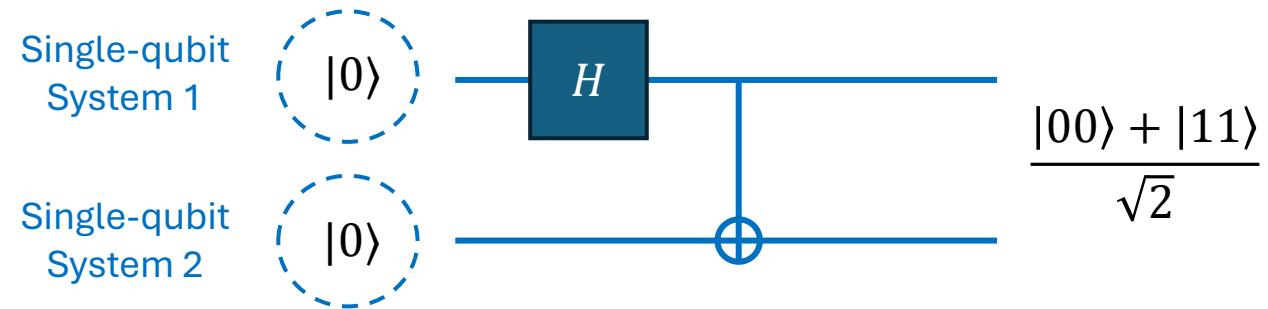


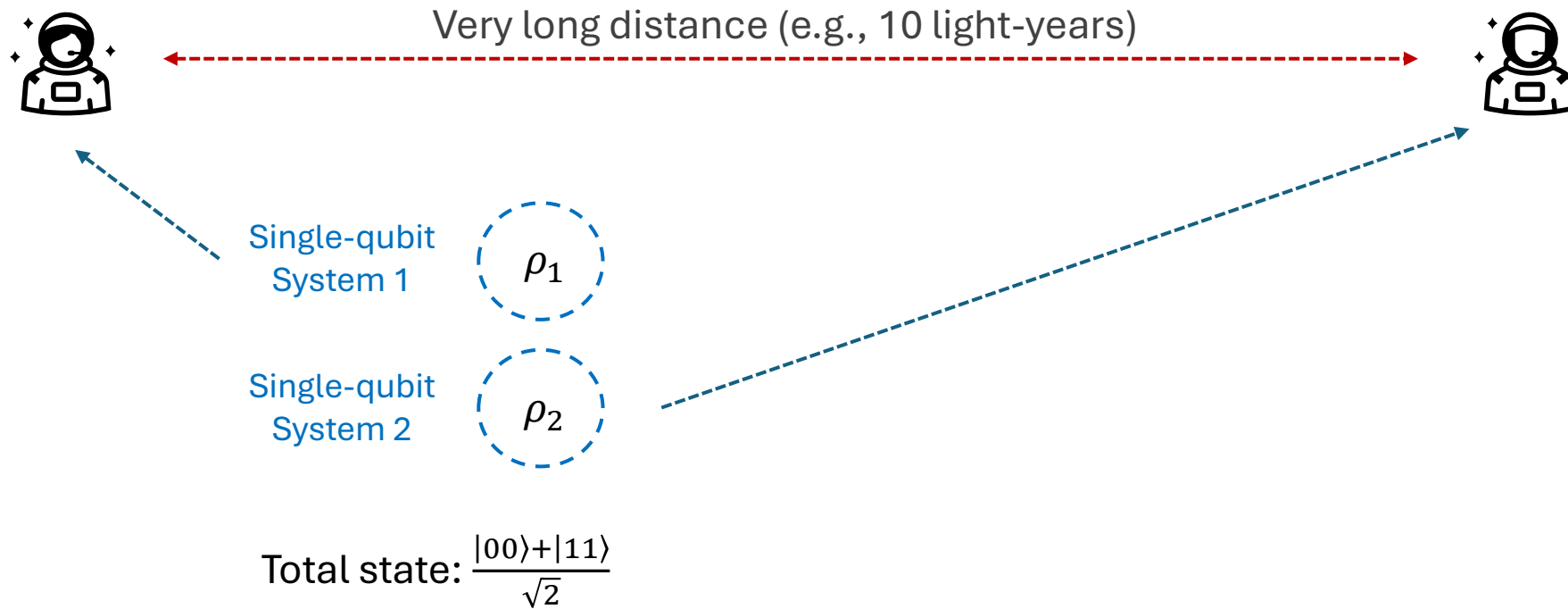
# Quantum Computing

- Lecture 9 (June 4, 2025)
- Today:
  - Superdense coding
  - Quantum teleportation

