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CLOSED-LOOP CONTROL WITH DELAYED INFORMATION

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Abstract

The theory of Markov Control Model with Perfect State Information (MCM-PSI) requires that the current state of the system is known to the decision maker at decision instants. Otherwise, one speaks of Markov Control Model with Imperfect State Information (MCM-ISI). In this article, we introduce a new class of MCM-ISI, where the information on the state of the system is delayed. Such an information structure is encountered, for instance, in high-speed data networks. In the first part of this article, we show that by enlarging the state space so as to include the last known state as well as all the decisions made during the travel time of the information, we may reduce a MCM-ISI to a MCM-PSI. In the second part of this paper, this result is applied to a flow control problem. Considered is a discrete time queueing model with Bernoulli arrivals and geometric services, where the intensity of the arrival stream is controlled. At the beginning of slot $t+1$, $t = 0, 1, 2, \ldots$, the decision maker has to select the probability of having one arrival in the current time slot from the set $\{p_1, p_2\}$, $0 \leq p_2 < p_1 \leq 1$, only on the basis of the queue-length and action histories in $[0, t]$. The aim is to optimize a discounted throughput/delay criterion. We show that there exists an optimal policy of a threshold type, where the threshold is seen to depend on the last action.

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1 Introduction

The theory of Markov Control Model with Perfect State Information (MCM-PSI) requires that the current state of the system is known to the decision maker at decision instants (Bertsekas [1], Ross [8], Schäl [9]). While this assumption may be realistic or may provide good approximations in some cases, it is no longer true, for instance, in modern communication networks. The reason is that the propagation delay in such networks is no longer negligible with respect to the packet transmission time. As a result, the state of the system may have changed considerably by the time it is known by the decision maker. This, in turn, may yield serious problems when closed-loop controls are used.

Control problems with imperfect state information have already received much attention in the literature. When the control is centralized the standard assumption in this setting is that the decision maker instead of having perfect knowledge of the state has only access to an observation of this state (see e.g., Bertsekas [1], Hernández-Lerma [3]). When the control is decentralized (i.e., there are several decision makers), the so-called Delayed Sharing of State Information (DSSI) pattern provides another instance of imperfect state information structure (Grizzle and Marcus [2], Hsu and Marcus [4], Schoute [10]). In that case, each decision maker informs the others about his observation with a delay. However, in each case the imperfect state information (or partially observable) problem is seen to reduce to a perfect state information (or completely observable) problem by enlarging the state space.

The first objective of this work is to introduce a new type of imperfect information structure, the so-called N-Step Delayed State Information (N-SDSI) structure. We assume a discrete-time MCM with a single decision maker. The N-SDSI pattern assumes that the state of the system at time $t \in \mathbb{N} := \{0, 1, 2, \ldots\}$ is not known by the decision maker until time $t + N$, where $N \geq 1$ is an arbitrary integer. However, we assume that all the past actions are known with no delay. Therefore, and as opposed to the DSSI pattern briefly described above, the N-SDSI pattern is such that the decision maker has no information whatsoever on the current state of the system. Alike the previous studies on imperfect state information, we show that this MCM-ISI can be converted into a MCM-PSI by enlarging the state space. However, because of the simple structure of the N-SDSI pattern the new state of the system remains easy to handle.

The second objective is to solve a flow control problem with a 1-SDSI pattern (see Stidham [11] for a survey on flow control models). We consider a discrete time queueing model with Bernoulli arrivals and geometric services, where the intenstiy of the arrival stream is controlled. At the beginning of slot $t + 1$, $t \in \mathbb{N}$, the decision maker has to select the probability of having one arrival in the current time slot from the set $\{p_1, p_2\}$, $0 \leq p_2 < p_1 \leq 1$, only on the basis of the queue-length and action histories in $[0, t]$. We show that there exists a policy of a threshold type that optimizes a discounted throughput/delay criterion over an infinite horizon. More precisely, we show that there exists a mapping $l : \{p_1, p_2\} \to \mathbb{N} \cup \{+\infty\}$ such that the probability p_1 (resp. p_2) is chosen at time $t + 1$ if $x < l(a)$ (resp. $x \geq l(a)$), where x is the queue-length at time t and a is the action chosen at time t, respectively. We further show that $l(p_1) \leq l(p_2) \leq l(p_1) + 1$.

The paper is organized as follows. In Section 2, we briefly recall the basic features of a MCM-PSI. Then, the N-SDSI structure is introduced, and we show how the related MCM-ISI can be transformed into a MCM-PSI. These results are then used in Section 3 to solve the flow control problem in the case when $N = 1$. Concluding remarks are given in Section 4.

2 Markov Control Model with Delayed Information

Let $\mathbb{R}$ denote the set of all real numbers. Let g be any mapping defined on some set E such that $E = E_1 \times E_2$. With a slight abuse of notation, $g(e_1, e_2)$ will stand for $g(e)$ for any $e = (e_1, e_2) \in E$, $e_i \in E_i$, $i = 1, 2$.

2.1 Markov Control Model with Perfect State Information

A MCM-PSI (see Schäl [9]) is the collection of the following objects: a state space $\mathbf{X}$, an action space $\mathbf{A}$, a transition law q and a cost function $C : \mathbf{X} \times \mathbf{A} \to \mathbb{R}$. In the following, we shall assume that the set $\mathbf{X}$ is denumerable and that the set $\mathbf{A}$ is finite (a more general setting can be found in Hernández-Lerma [3] and in Schäl [9]). We shall also assume without loss of generality that in every state all the actions in $\mathbf{A}$ are available to the decision maker.

Let $\mathcal{P}(\mathbf{X})$ and $\mathcal{P}(\mathbf{A})$ be the set of all probability measures on the Borel σ-algebra of $\mathbf{X}$ and $\mathbf{A}$, respectively.

The transition law q is a transition probability $q : \mathbf{X} \times \mathbf{A} \to \mathcal{P}(\mathbf{X})$, where $q(\bullet \mid x, a)$ is the distribution of the next state visited by the system if the system is in state $x \in \mathbf{X}$ and action $a \in \mathbf{A}$ is taken.

A control policy (or simply a policy) u is defined as a sequence of conditional probabilities $u_t : H_t \to \mathcal{P}(\mathbf{A})$, $H_0 := \mathbf{X}$ and $H_{t+1} = H_t \times (\mathbf{A} \times \mathbf{X})$ for all $t \in \mathbb{N}$, such that $u_t(h_t; \bullet)$ assigns probability one to the set $\mathbf{A}$ for all $h_t \in H_t$, $t \in \mathbb{N}$. Let U be the class of all policies.

