

# Probability

Notes taken by Runqiu Ye

Lectures by Konstantin Tikhomirov

Carnegie Mellon University

Spring 2026

## Contents

<b>1 Measure theory review</b>	<b>3</b>
1.1 Measurable space and mapping . . . . .	3
1.2 Measure space . . . . .	4
1.3 $\pi$ - $\lambda$ theorem . . . . .	4

# 1 Measure theory review

## 1.1 Measurable space and mapping

**Definition** ( $\sigma$ -field). A set of sets  $\Sigma \subset 2^\Omega$  is a  $\sigma$ -field if

- $\emptyset \in \Sigma$ ,
- If  $A \in \Sigma$ , then  $A^c \in \Sigma$ ,
- If  $\{A_i\}_{i=1}^\infty \subset \Sigma$ , then  $\bigcup_{i=1}^\infty A_i \in \Sigma$ .

The pair  $(\Omega, \Sigma)$  is called a measurable space.

**Definition** (atom). Let  $\Sigma$  be a  $\sigma$ -field. Say  $A \in \Sigma$  is an atom if for all  $B \in \Sigma$  either  $A \subset B$  or  $A \cap B = \emptyset$ .

**Proposition.** For all  $\omega \in \Omega$ , there exists atom  $A \in \Sigma$  containing  $\omega$  if  $\Omega$  is finite or countable.

*Proof.* Define  $\tilde{A} = \bigcap \{B \in \Sigma : \omega \in B\}$ . We can check that  $\tilde{A} \in \Sigma$  and  $\tilde{A}$  is an atom containing  $\omega$ .  $\square$

**Corollary.** If  $\Omega$  is finite or countable, there exists a partition  $\Omega = \bigsqcup_i \Omega_i$ , where each  $\Omega_i$  is an atom of  $\Sigma$ . With this partition,  $\Sigma$  is just the power set with respect to  $\{\Omega_i\}_i$ .

**Definition.** If  $F \subset 2^\Omega$ , then the  $\sigma$ -field generated by  $F$  is the smallest  $\sigma$ -field containing all elements of  $F$ . Write this  $\sigma$ -field as  $\sigma(F)$ .

**Example.** Let  $\Omega = \{1, 2, 3, 4, 5\}$  and  $F = \{\{2, 3\}, \{3, 4\}\}$ . Construct  $\sigma$ -field  $\Sigma$  generated by  $F$ .  $\Sigma$  is all possible union of sets from the collection  $\{\{2\}, \{3\}, \{4\}, \{1, 5\}\}$ .

**Definition** (measurable mapping). Given two measurable spaces  $(\Omega, \Sigma)$  and  $(\tilde{\Omega}, \tilde{\Sigma})$ . Then  $f : \Omega \rightarrow \tilde{\Omega}$  is measurable if  $f^{-1}(B) \in \Sigma$  for all  $B \in \tilde{\Sigma}$ .

**Definition** (Borel  $\sigma$ -field). Let  $(T, \tau)$  be a topological space. Then the Borel  $\sigma$ -field  $\mathcal{B}(T, \tau)$  is defined as the smallest  $\sigma$ -field containing all open sets.

**Definition** (product measurable space). Given two measurable spaces  $(\Omega, \Sigma)$  and  $(\tilde{\Omega}, \tilde{\Sigma})$ . We can define the product measurable space as follows: let the ground set be  $\Omega \times \tilde{\Omega}$ , and let  $\Sigma \otimes \tilde{\Sigma}$  be the smallest  $\sigma$ -field containing all rectangles  $B \times \tilde{B}$  where  $B \in \Sigma$  and  $\tilde{B} \in \tilde{\Sigma}$ .

More generally, let  $\Lambda$  be an index set and  $(\Omega_\lambda, \Sigma_\lambda)_{\lambda \in \Lambda}$ . Define the product  $\sigma$ -field  $\bigotimes_{\lambda \in \Lambda} \Sigma_\lambda$  be the smallest  $\sigma$ -field containing all elements in the form of  $\prod_{\lambda \in \Lambda} B_\lambda$  where  $B_\lambda \in \Sigma_\lambda$  and  $B_\lambda = \Omega_\lambda$  for all but countably many indices.

**Proposition.** Let  $(\Omega_i, \Sigma_i)_{i=1}^n$  be measurable spaces and  $(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$  be the product space. Let  $(\Omega, \Sigma)$  be the domain and  $f = (f_1, \dots, f_n) : (\Omega, \Sigma) \rightarrow (\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \Sigma_i)$ . Suppose  $f$  is measurable, then every coordinate projection  $f_i : \Omega \rightarrow \Omega_i$  is measurable.

This is also true for arbitrary index set.

**Proposition.** If  $f : (\Omega, \Sigma) \rightarrow (\Omega_f, \Sigma_f)$  and  $g : (\Omega, \Sigma) \rightarrow (\Omega_g, \Sigma_g)$ , then the concatenation  $(f, g)$  is measurable w.r.t. the product space  $(\Omega_f \times \Omega_g, \Sigma_f \otimes \Sigma_g)$ .