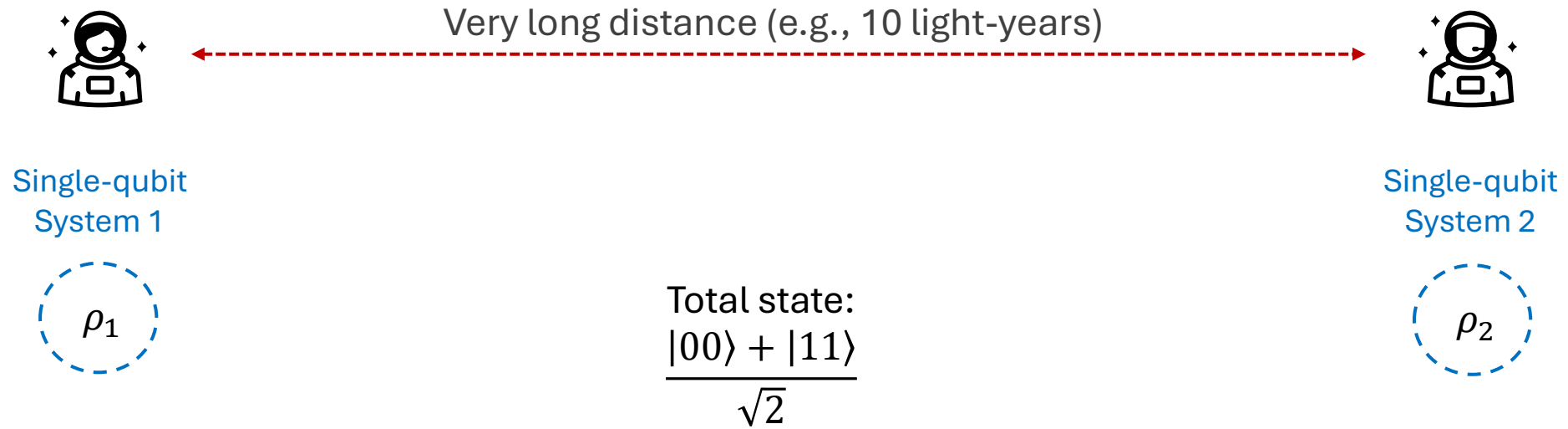
# Action at a Distance



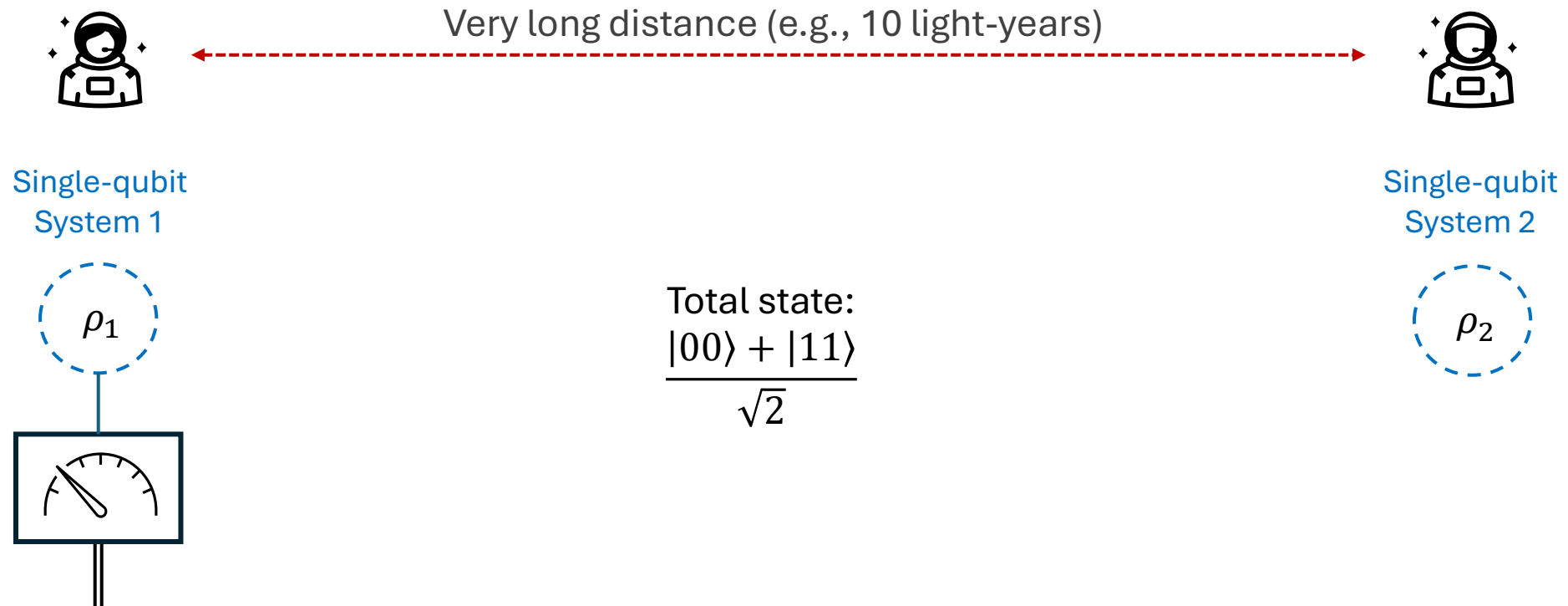
# Action at a Distance



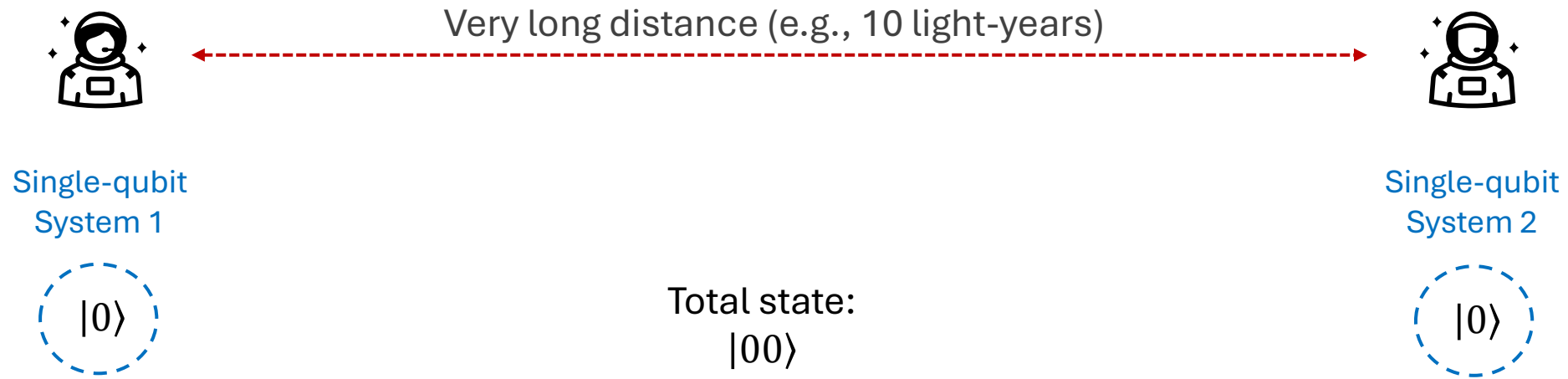
# Action at a Distance



# Action at a Distance



# Action at a Distance



- Application: Superdense coding

# Pauli Matrices

- Pauli matrices:

$$\mathbf{X} = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(Pauli-**X**)

$$\mathbf{Y} = \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

(Pauli-**Y**)

$$\mathbf{Z} = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(Pauli-**Z**)