A *stationary* policy is a policy such that $u_t(h_t; \bullet)$ is concentrated at the point $\alpha(x_t)$ for all $h_t = (x_0, a_0, \ldots, x_{t-1}, a_{t-1}, x_t) \in H_t$, $t \in \mathbb{N}$, where α is a measurable mapping from $\mathbf{X}$ to $\mathbf{A}$. In other words, a stationary policy is a nonrandomized policy that only depends on the current state of the system.

Given some initial distribution p_0 on $\mathbf{X}$ and some transition law q, any policy u defines a probability measure on the product space $(\mathbf{X} \times \mathbf{A})^\infty$ endowed with the product σ-algebra (see Schäl [9]). On this probability space, let us define the random variables X_t and A_t that describe the state of the system at time t and the action taken at time t, respectively, for all $t \in \mathbb{N}$.

A standard objective in this setting is to solve the following control problem (e.g., see Ross [8]):

$(\mathbf{P}_\beta)$: Minimize $V_\beta(x, u)$ over U for all $x \in \mathbf{X}$, where

$$V_\beta(x, u) := E^u \left[\sum_{t \in \mathbb{N}} \beta^t \, C(X_t, A_t) \mid X_0 = x \right],$$

$u \in U$, where β is a discount factor in $(0, 1)$. The quantity $V_\beta(x, u)$ is the expected β-discounted cost incurred over an infinite horizon. Let $V_\beta(x) := \inf_{u \in U} V_\beta(x, u)$.

Let $\mathcal{H}$ be the set of all real-valued function on $\mathbf{X}$. The solution to the problem $\mathbf{P}_\beta$ may be obtained by using the following basic result of the MCM-PSI theory (cf. Bertsekas[1, Chapter 4]):

Theorem 2.1 *Assume that the cost function C is nonnegative. Then, for every $\beta \in (0,1)$, V_β satisfies the Dynamic Programming (DP) equation*

$$V_\beta(x) = T_\beta\, V_\beta(x) \tag{1}$$

for all $x \in \mathbf{X}$, where the DP operator $T_\beta : \mathcal{H} \to \mathcal{H}$ is given by

$$T_\beta f(\bullet) := \min_{a \in \mathbf{A}} \left\{ C(\bullet, a) + \beta \sum_{x' \in \mathbf{X}} f(x')\, q(x' \,|\, \bullet, a) \right\}. \tag{2}$$

Further, the stationary policy which selects an action minimizing the right-hand side of (1) for all $x \in \mathbf{X}$ solves the problem $\mathbf{P}_\beta$.

2.2 The N-SDSI Pattern

We now consider the case when the state of the system at time t is not known to the decision maker until time $t + N$ for all $t \in \mathbb{N}$, where N is a given integer. However, we will still assume that all past actions are known without delay at any decision epoch.

It is worth noting that this setting differs from the imperfect information setting usually encountered in the literature (see Bertsekas [1, Chapter 3] and Hernández-Lerma [3, Chapter 4]), where at each stage t the decision maker receives *some observation* about the current state of the system, that may be corrupted by some random disturbance.

Alike to the standard imperfect information setting we shall show in this section that, by enlarging the state space so as to include the last known state and the successive actions taken during the delay period, the control problem with the N-SDSI pattern can be reformulated into the MCM-PSI framework.

Define $\mathbf{Y} := \mathbf{X} \times \mathbf{A}^N$. We first introduce the notion of N-SDSI policies.

Definition 2.1 *A N-SDSI-policy u is defined as a sequence of transition probabilities $u_t : K_t \to \mathcal{P}(\mathbf{A})$ with $K_N := \mathbf{Y}$ and $K_{t+1} = K_t \times (\mathbf{A} \times \mathbf{X})$ for all $t > N$, such that $u_t(k_t; \bullet)$ assigns probability one to the set $\mathbf{A}$ for any $k_t \in K_t$, $t = N, N+1, \ldots$*

Let U_N be the set of all N-SDSI-policies.

A natural objective is then to minimize over U_N the cost criterion

$$V_\beta^N(y, u) := E^u \left[\sum_{t \geq N} \beta^{t-N}\, C'(Z_{t-N}, A_t) \,|\, Z_0 = y \right], \tag{3}$$

for all $y \in \mathbf{Y}$, where

$$Z_t := (X_t, A_t, A_{t+1}, \ldots, A_{t+N-1}), \quad t \in \mathbb{N},$$

and where C' is a measurable mapping from $\mathbf{Y} \times \mathbf{A}$ to $\mathbb{R}$. In (3) E_u is the expectation operator associated with the probability space defined by a policy $u \in U_N$, given some initial distribution p on $\mathbf{Y}$ and the transition law q.

Let us now show that the solution to the MCM-ISI problem (3) may be obtained by solving a related MCM-PSI problem.

This MCM-PSI is constructed as follows (see Section 2.1). Let $\mathbf{Y}$, $\mathbf{A}$, q', C' and U' be its state space, action space, transition law, cost function and set of policies, respectively, where

$$q'(y' \mid y; a) := \mathbf{1}\,\{(a'_1, a'_2, \ldots, a'_N) = (a_2, \ldots, a_N, a)\}\; q(x' \mid x; a_1), \tag{4}$$

for all $y = (x, a_1, a_2, \ldots, a_N) \in \mathbf{Y}$, $y' = (x', a'_1, a'_2, \ldots, a'_N) \in \mathbf{Y}$, $a \in \mathbf{A}$.

Given the initial distribution p on $\mathbf{Y}$ and the transition law q' (or, equivalently, see (4), given the transition law q), any policy $u \in U'$ defines a probability measure on $(\mathbf{Y} \times \mathbf{A})^\infty$ endowed with the product σ-algebra. Let Y_t and A'_t be random variables describing the state of this new system at time t and the action taken at time t, respectively, for all $t \in \mathbb{N}$.

For any vector $k_t \in \mathbf{Y} \times (\mathbf{A} \times \mathbf{X})^t$, $t \in \mathbb{N}$, such that

$$k_t = (x_0, a_0, \ldots, a_{N-1}, a_N, x_1, \ldots, a_{N+t-1}, x_t),$$

define the mappings $g_t : \mathbf{Y} \times (\mathbf{A} \times \mathbf{X})^t \to \mathbf{Y} \times (\mathbf{A} \times \mathbf{Y})^t$ as

$$g_t(k_t) = (y_0, a_0, \ldots, y_{t-1}, a_{t-1}, y_t),$$

with $y_i = (x_i, a_i, a_{i+1}, \ldots, a_{i+N-1})$ for $i = 0, 1, \ldots, t$.

