

Probability

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1 Measure theory review

1.1 Measurable space and mapping

Definition (σ -field). A set of sets $\Sigma \subset 2^\Omega$ is a σ -field if

- $\emptyset \in \Sigma$,
- If $A \in \Sigma$, then $A^c \in \Sigma$,
- If $\{A_i\}_{i=1}^\infty \subset \Sigma$, then $\bigcup_{i=1}^\infty A_i \in \Sigma$.

The pair (Ω, Σ) is called a measurable space.

Definition (atom). Let Σ be a σ -field. Say $A \in \Sigma$ is an atom if for all $B \in \Sigma$ either $A \subset B$ or $A \cap B = \emptyset$.

Proposition. For all $\omega \in \Omega$, there exists atom $A \in \Sigma$ containing ω if Ω is finite or countable.

Proof. Only prove this for Ω finite. Define $\tilde{A} = \bigcap \{B \in \Sigma : \omega \in B\}$. □

Corollary. If Ω is finite or countable, there exists a partition $\Omega = \bigsqcup_i \Omega_i$, where each Ω_i is an atom of Σ . With this partition, Σ is just the power set with respect to $\{\Omega_i\}_i$.

Definition. If $F \subset 2^\Omega$, then the σ -field generated by F is the smallest σ -field containing all elements of F .

Example. Let $\Omega = \{1, 2, 3, 4, 5\}$ and $F = \{\{2, 3\}, \{3, 4\}\}$. Construct σ -field Σ generated by F . Σ is all possible union of sets from the collection $\{\{2\}, \{3\}, \{4\}, \{1, 5\}\}$.

Definition (measurable mapping). Given two measurable spaces (Ω, Σ) and $(\tilde{\Omega}, \tilde{\Sigma})$. Then $f : \Omega \rightarrow \tilde{\Omega}$ is measurable if $f^{-1}(B) \in \Sigma$ for all $B \in \tilde{\Sigma}$.

Definition (Borel σ -field). Let (T, τ) be a topological space. Then the Borel σ -field $\mathcal{B}(T, \tau)$ is defined as the smallest σ -field containing all open sets.

Definition (product measurable space). Given two measurable spaces (Ω, Σ) and $(\tilde{\Omega}, \tilde{\Sigma})$. We can define the product measurable space as follows: let the ground set be $\Omega \times \tilde{\Omega}$, and let $\Sigma \otimes \tilde{\Sigma}$ be the smallest σ -field containing all rectangles $B \times \tilde{B}$ where $B \in \Sigma$ and $\tilde{B} \in \tilde{\Sigma}$.

More generally, let Λ be an index set and $(\Omega_\lambda, \Sigma_\lambda)_{\lambda \in \Lambda}$. Define the product σ -field $\bigotimes_{\lambda \in \Lambda} \Sigma_\lambda$ be the smallest σ -field containing all elements in the form of $\prod_{\lambda \in \Lambda} B_\lambda$ where $B_\lambda \in \Sigma_\lambda$ and $B_\lambda = \Omega_\lambda$ for all but countably many indices.

Proposition. Let $(\Omega_i, \Sigma_i)_{i=1}^n$ be measurable spaces and $(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$ be the product space. Let (Ω, Σ) be the domain and $f = (f_1, \dots, f_n) : (\Omega, \Sigma) \rightarrow (\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$. Suppose f is measurable, then every coordinate projection $f_i : \Omega \rightarrow \Omega_i$ is measurable.

This is also true for arbitrary index set.

1.2 Measure space

Definition (measure). Let (Ω, Σ) be a measurable space. Then $\mu : \Sigma \rightarrow [0, \infty]$ is a measure if

- $\mu(\emptyset) = 0$.
- If $A_i \in \Sigma$ is pairwise disjoint then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

Proposition (continuity of measure). If $A_1 \subset A_2 \subset \dots$ is a nested sequence of elements of Σ and μ be any measure on (Ω, Σ) . Then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i).$$

If $A_1 \supset A_2 \supset \dots$ is a nested sequence of elements of Σ and $\mu(A_n) < \infty$ for some n . Then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i).$$