- Some facts:
  - Pauli-**X** is the qNOT gate (in the computational basis)
  - $\sigma_j^2 = I$  for  $j = 1, 2, 3$

# Pauli Matrices

- (Extended) Pauli matrices:

$$\begin{array}{llll} \mathbf{I} = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \mathbf{X} = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mathbf{Y} = \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & \mathbf{Z} = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ & \text{(Pauli-}\mathbf{X}\text{)} & \text{(Pauli-}\mathbf{Y}\text{)} & \text{(Pauli-}\mathbf{Z}\text{)} \end{array}$$

- Some facts:
  - Pauli- $X$  is the qNOT gate (in the computational basis)
  - $\sigma_j^2 = I$  for  $j = 0,1,2,3$



# Superdense Coding

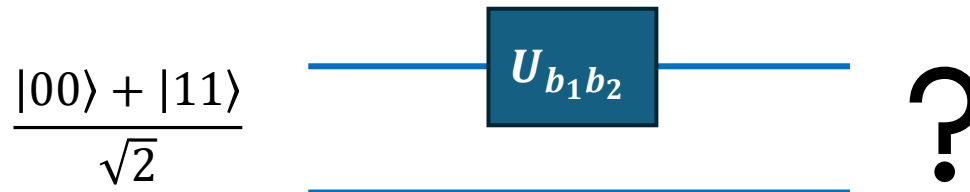
- Consider the four matrices

$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let's define:

$$U_{b_1 b_2} := Z^{b_2} X^{b_1} \begin{cases} U_{00} = I \\ U_{01} = X \\ U_{10} = Z \\ U_{11} = Z \cdot X \end{cases}$$

- Small Exercise:



# Superdense Coding

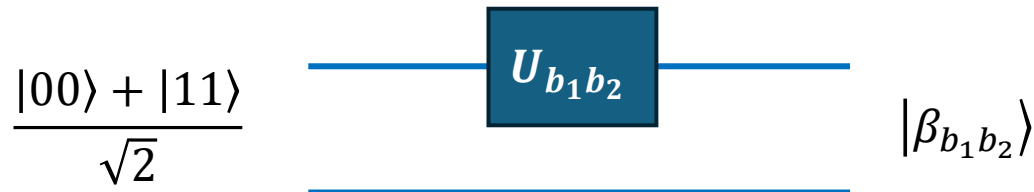
- Consider the four matrices

$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let's define:

$$U_{b_1 b_2} := Z^{b_2} X^{b_1} \begin{cases} U_{00} = I \\ U_{01} = X \\ U_{10} = Z \\ U_{11} = Z \cdot X \end{cases}$$

- Small Exercise:



# Superdense Coding

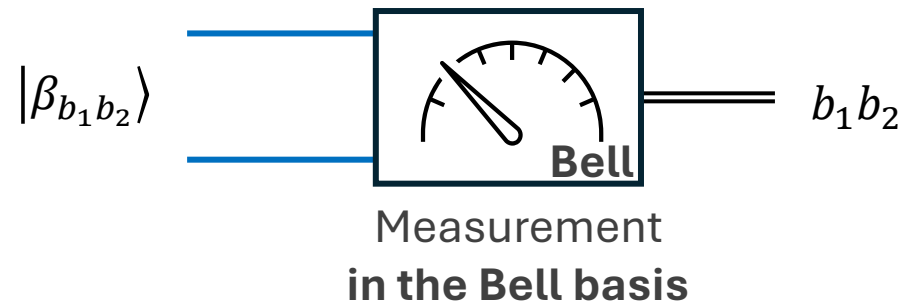
- Consider the four matrices

$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let's define:

$$U_{b_1 b_2} := Z^{b_2} X^{b_1} \begin{cases} U_{00} = I \\ U_{01} = X \\ U_{10} = Z \\ U_{11} = Z \cdot X \end{cases}$$

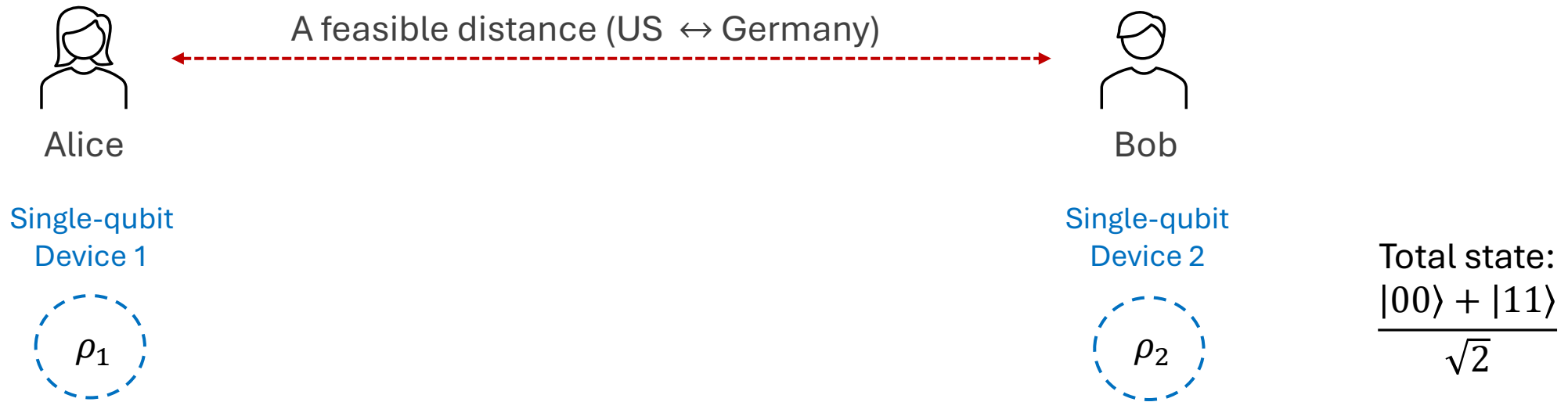
- Small Exercise:



