\mu = \int_{E^c} \operatorname{Re}(f) - |f| d\mu \leq 0.$$

Since $E \in \mathfrak{M}$ is arbitrary, $\operatorname{Re}(f) = |f|$ and $\operatorname{Im}(f) = 0$ a.e. It follows that $d\nu = f d\mu = |f| d\mu = d|\nu|$. \square