The following lemma holds:

Lemma 2.1 *Let $u' = \{u'_t\}_{t \in \mathbb{N}} \in U'$ and $u = \{u_{t+N}\}_{t \in \mathbb{N}} \in U_N$ be such that*

$$u'_t(\bullet \mid g_t(k_t)) = u'_{t+N}(\bullet \mid k_t), \tag{5}$$

for all $k_t \in \mathbf{Y} \times (\mathbf{A} \times \mathbf{X})^t$, $t \in \mathbb{N}$. Then, the processes $\{Y_t, A'_t; t \in \mathbb{N}\}$ and $\{Z_t, A_{t+N}; t \in \mathbb{N}\}$ have the same distribution under policies u' and u, respectively.

Proof. The proof is left to the reader (hint: show by induction in t that $\{Y_s, A'_s; 0 \le s \le t\}$ and $\{Z_s, A_{s+N}; 0 \le s \le t\}$ have the same distribution under policies u' and u, respectively, for all $t \in \mathbb{N}$). ∎

Applying Theorem 2.1 to the MCM-PSI $\{Y_t, t \in \mathbb{N}\}$, it is seen that there exists a stationary policy $u^* = \{u^*_t\}_{\in \mathbb{N}}$ in U' that solves the following problem:

($\mathbf{Q}_\beta$): Minimize $J_\beta(y,u)$ over U' for all $y \in \mathbf{Y}$, where

$$J_\beta(y,u) := E^u\left[\sum_{t\in\mathbb{N}} \beta^t\, C'(Y_t, A'_t)\,|\, Y_0 = y\right], \tag{6}$$

$u \in U'$. Let $J_\beta(y) := \inf_{u\in U'} J_\beta(y,u)$.

Let α^* be a measurable mapping from $\mathbf{Y}$ to $\mathbf{A}$ such that

$$u_t^*(\bullet \,|\, y_0, a_0, \ldots, y_t) = \mathbf{1}(\alpha^*(y_t) = \bullet), \quad t \in \mathbb{N}. \tag{7}$$

We have the following proposition:

Proposition 2.1 *The policy $v^* = \{v_{t+N}^*\}_{t\in\mathbb{N}} \in U_N$ such that*

$$v_t^*(\bullet \,|\, k_t) = \mathbf{1}(\alpha^*(x_t, a_t, \ldots, a_{t+N-1}) = \bullet), \tag{8}$$

for all $k_t \in \mathbf{Y} \times (\mathbf{A}\times\mathbf{X})^t$ with

$$k_t = (x_0, a_0, \ldots, a_{N-1}, a_N, x_1, \ldots, a_{t+N-1}, x_t),$$

minimizes $V_\beta^N(y,u)$ over U_N, for all $y \in \mathbf{Y}$.

Proof. Let u be an arbitrary policy in U_N. Let $u' \in U'$ be such that (5) holds for all $t \in N$. Then, by Lemma 2.1, (3), (6),

$$J_\beta(y,u') = V_\beta^N(y,u),$$

for all $y \in \mathbf{Y}$.

On the other hand,

$$J_\beta(y,u^*) \le J_\beta(y,u'),$$

for all $y \in \mathbf{Y}$, since u^* solves the problem $\mathbf{Q}_\beta$.

It remains to show that $J_\beta(y,u^*) = V_\beta^N(y,v^*)$ for all $y \in \mathbf{Y}$. This result again follows from Lemma 2.1 since it is easily seen from (7) and (8) that policies u^* and v^* satisfy (5). ∎

It other words, we have shown that a policy that minimizes the cost $V_\beta^N(\bullet, u)$ over U_N may be obtained by solving the problem $\mathbf{Q}_\beta$.

To conclude this section, let us briefly discuss the problem of minimizing the β-discounted cost over all stages $t \in \mathbb{N}$.

Assume that the initial distribution p_0 is known. Then, once J_β is known, the following optimization problem may be addressed:

$$\min_{i_0,i_1,\ldots,i_{N-1}} \left\{ \sum_{x\in\mathbb{N}} p_0(x) \left[\sum_{t=0}^{N-1} \beta^t\, E[\tau(X_t,A_t)|\, X_0 = x, A_j = i_j, 0 \le j \le N-1] + \beta^N J_\beta(x, i_0, \ldots, i_{N-1})\right]\right\},$$

where τ is a measurable mapping from $\mathbf{Y}$ to $\mathbb{R}$.

This optimization problem will not be investigated in this article. In the remainder of this paper, we will assume that the initial distribution p_0 is not known.

3 A Flow Control Problem with Delayed Information

The results in Section 2.1 are illustrated through the study of a simple flow control problem with the 1-SDSI pattern (i.e., $N = 1$). Throughout this section β is a fixed number in $(0, 1)$.

3.1 The Queueing Model

Considered is a discrete-time single-server queueing model. The service times form a sequence of i.i.d. random variables distributed according to a geometric distribution with known parameter $0 \le b \le 1$. We assume that at most one customer may join the system in every time slot. This arrival (if any) is assumed to occur at the beginning of the time slot (synchronized arrivals). At the beginning of each time slot, the decision maker chooses in the set $\mathbf{A} := \{p_1, p_2\}$, $0 \le p_2 < p_1 \le 1$, the probability of having one arrival in this time slot. Therefore, if action p_i is chosen at time $t \in \mathbb{N}$ then a customer will enter the system at time t with the probability p_i, $i = 1, 2$. We further assume that a customer that enters an empty system may leave the system (with the probability b) at the end of this same time slot.

Let $X_t \in \mathbb{N}$ denote the number of customers in the system at time t, $t \in \mathbb{N}$. We assume that the state of the system is known with a delay of one unit of time or, equivalently, we assume that the 1-SDSI pattern is used.

Our objective is to minimize, over U_1, the cost criterion $V_\beta^1(\bullet, u)$ defined in (3) in the case when

$$C'(y, A) := c(x) + \gamma\, a, \tag{9}$$

for all $y = (x, a) \in \mathbf{Y} = \mathbb{N} \times \mathbf{A}$, $A \in \mathbf{A}$. In (9), $c(x)$ is any real-valued *nondecreasing convex* function on $\mathbb{N}$ and γ is an arbitrary constant. Observe that the assumptions placed on c ensure that C' is bounded below. Therefore, we shall assume in the following that without loss of generality the cost function C' is nonnegative. The choice of the cost function C' is discussed below.

From the results in Section 2.2, we know that the solution to this control problem is obtained by solving the problem $\mathbf{Q}_\beta$, defined in (6), where the transition law q' is given by (see (4))

$$q'(y' \mid y; A) = \begin{cases} \bar{a}b, & \text{if } x \ge 1,\ x' = x - 1; \\ ab + \bar{a}\bar{b}, & \text{if } x \ge 1,\ x' = x; \\ a\bar{b}, & \text{if } x \ge 1,\ x' = x + 1; \\ 1 - a\bar{b}, & \text{if } x = x' = 0; \\ a\bar{b}, & \text{if } x = 0,\ x' = 1; \\ 0, & \text{if } |x - x'| > 1, \end{cases} \tag{10}$$

if $y' = (x', a') \in \mathbf{Y}$ is such that $a' = A$, and zero otherwise.