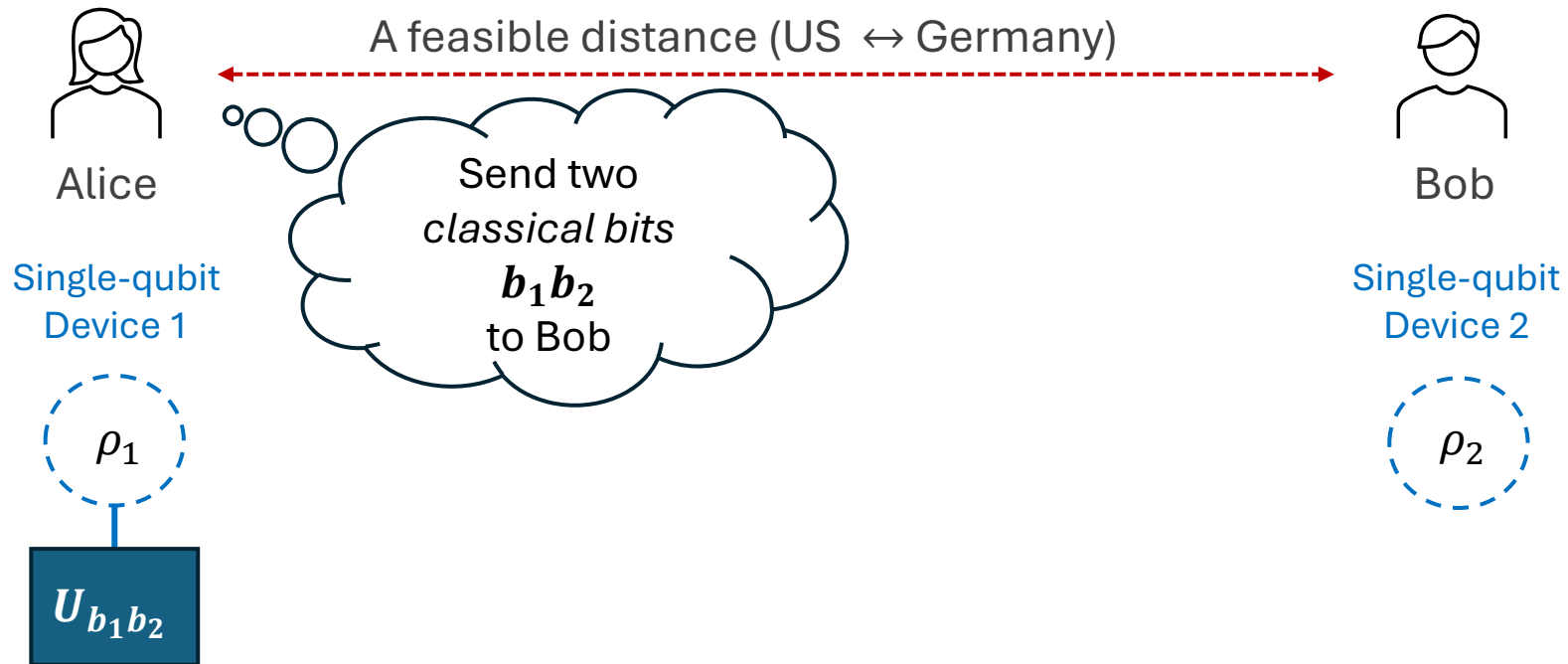
**The Bell basis:**

$$\begin{cases} M_{00}: |\beta_{00}\rangle\langle\beta_{00}|, \\ M_{01}: |\beta_{01}\rangle\langle\beta_{01}|, \\ M_{10}: |\beta_{10}\rangle\langle\beta_{10}|, \\ M_{11}: |\beta_{11}\rangle\langle\beta_{11}|, \end{cases}$$