*Proof.* Let  $A \times B$  be such that  $A \in \Sigma_f$  and  $B \in \Sigma_g$ . Then the preimage

$$(f, g)^{-1}(A \times B) = f^{-1}(A) \cap g^{-1}(B) \in \Sigma.$$

By definition, the product  $\sigma$ -field is generated by rectangles, so the proof is complete.  $\square$

## 1.2 Measure space

**Definition (measure).** Let  $(\Omega, \Sigma)$  be a measurable space. Then  $\mu : \Sigma \rightarrow [0, \infty]$  is a measure if

- $\mu(\emptyset) = 0$ .
- If  $A_i \in \Sigma$  is pairwise disjoint then  $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ .

**Proposition (continuity of measure).** If  $A_1 \subset A_2 \subset \dots$  is a nested sequence of elements of  $\Sigma$  and  $\mu$  be any measure on  $(\Omega, \Sigma)$ . Then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i).$$

If  $A_1 \supset A_2 \supset \dots$  is a nested sequence of elements of  $\Sigma$  and  $\mu(A_n) < \infty$  for some  $n$ . Then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i).$$

**Definition.** Let  $(\Omega, \Sigma, \mu)$  be a measure space.

Say  $\mu$  is  $\sigma$ -finite if there is a representation  $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$  where  $\Omega_i \in \Sigma$  and  $\mu(\Omega_i) < \infty$ .

Say  $\mu$  is a probability measure if  $\mu(\Omega) = 1$ .

**Definition (completion of measure space).** Let  $(\Omega, \Sigma, \mu)$  be a measure space. Let

$$\tilde{\Sigma} = \{A \cup B : A \in \Sigma, B \subset \Omega, \text{there exists } C \in \Sigma \text{ with } \mu(C) = 0 \text{ and } B \subset C\}.$$

We can check  $\tilde{\Sigma}$  is a  $\sigma$ -field. If  $\tilde{\mu}$  is a measure on  $(\Omega, \tilde{\Sigma})$  which agrees with  $\mu$  on  $\Sigma$ , then  $(\Omega, \tilde{\Sigma}, \tilde{\mu})$  is called a completion of  $(\Omega, \Sigma, \mu)$ .

## 1.3 $\pi$ - $\lambda$ theorem

**Definition ( $\pi$ -system).** Let  $\Omega$  be a set and  $\mathcal{P}$  be a collection of subsets of  $\Omega$ . Then  $\mathcal{P}$  is a  $\pi$ -system if it is closed with respect to taking finite intersections. That is,  $A, B \in \mathcal{P}$  implies  $A \cap B \in \mathcal{P}$ .

**Example.** On the real line  $\mathbb{R}$ , both  $\mathcal{P}_1 = \{(a, b) : a < b\}$  and  $\mathcal{P}_2 = \{(-\infty, a] : a \in \mathbb{R}\}$  are  $\pi$ -systems.

**Definition ( $\lambda$ -system).** Let  $\Omega$  be a set and  $\mathcal{L}$  be a collection of subsets of  $\Omega$ . Say  $\mathcal{L}$  is a  $\lambda$ -system if

- $\emptyset \in \mathcal{L}$ ,
- $A \in \mathcal{L}$  implies  $A^c \in \mathcal{L}$ ,

- for all countable collection of disjoint elements  $A_i \in \mathcal{L}$ , we have  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$ .

**Theorem** ( $\pi$ - $\lambda$  theorem). Let  $\Omega$  be a set,  $\mathcal{P}$  be a  $\pi$ -system and  $\mathcal{L}$  be a  $\lambda$ -system. Also suppose  $\mathcal{P} \subset \mathcal{L}$ , then  $\sigma(\mathcal{P}) \subset \mathcal{L}$ .

*Proof.* Let  $\ell(\mathcal{P})$  be the smallest  $\lambda$ -system on  $\Omega$  containing  $\mathcal{P}$ . The goal is to show that  $\ell(\mathcal{P})$  is a  $\sigma$ -field. We need to show that if  $A_i \in \ell(\mathcal{P})$  for  $1 \leq i < \infty$ , then  $\bigcup_{i=1}^{\infty} A_i \in \ell(\mathcal{P})$ . Note that

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left( A_i \setminus \bigcup_{j=1}^{i-1} A_j \right),$$

so it suffices to show that  $A, B \in \ell(\mathcal{P})$  implies  $A \cap B \in \ell(\mathcal{P})$ .

For  $A \in \ell(\mathcal{P})$  we define

$$W_A = \{B \subset \Omega : A \cap B \in \ell(\mathcal{P})\}.$$

It can be directly verified that  $W_A$  is a  $\lambda$ -system.

Take  $A \in \mathcal{P}$ , then for any  $B \in \mathcal{P}$  we have  $A \cap B \in \mathcal{P} \subset \ell(\mathcal{P})$ . Hence,  $\mathcal{P} \subset W_A$  and thus  $\ell(\mathcal{P}) \subset W_A$  for all  $A \in \mathcal{P}$ , as  $\ell(\mathcal{P})$  is the smallest  $\lambda$ -system on  $\Omega$  containing  $\mathcal{P}$ . Now take  $A \in \ell(\mathcal{P})$ , we have  $A \in W_B$  for all  $B \in \mathcal{P}$ . It follows that  $A \cap B \in \ell(\mathcal{P})$  and thus  $B \in W_A$ . Hence similarly  $\ell(\mathcal{P}) \subset W_A$  for all  $A \in \ell(\mathcal{P})$ .

Now for any pair  $B, C \in \ell(\mathcal{P})$ , we have  $C \in W_B$  and thus  $B \cap C \in \ell(\mathcal{P})$ . This completes the proof.  $\square$