Let us now comment on the definition of the cost C' (see (9)). If c is nonnegative and if $\gamma < 0$ then the cost C' reduces to a cost frequently encountered in the literature on flow control models with completely observable states (see for instance Stidham [11]). In that case, $c(x)$ can be interpreted as a holding cost per unit time, and γa as a reward related to the acceptance of an incoming customer. Then, the problem $\mathbf{Q}_\beta$ involves a trade-off between low expected response time and low throughput on one hand (e.g., always choose action p_2) and high expected response time and high throughput on the other hand (e.g., always choose action p_1). A similar problem in the case of completely observable states was addressed by Ma and Makowski [6] under the additional requirements that the expected response time must be bounded from above. Also note that the problem $\mathbf{Q}_\beta$ becomes trivial when c is nonnegative and $\gamma \geq 0$. In that case, the optimal action is always to choose p_2.

The remainder of Section 3 is devoted to identifying a stationary policy in U' that solves the problem $\mathbf{Q}_\beta$ in the case when the cost function and the transition law are given by (9) and (10), respectively.

3.2 Preliminary Results

Further notation are needed at this point.

Let $\mathcal{K}$ be the set of all real-valued functions on $\mathbf{Y}$. Let $S_A : \mathcal{K} \times \mathbf{Y} \to \mathbb{R}$ be the operator defined as

$$S_A(f, y) := \sum_{y' \in \mathbf{Y}} f(y')\, q'(y' \mid y; A), \tag{11}$$

for all $f \in \mathcal{K}$, $y \in \mathbf{Y}$, $A \in \mathbf{A}$.

It is easily seen from (10) and (11) that

$$\begin{aligned} S_A(f, 0, a) &:= (1 - a\bar{b})\, f(0, A) + a\bar{b} f(1, A); && (12) \\ S_A(f, x, a) &:= \bar{a}b f(x-1, A) + (ab + \bar{a}\bar{b}) f(x, A) + a\bar{b} f(x+1, A), && (13) \end{aligned}$$

for all $x \geq 1$, $a \in \mathbf{A}$, $A \in \mathbf{A}$.

Let $T_\beta : \mathcal{K} \to \mathcal{K}$ be the DP operator associated with the problem $\mathbf{Q}_\beta$. It is easily seen from (2), (9) and (11) that

$$T_\beta f(y) = c(x) + \gamma\, a + \beta \min\{S_{p_1}(f, x, a), S_{p_2}(f, x, a)\}, \tag{14}$$

for all $y = (x, a) \in \mathbf{Y}$.

It is usually difficult to directly determine the optimal policy from the DP equation $J_\beta = T_\beta J_\beta$ (see (1)). An alternative approach is to use the well known *value iteration* algorithm (see Ross [8]).

A key result of this approach is the following (see Bertsekas [1, Sec. 5.4, Prop. 12]):

Proposition 3.1 *Let f_0 be the zero function on $\mathcal{K}$. Then,*

$$\lim_{n \to \infty} T_\beta^n f_0(y) = J_\beta(y), \quad y \in \mathbf{Y}. \tag{15}$$

Proposition (3.1) will be used in the proof of Theorem 3.1.

3.3 The Value Iteration Algorithm Approach

The following notation will be used throughout this section. We shall say that $f \in \mathcal{K}$ satisfies assumption

R1 if $f(x,p_1) - f(x,p_2)$ is monotone increasing in x (i.e., f is supermodular); equivalently,

$$f(x,p_1) - f(x-1,p_1) \geq f(x,p_2) - f(x-1,p_2), \quad x \geq 1;$$

R2 if $f(x+1,p_2) - f(x,p_1)$ is monotone increasing in x; equivalently,

$$f(x,p_1) - f(x-1,p_1) \leq f(x+1,p_2) - f(x,p_2), \quad x \geq 1.$$

A mapping $h : \mathbb{N} \to \mathbb{R}$ will be said to satisfy assumption

R3 if $h(x)$ is integer-convex in x; equivalently,

$$h(x+1) - h(x) \geq h(x) - h(x-1), \quad x \geq 1;$$

R4 if $h(1) \geq h(0)$.

The proof of the main result (see Theorem 3.1) relies upon the following technical lemmas:

Lemma 3.1 *Let $h : \mathbb{N} \to \mathbb{R}$ be a nondecreasing function. Then, for all $x \geq 1$,*

$$\begin{aligned} &\bar{p}_2 b\, h(x) + (p_2 b + \bar{p}_2 \bar{b})\, h(x+1) + p_2 \bar{b}\, h(x+2) \\ &\quad \geq \quad \bar{p}_1 b\, h(x-1) + (p_1 b + \bar{p}_1 \bar{b})\, h(x) + p_1 \bar{b}\, h(x+1). \end{aligned}$$

Proof. For $i = 1, 2$, $x \geq 1$, define

$$F_i(x) := \bar{p}_i b\, h(x-1) + (p_i b + \bar{p}_i \bar{b})\, h(x) + p_i \bar{b}\, h(x+1).$$

We must show that $F_2(x+1) \geq F_1(x)$ for $x \geq 1$. We have

$$\begin{aligned} F_2(x+1) - F_1(x) \quad &= \quad p_2 \bar{b}\, [h(x+2) - h(x+1)] + (1 - (p_1 \bar{b} + \bar{p}_2 b))\, [h(x+1) - h(x)] \\ &\quad\ + \bar{p}_1 b\, [h(x) - h(x-1)]. \end{aligned}$$

The proof is then concluded by using the increasingness of h together with the fact that $p_1 \bar{b} + \bar{p}_2 b \leq 1$. ∎

Lemma 3.2 *Let $f \in \mathcal{K}$ be such that f satisfies* **R1** *and* **R2**. *Then, $f(\bullet, a)$ satisfies* **R3** *for all $a \in$* **A**.