# Superdense Coding



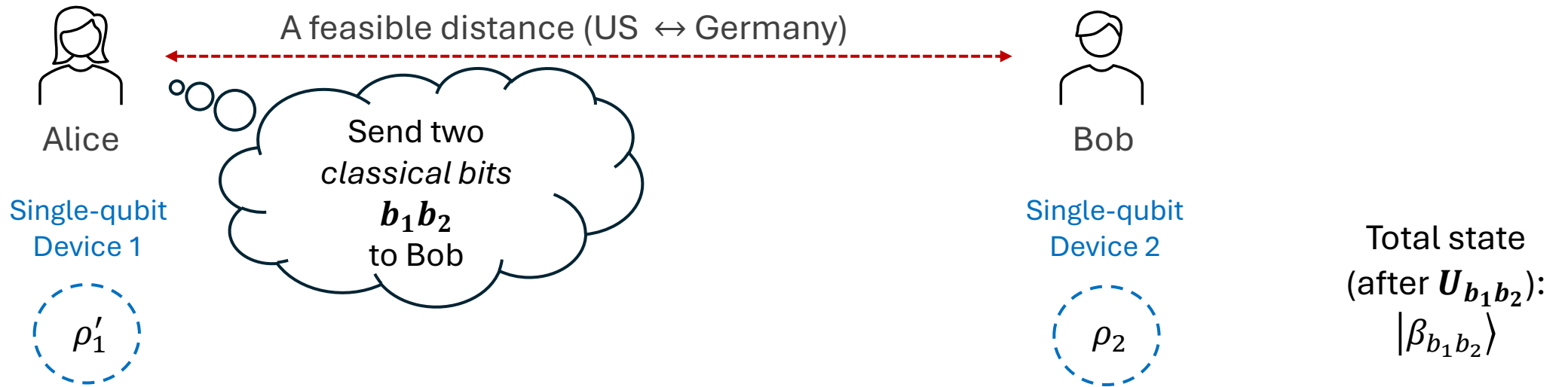
# Superdense Coding



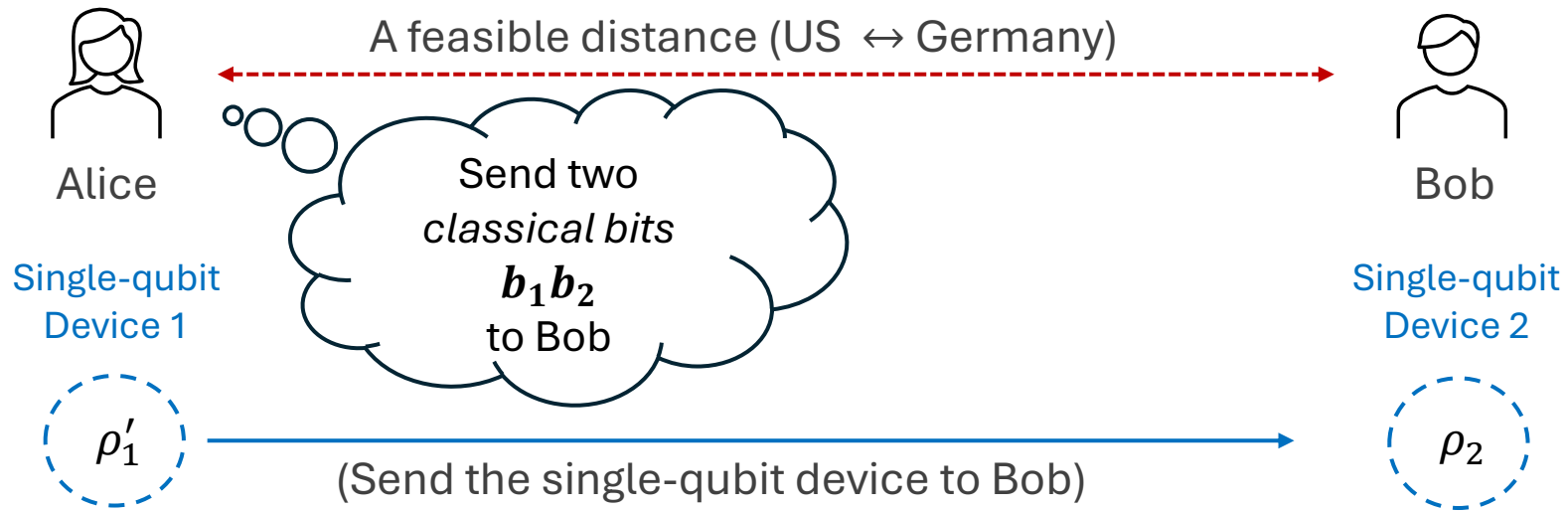
Total state  
(before  $U_{b_1 b_2}$ ):

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

# Superdense Coding



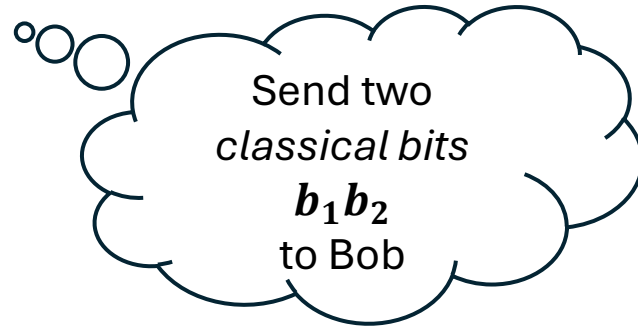
# Superdense Coding



Total state  
(after  $U_{b_1b_2}$ ):  
 $|\beta_{b_1b_2}\rangle$

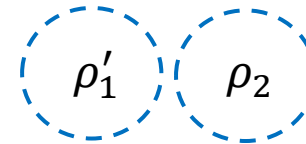
# Superdense Coding

Alice



Bob

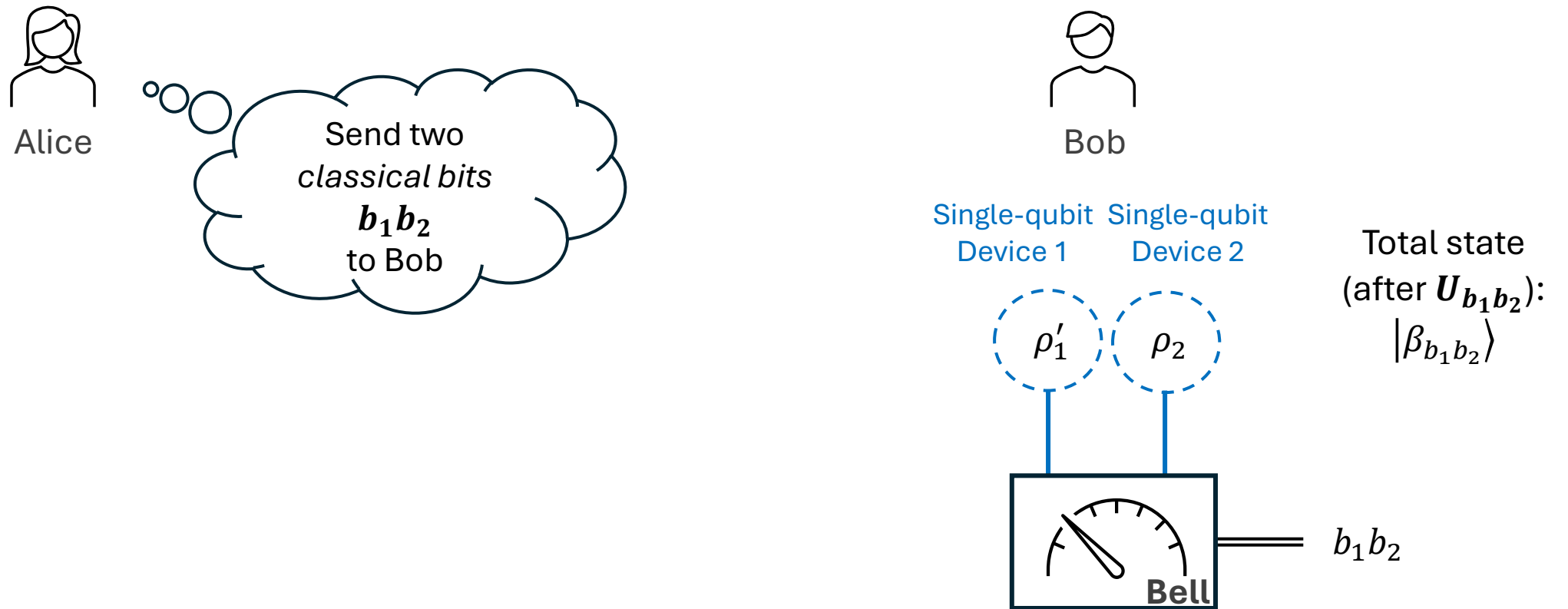
Single-qubit Device 1    Single-qubit Device 2