Proof. Let $f \in \mathcal{K}$ be such that f satisfies **R1** and **R2**. We have, for $x \geq 1$,

$$\begin{aligned} & f(x+1,p_1) - f(x,p_1) - [f(x,p_1) - f(x-1,p_1)] \\ \geq\ & f(x+1,p_1) - f(x,p_1) - [f(x+1,p_2) - f(x,p_2)], \\ \geq\ & 0, \end{aligned}$$

where the first (resp. second) inequality follows since f satisfies **R2** (resp. **R1**). On the other hand,

$$\begin{aligned} & f(x+1,p_2) - f(x,p_2) - [f(x,p_2) - f(x-1,p_2)] \\ \geq\ & f(x,p_1) - f(x-1,p_1) - [f(x,p_2) - f(x-1,p_2)], \\ \geq\ & 0, \end{aligned}$$

where again the first (resp. second) inequality follows since f satisfies **R2** (resp. **R1**). This establishes the proof. ∎

Lemma 3.3 *Let $f \in \mathcal{K}$ be such that $f(\bullet, a)$ satisfies* **R3** *and* **R4** *for all $a \in \mathbf{A}$. Then, $S_A(f,1,a) \geq S_A(f,0,a)$ for all $A \in \mathbf{A}$, $a \in \mathbf{A}$.*

Proof. Let $f \in \mathcal{K}$ be such that $f(\bullet, a)$ satisfies **R3** and **R4** for all $a \in \mathbf{A}$. From the definition of $S_A(f,x,a)$ (see (12)-(13)) it is seen that for $A \in \mathbf{A}$, $a \in \mathbf{A}$,

$$\begin{aligned} S_A(f,1,a) - S_A(f,0,a) &= a\bar{b}\,[f(2,A) - f(1,A) - (f(1,A) - f(0,A))] \\ &\quad + (1 - \bar{a}b)\,[f(1,A) - f(0,A)]\,, \end{aligned} \tag{16}$$

which is nonnegative since $f(\bullet, a)$ satisfies **R3** and **R4**, which concludes the proof. ∎

Lemma 3.4 *Let $f \in \mathcal{K}$ be such that f satisfies* **R1**. *Then, $S_{(\bullet)}(f, \bullet, a)$ satisfies* **R1** *for all $a \in \mathbf{A}$.*

Proof. Let $f \in \mathcal{K}$ be such that f satisfies **R1** and fix $a \in \mathbf{A}$.

Since f satisfies **R1**, it is seen for $x \geq 2$ that

$$\begin{aligned} S_{p_1}(f,x,a) - S_{p_1}(f,x-1,a) &= \bar{a}b\,[f(x-1,p_1) - f(x-2,p_1)] \\ &\quad + (ab + \bar{a}\bar{b})[f(x,p_1) - f(x-1,p_1)] \\ &\quad + a\bar{b}\,[f(x+1,p_1) - f(x,p_1)], \\ &\geq \bar{a}b\,[f(x-1,p_2) - f(x-2,p_2)] \\ &\quad + (ab + \bar{a}\bar{b})[f(x,p_2) - f(x-1,p_2)] \\ &\quad + a\bar{b}\,[f(x+1,p_2) - f(x,p_2)], \\ &= S_{p_2}(f,x,a) - S_{p_2}(f,x-1,a). \end{aligned}$$

On the other hand, using (16) we see that

$$
\begin{aligned}
S_{p_1}(f,1,a) - S_{p_1}(f,0,a) &= a\bar{b}\,[f(2,p_1) - f(1,p_1)] + (1 - \bar{a}b - a\bar{b})\,[f(1,p_1) - f(0,p_1)], \\
&\geq a\bar{b}\,[f(2,p_2) - f(1,p_2)] + (1 - \bar{a}b - a\bar{b})\,[f(1,p_2) - f(0,p_2)], \\
&= S_{p_2}(f,1,a) - S_{p_2}(f,0,a),
\end{aligned}
$$

by using the assumption that f satisfies **R1** and the fact that $\bar{a}b + a\bar{b} \leq 1$. ∎

Lemma 3.5 *Let $f \in \mathcal{K}$ and $A \in \mathbf{A}$ be such that (i) f satisfies* **R1** *and* **R2**, *and (ii) $f(\bullet, A)$ satisfies* **R4**. *Then, $S_A(f, \bullet, \bullet)$ satisfies* **R1** *and* **R2**.

Proof. Let $f \in \mathcal{F}$ and $A \in \mathbf{A}$ be such that conditions (i) and (ii) in the lemma hold. We first establish that $S_A(f,\bullet,\bullet)$ satisfies **R1**. For $x \geq 1$, we have

$$
S_A(f,x,p_1) - S_A(f,x,p_2) = (p_1 - p_2)\left\{\bar{b}\,[f(x+1,A) - f(x,A)] + b\,[f(x,A) - f(x-1,A)]\right\},
$$

and the latter is increasing in x since $p_1 > p_2$ and since by Lemma 3.2 $f(\bullet, a)$ satisfies **R3** for all $a \in \mathbf{A}$. It remains to show that the monotonicity property also holds at the boundary. We have

$$
\begin{aligned}
&S_A(f,1,p_1) - S_A(f,1,p_2) - [S_A(f,0,p_1) - S_A(f,0,p_2)] \\
&\quad= (p_1 - p_2)\left[b\,(f(1,A) - f(0,A)) + \bar{b}\,(f(2,A) - f(1,A) - (f(1,A) - f(0,A))\right],
\end{aligned}
$$

which is nonnegative since $f(\bullet, A)$ satisfies **R4** and since (see Lemma 3.2) $f(\bullet, a)$ satisfies **R3** for all $a \in \mathbf{A}$. This completes the proof that $S_A(f,\bullet,\bullet)$ satisfies **R1**.

Let us now show that $S_A(f,\bullet,\bullet)$ satisfies **R2**. For $x \geq 2$, it is easily seen that

$$
\begin{aligned}
&S_A(f,x,p_1) - S_A(f,x-1,p_1) - [S_A(f,x+1,p_2) - S_A(f,x,p_2)] \\
&\quad= \bar{p}_1 b\,[f(x-1,A) - f(x-2,A)] + (p_1 b + \bar{p}_1\bar{b})\,[f(x,A) - f(x-1,A)] \\
&\qquad + p_1\bar{b}\,[f(x+1,A) - f(x,A)] - \bar{p}_2 b\,[f(x,A) - f(x-1,A)] \\
&\qquad - (p_2 b + \bar{p}_2\bar{b})\,[f(x+1,A) - f(x,A)] - p_2\bar{b}\,[f(x+2,A) - f(x+1,A)], \\
&\quad= -p_2\bar{b}\,[f(x+2,A) - f(x+1,A) - (f(x+1,A) - f(x,A))] \\
&\qquad - (1 - p_1\bar{b} - \bar{p}_2 b)\,[f(x+1,A) - f(x,A) - (f(x,A) - f(x-1,A))] \\
&\qquad - \bar{p}_1 b\,[f(x,A) - f(x-1,A) - (f(x-1,A) - f(x-2,A))], \\
&\quad\leq 0,
\end{aligned}
$$

where the above inequality again follows from the fact that $f(\bullet, a)$ satisfies **R3** for all $a \in \mathbf{A}$. It remains to examine the case when $x = 1$. We have

$$
\begin{aligned}
&S_A(f,1,p_1) - S_A(f,0,p_1) - [S_A(f,2,p_2) - S_A(f,1,p_2)] \\
&\quad= -p_2\bar{b}\,[f(3,A) - f(2,A) - (f(2,A) - f(1,A))] \\
&\qquad - (1 - p_1\bar{b} - \bar{p}_2 b)\,[f(2,A) - f(1,A) - (f(1,A) - f(0,A))] \\
&\qquad - \bar{p}_1 b\,[f(1,A) - f(0,A)],
\end{aligned}
$$

which is nonnegative since $f(\bullet, A)$ satisfies both conditions **R3** by Lemma 3.2 and **R4** by assumption. Hence, $S_A(f, \bullet, \bullet)$ satisfies **R2**, which concludes the proof. ∎

Lemma 3.6 *Assume that f satisfies* **R1**. *Then, for every $a \in \mathbf{A}$, (i) there exists $l(a) \in \mathbb{N} \cup \{+\infty\}$ such that*

$$\begin{aligned} S_{p_1}(f,x,a) - S_{p_2}(f,x,a) &< 0, \quad \textit{for } 0 \le x < l(a); \\ S_{p_1}(f,x,a) - S_{p_2}(f,x,a) &\ge 0, \quad \textit{for } x \ge l(a). \end{aligned}$$

Moreover, (ii) $l(p_1) \le l(p_2)$ and (iii) $l(p_2) \le l(p_1) + 1$.

Proof. Let $f \in \mathcal{K}$ be such that f satisfies **R1**.

The property (i) is a direct consequence of the fact that $S_{(\bullet)}(f, \bullet, a)$ satisfies **R1** for all $a \in \mathbf{A}$ (see Lemma 3.4), where

$$l(a) := \inf\{x \ge 0 : S_{p_1}(f,x,a) \ge S_{p_2}(f,x,a)\}, \;\; a \in \mathbf{A}.$$

Let us now turn to the proof of (ii). For $x \ge 1$ we have

$$\begin{aligned} & S_{p_1}(f,x,p_1) - S_{p_2}(f,x,p_1) - [S_{p_1}(f,x,p_2) - S_{p_2}(f,x,p_2)] \\ &= \bar{p}_1 b\,[f(x-1,p_1) - f(x-1,p_2)] + (p_1 b + \bar{p}_1 \bar{b})[f(x,p_1) - f(x,p_2)] \\ &\quad + p_1 \bar{b}\,[f(x+1,p_1) - f(x+1,p_2)] - \bar{p}_2 b\,[f(x-1,p_1) - f(x-1,p_2)] \\ &\quad - (p_2 b + \bar{p}_2 \bar{b})[f(x,p_1) - f(x,p_2)] - \; p_2 \bar{b}\,[f(x+1,p_1) - f(x+1,p_2)], \\ &= (p_1 - p_2)\,\{\bar{b}\,[f(x+1,p_1) - f(x+1,p_2) - (f(x,p_1) - f(x,p_2))] \\ &\quad + b\,[f(x,p_1) - f(x,p_2) - (f(x-1,p_1) - f(x-1,p_2))]\}, \\ &\ge 0, \end{aligned} \tag{17}$$

since $p_1 > p_2$ and since f satisfies **R1**.

For $x = 0$ we have

$$\begin{aligned} & S_{p_1}(f,0,p_1) - S_{p_2}(f,0,p_1) - [S_{p_1}(f,0,p_2) - S_{p_2}(f,0,p_2)] \\ &= (p_1 - p_2)\,\bar{b}\,[f(1,p_1) - f(1,p_2) - (f(0,p_1) - f(0,p_2))], \\ &\ge 0, \end{aligned} \tag{18}$$

since f satisfies **R1**. Therefore, it is seen from (17) and (18) that $S_{p_1}(f,x,p_1) - S_{p_2}(f,x,p_1) \ge S_{p_1}(f,x,p_2) - S_{p_2}(f,x,p_2)$ for all $x \ge 0$, which proves (ii).

It remains to establish (iii). For $x \ge 1$, we have

$$\begin{aligned} & S_{p_1}(f,x,p_1) - S_{p_2}(f,x,p_1) - [S_{p_1}(f,x+1,p_2) - S_{p_2}(f,x+1,p_2)] \\ &= \bar{p}_1 b\,[f(x-1,p_1) - f(x-1,p_2] + (p_1 b + \bar{p}_1 \bar{b})[f(x,p_1) - f(x,p_2)] \\ &\quad + p_1 \bar{b}\,[f(x+1,p_1) - f(x+1,p_2)] - \bar{p}_2 b\,[f(x,p_1) - f(x,p_2)] \\ &\quad - (p_2 b + \bar{p}_2 \bar{b})[f(x+1,p_1) - f(x+1,p_2)] - \; p_2 \bar{b}\,[f(x+2,p_1) - f(x+2,p_2)] \\ &\le 0. \end{aligned} \tag{19}$$

The above inequality is obtained by applying Lemma 3.1 with $h(x) := f(x, p_1) - f(x, p_2)$ and by noting that h is nondecreasing since f satisfies **R1**. On the other hand,

$$\begin{aligned}
& S_{p_1}(f, 0, p_1) - S_{p_2}(f, 0, p_1) - [S_{p_1}(f, 1, p_2) - S_{p_2}(f, 1, p_2)] \\
&= -p_2\bar{b}\,[f(2, p_1) - f(2, p_2) - (f(1, p_1) - f(1, p_2))] \\
&\quad -(1 - p_1\bar{b} - \bar{p}_2 b)\,[f(1, p_1) - f(1, p_2) - (f(0, p_1) - f(0, p_2))], \\
&\leq 0, \qquad (20)
\end{aligned}$$

since f satisfies **R1**. Combining (19) and (20) yields $l(p_2) \leq l(p_1) + 1$, which completes the proof. ∎

Lemma 3.7 *Let $f \in \mathcal{K}$ be such that f satisfies* **R1** *and* **R2**, *and $f(\bullet, a)$ satisfies* **R4** *for all $a \in \mathbf{A}$. Then, $T_\beta f$ satisfies* **R1** *and* **R2**, *and $T_\beta f(\bullet, a)$ satisfies* **R4** *for all $a \in \mathbf{A}$.*

Proof. Let $f \in \mathcal{K}$ be such that f satisfies **R1** and **R2**, and $f(\bullet, a)$ satisfies **R4** for all $a \in \mathbf{A}$.

We first establish that $T_\beta f$ satisfies **R1**. This is equivalent to showing that

$$\begin{aligned}
G(x) &:= \beta^{-1}\left(T_\beta f(x, p_1) - T_\beta f(x, p_2) - [T_\beta f(x+1, p_1) - T_\beta f(x+1, p_2)]\right), \\
&= \min\{S_{p_1}(f, x, p_1), S_{p_2}(f, x, p_1)\} - \min\{S_{p_1}(f, x, p_2), S_{p_2}(f, x, p_2)\} \\
&\quad - [\min\{S_{p_1}(f, x+1, p_1), S_{p_2}(f, x+1, p_1)\} - \min\{S_{p_1}(f, x+1, p_2), S_{p_2}(f, x+1, p_2)\}], \qquad (21)
\end{aligned}$$

is nonpositive for all $x \geq 0$.