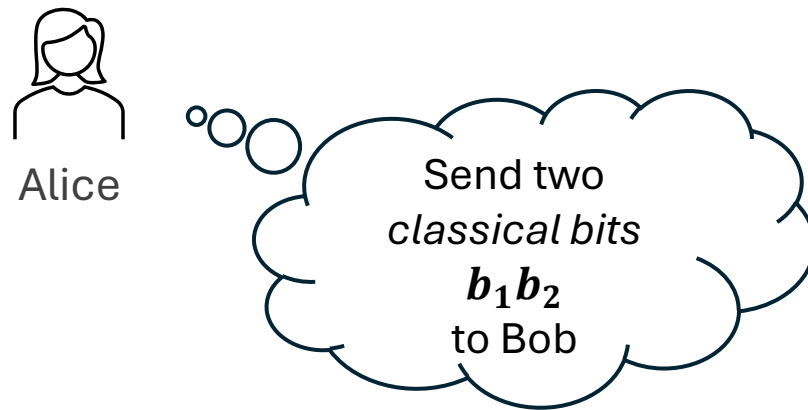
Total state  
(after  $U_{b_1 b_2}$ ):  
 $|\beta_{b_1 b_2}\rangle$



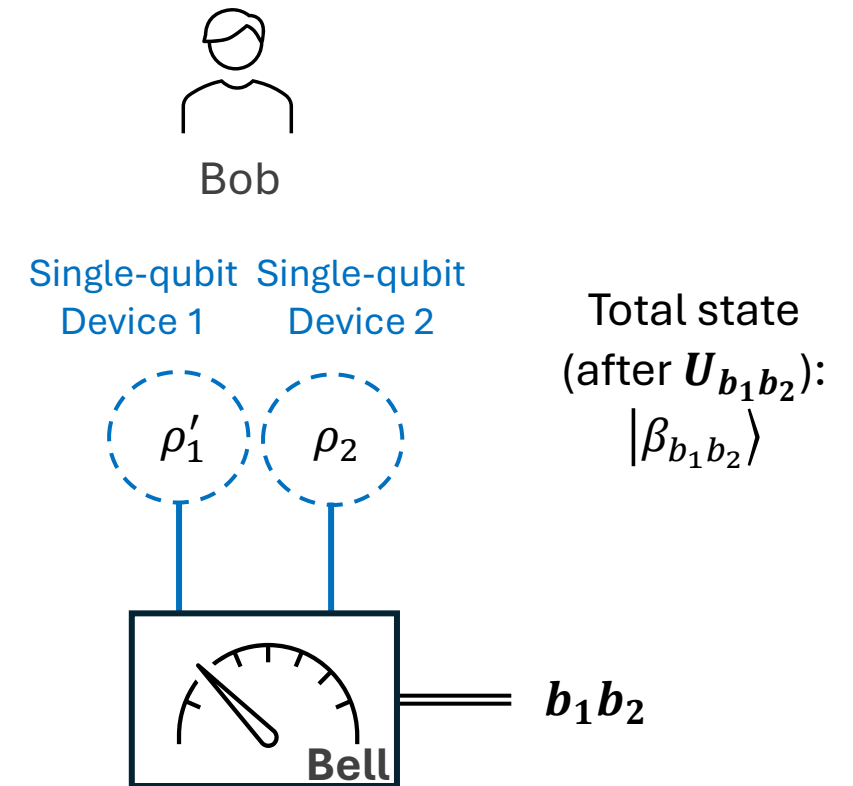
# Superdense Coding



# Superdense Coding

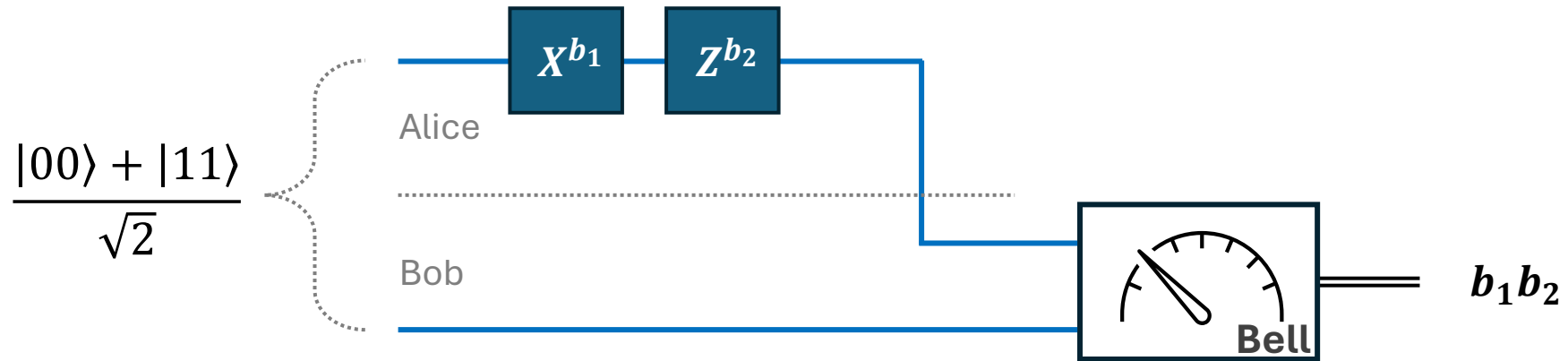


- **One** physical qubit can transmit **two** classical bits of information
  - (Require prior entanglement)
- **Superdense coding:** One qubit “encodes” two classical bits...