Recall the definition of $l(p_i)$ for $i = 1, 2$ (see Lemma 3.6). Also recall that $l(p_1) \leq l(p_2) \leq l(p_1) + 1$.

We shall distinguish the following cases:

(1) $x \geq l(p_2)$;

(2) $\max(0, l(p_1) - 1) \leq x < l(p_2)$;

(3) $x \leq \max(0, l(p_1) - 2)$.

Case (1): $x \geq l(p_2)$.

We have, cf. (21),

$$\begin{aligned}
G(x) &= \min\{S_{p_1}(f, x, p_1), S_{p_2}(f, x, p_1)\} - S_{p_2}(f, x, p_2) - [S_{p_2}(f, x+1, p_1) - S_{p_2}(f, x+1, p_2)], \\
&\leq S_{p_2}(f, x, p_1) - S_{p_2}(f, x, p_2) - [S_{p_2}(f, x+1, p_1) - S_{p_2}(f, x+1, p_2)],
\end{aligned}$$

which is nonpositive since $S_{p_2}(f, \bullet, \bullet)$ satisfies **R1** from Lemma 3.5.

Case (2): $\max(0, l(p_1) - 1) \leq x < l(p_2)$.

We have for all $x \in \mathbb{N}$, cf. (21),

$$\begin{aligned} G(x) &= \min\{S_{p_1}(f,x,p_1), S_{p_2}(f,x,p_1)\} - S_{p_1}(f,x,p_2) \\ &\quad - [S_{p_2}(f,x+1,p_1) - \min\{S_{p_1}(f,x+1,p_2), S_{p_2}(f,x+1,p_2)\}]. \end{aligned} \tag{22}$$

For $x \geq 1$, we have from (13), (22),

$$\begin{aligned} G(x) &\leq S_{p_1}(f,x,p_1) - S_{p_1}(f,x,p_2) - [S_{p_2}(f,x+1,p_1) - S_{p_2}(f,x+1,p_2)], \\ &= (p_1 - p_2)\{b\,[f(x,p_2) - f(x-1,p_1) - (f(x+1,p_2) - f(x,p_1))] \\ &\quad + \bar{b}\,[f(x+1,p_2) - f(x,p_1) - (f(x+2,p_2) - f(x+1,p_1))]\}, \end{aligned}$$

which is nonpositive since f satisfies **R2** and since $p_1 > p_2$.

On the other hand, we see from (12), (13), (22) that

$$\begin{aligned} G(0) &\leq S_{p_1}(f,0,p_1) - S_{p_1}(f,0,p_2) - [S_{p_2}(f,1,p_1) - S_{p_2}(f,1,p_2)], \\ &= -(p_1-p_2)\{b\,[f(1,p_2) - f(0,p_2)] + \bar{b}\,[f(2,p_2) - f(1,p_2) - (f(1,p_1) - f(0,p_1)]\}, \end{aligned}$$

which is nonpositive since f satisfies **R2** and since $f(\bullet,a)$ satisfies **R4** for all $a \in \mathbf{A}$.

Case (3): $x \leq \max(0, l(p_1) - 2)$.

We obtain from (21)

$$G(x) = S_{p_1}(f,x,p_1) - S_{p_1}(f,x,p_2) - [S_{p_1}(f,x+1,p_1) - S_{p_1}(f,x+1,p_2)],$$

which is nonpositive for all $x \in \mathbb{N}$ since $S_{p_1}(f,\bullet,\bullet)$ satisfies **R1** by Lemma 3.5. This concludes the proof that $G(x) \leq 0$ for all $x \in \mathbb{N}$. Hence, $T_\beta f$ satisfies **R1**.

Let us now show that $T_\beta f(\bullet,a)$ satisfies **R4**. We have, cf. (14),

$$\begin{aligned} \beta^{-1}[T_\beta f(1,a) - T_\beta f(0,a)] &= \beta^{-1}(c(1) - c(0)) + \min\{S_{p_1}(f,1,a), S_{p_2}(f,1,a)\} \\ &\quad - \min\{S_{p_1}(f,0,a), S_{p_2}(f,0,a)\}, \quad a \in \mathbf{A}. \end{aligned} \tag{23}$$

Since $S_A(f,1,a) \geq S_A(f,0,a)$ by Lemma 3.3 and since $c(1) \geq c(0)$ by assumption, we readily deduce from (23) that $T_\beta f(\bullet,a)$ satisfies **R4**.

It remains to establish that $T_\beta f$ satisfies **R2**. This amounts to showing that

$$H(x) := \beta^{-1}(T_\beta f(x+1,p_2) - T_\beta f(x,p_2) - [T_\beta f(x,p_1) - T_\beta f(x-1,p_1)]),$$

is nonnegative for all $x \geq 1$.

We have for $x \geq 1$,

$$H(x) = \beta^{-1}[c(x+1) - c(x) - (c(x) - c(x-1))]$$

$$
\begin{aligned}
&+\min\left\{S_{p_1}(f,x+1,p_2),S_{p_2}(f,x+1,p_2)\right\}-\min\left\{S_{p_1}(f,x,p_2),S_{p_2}(f,x,p_2)\right\}\\
&-\left[\min\left\{S_{p_1}(f,x,p_1),S_{p_2}(f,x,p_1)\right\}-\min\left\{S_{p_1}(f,x-1,p_1),S_{p_2}(f,x-1,p_1)\right\}\right],\\
\geq\ &\min\left\{S_{p_1}(f,x+1,p_2),S_{p_2}(f,x+1,p_2)\right\}-\min\left\{S_{p_1}(f,x,p_2),S_{p_2}(f,x,p_2)\right\}\\
&-\left[\min\left\{S_{p_1}(f,x,p_1),S_{p_2}(f,x,p_1)\right\}-\min\left\{S_{p_1}(f,x-1,p_1),S_{p_2}(f,x-1,p_1)\right\}\right],
\end{aligned}
\tag{24}
$$

where (24) follows from the convexity of the cost function c.

For $x \geq l(p_1)+1$, we have from (24)

$$
\begin{aligned}
H(x) &\geq S_{p_2}(f,x+1,p_2)-S_{p_2}(f,x,p_2)-\left[S_{p_2}(f,x,p_1)-S_{p_2}(f,x-1,p_1)\right],\\
&\geq 0.
\end{aligned}
$$

The first inequality holds because $l(p_2) \leq l(p_1)+1$. The last inequality holds because $S_{p_2}(f,\bullet,\bullet)$ satisfies **R2** (cf. Lemma 3.5).

It remains to cover the case when $1 \leq x \leq l(p_1)$.

Assume first that $l(p_1)=1$. We have for $x=1$, cf. (24),

$$
\begin{aligned}
H(1) &\geq S_{p_2}(f,2,p_2)-\min\{S_{p_1}(f,1,p_2),S_{p_2}(f,1,p_2)\}\\
&\quad-\left[\min\{S_{p_1}(f,1,p_1),S_{p_2}(f,1,p_1)\}-S_{p_1}(f,0,p_1)\right],\\
&\geq S_{p_2}(f,2,p_2)-S_{p_1}(f,1,p_2)-\left[S_{p_2}(f,1,p_1)-S_{p_1}(f,0,p_1)\right],\\
&= p_2\bar{b}\,[f(3,p_2)-f(2,p_1)-(f(2,p_2)-f(1,p_1))]+(1-p_1\bar{b}-\bar{p}_2 b)\,[f(2,p_2)-f(1,p_1)\\
&\quad-(f(1,p_2)-f(0,p_1))]+\bar{p}_1 b\,[f(1,p_2)-f(0,p_2)],
\end{aligned}
$$

which is nonnegative since f satisfies **R2** and since $f(\bullet,a)$ satisfies **R4**.

Assume now that $l(p_1)\geq 2$. For $1\leq x< l(p_1)-1$, it is easily seen from (24) that

$$
H(x)\geq S_{p_1}(f,x+1,p_2)-S_{p_1}(f,x,p_2)-\left[S_{p_1}(f,x,p_1)-S_{p_1}(f,x-1,p_1)\right],
$$

which is nonnegative since $S_{p_1}(f,\bullet,\bullet)$ satisfies **R2** (cf. Lemma 3.5).

For $x=l(p_1)$, we have

$$
\begin{aligned}
H(x) &\geq S_{p_2}(f,x+1,p_2)-S_{p_1}(f,x,p_2)-[S_{p_2}(f,x,p_1)-S_{p_1}(f,x-1,p_1)],\\
&= \bar{p}_2 b\,[f(x,p_2)-f(x-1,p_1)]+(p_2 b+\bar{p}_2\bar{b})\,[f(x+1,p_2)-f(x,p_1)]\\
&\quad+p_2\bar{b}\,[f(x+2,p_2)-f(x+1,p_1)]-\bar{p}_1 b\,[f(x-1,p_2)-f(x-2,p_1)]\\
&\quad-(p_1 b+\bar{p}_1\bar{b})\,[f(x,p_2)-f(x-1,p_1)]-p_1\bar{b}\,[f(x+1,p_2)-f(x,p_1)],
\end{aligned}
$$

which is nonnegative by Lemma 3.1 with $h(x):=f(x+1,p_2)-f(x,p_1)$ (note that h is nondecreasing since f satisfies **R2**).

Assume now that $x = l(p_1) - 1$. When $l(p_1) = l(p_2)$ this case is seen to reduce to the case $x = l(p_1)$. When $l(p_2) = l(p_1) + 1$, we have

$$\begin{aligned} H(x) &\geq S_{p_1}(f, x+1, p_2) - \min\{S_{p_1}(f, x, p_2), S_{p_2}(f, x, p_2)\} \\ &\quad - [\min\{S_{p_1}(f, x, p_1), S_{p_2}(f, x, p_1)\} - S_{p_1}(f, x-1, p_1)], \\ &\geq S_{p_1}(f, x+1, p_2) - S_{p_1}(f, x, p_2) - [S_{p_1}(f, x, p_1) - S_{p_1}(f, x-1, p_1)], \end{aligned}$$

which is nonnegative since $S_{p_1}(f, \bullet, \bullet)$ satisfies **R2**, which concludes the proof. ∎

We are now ready to prove the main result of this article.

Theorem 3.1 *There exists an optimal threshold policy for the problem* $\mathbf{Q}_\beta$. *More precisely, for every* $a \in \mathbf{A}$ *there exists* $l(a) \in \mathbb{N} \cup \{+\infty\}$ *such that when the system is in state* $y = (x, a)$, $x \in \mathbb{N}$, *then the optimal action is* p_1 *if* $x < l(a)$, *and* p_2 *otherwise. Further,*

$$l(p_1) \leq l(p_2) \leq l(p_1) + 1.$$

Proof. Define $\mathcal{G} \subset \mathcal{K}$ to be the set of all real-valued functions on $\mathbf{Y}$ such that f satisfies **R1** and **R2**, and $f(\bullet, a)$ satisfies **R4** for all $a \in \mathbf{A}$. Since the zero function f_0 on $\mathcal{K}$ belongs to $\mathcal{G}$, it is seen from Lemma 3.7 that

$$T_\beta^n f_0 \in \mathcal{G}, \quad n \geq 1. \tag{25}$$

Since $\mathcal{G}$ is closed under pointwise limits, we see from (25) and Proposition 3.1 that $J_\beta \in \mathcal{G}$, which in turn implies from Lemma 3.6 that for every $a \in \mathbf{A}$ there exists $l(a) \in \mathbb{N} \cup \{+\infty\}$ such that

$$\begin{aligned} S_{p_1}(J_\beta, x, a) - S_{p_2}(J_\beta, x, a) &< 0, \quad \text{for } x < l(a); \\ S_{p_1}(J_\beta, x, a) - S_{p_2}(J_\beta, x, a) &\geq 0, \quad \text{for } x \geq l(a). \end{aligned}$$

Hence, it is seen from (11), (14) and Theorem 2.1 that the optimal action when the system is in state $y = (x, a) \in \mathbf{Y}$ is p_1 if $x < l(a)$ and p_2 otherwise. The last part of the theorem also follows from Lemma 3.6. ∎

4 Concluding Remarks

The existence of an optimal policy of a threshold type has been shown for a simple flow control problem in the case when the decision maker has only access to delayed state information (the so-called 1-SDSI pattern). High-Speed Data Networks (HSDN's) provide a good instance where such N-SDSI information patterns are useful. Indeed, because of the very high throughputs that can be handled by HSDN's, the state of the network may have changed considerably by the time the messages on the state of the network reach the decision-maker. Therefore, only controls based on delayed information patterns may be used as far as closed-loop controls are concerned.

Extensions of our model to arbitrary values of N and to more general arrival processes (e.g., batch arrivals, Markov Arrival Processes, cf. Neuts [7], Lucantoni [5]) are however needed so as to capture the basic characteristics of the traffic in HSDN's (e.g., burstiness). These issues, as well as the long-run average version of the control problem considered in this article, are the object of ongoing research.

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