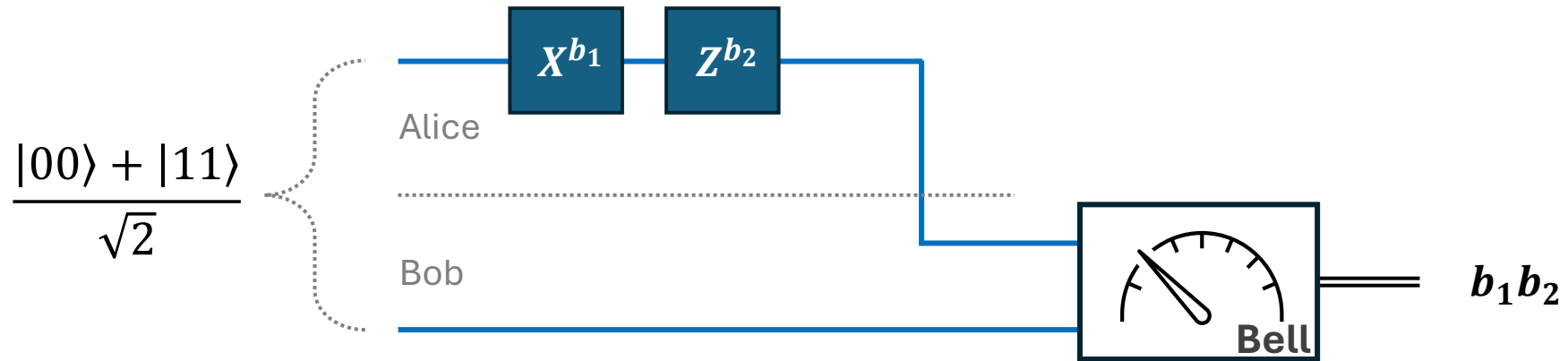
# Superdense Coding

- A more compact description of the experiment:



# Superdense Coding

- A more compact description of the experiment:



- Quick questions:
- (1) We wrote  $U_{b_1b_2} := Z^{b_2}X^{b_1}$  before, but why does X come first here?
- (2) How can we **transmit  $2n$  classical bits using only  $n$  qubits?**

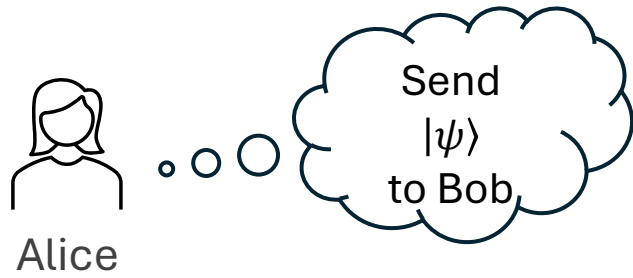
# Quantum Teleportation

- Superdense coding: Transmit classical bits via qubits
- Quantum teleportation: **Transmit qubits via classical bits**



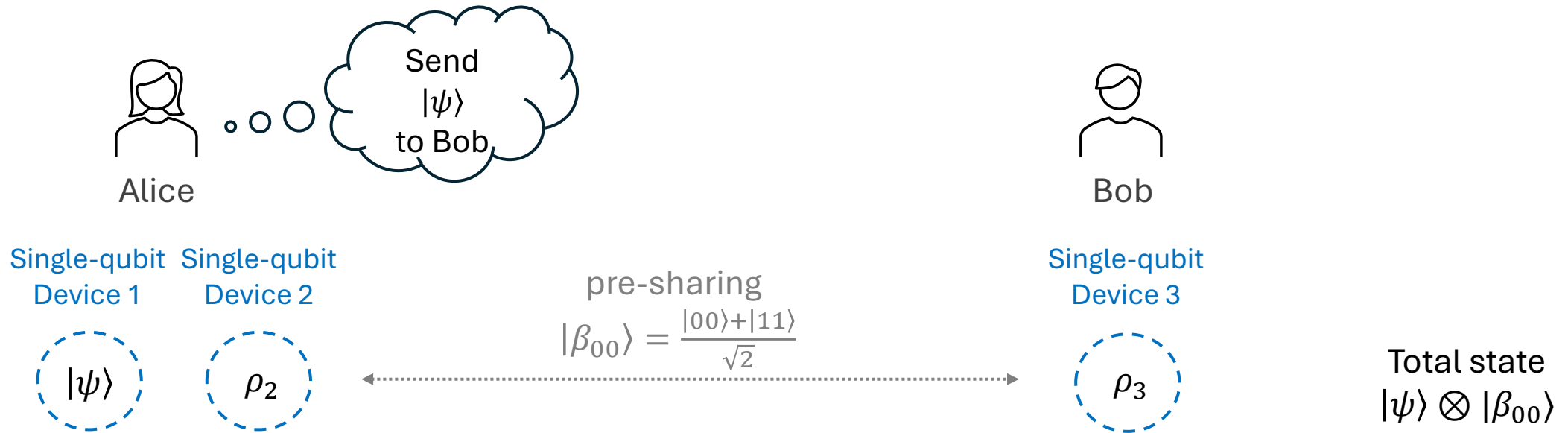
- A trivial approach (transportation): Physically deliver the device carrying  $|\psi\rangle$  to Bob
- But here we consider **Teleportation**: Transmit via (quantum) channel

# Quantum Teleportation

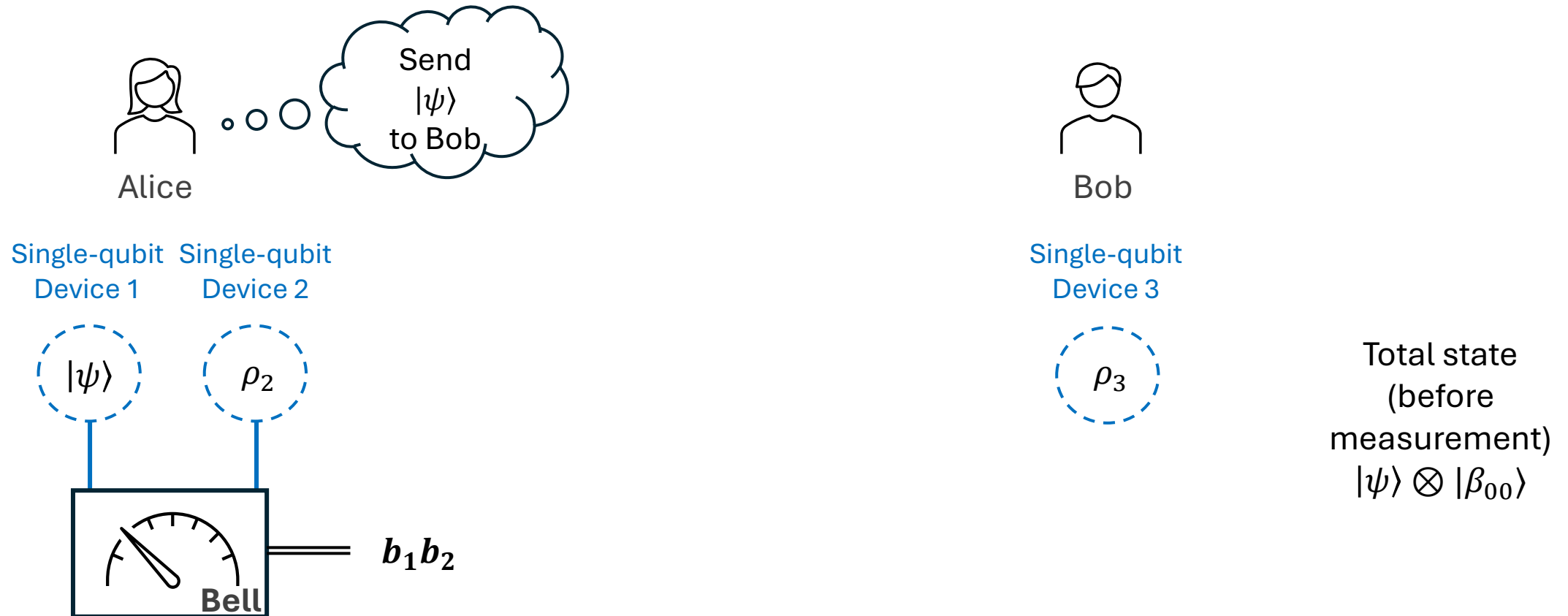


$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

# Quantum Teleportation

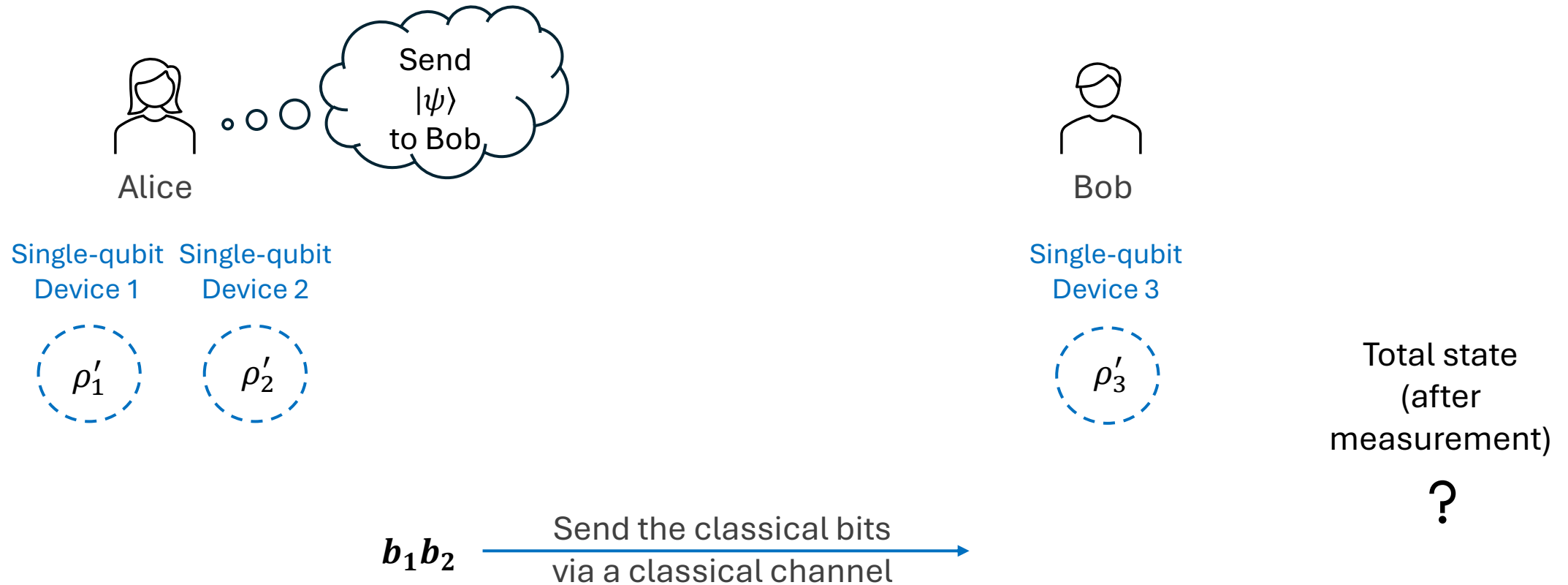


# Quantum Teleportation

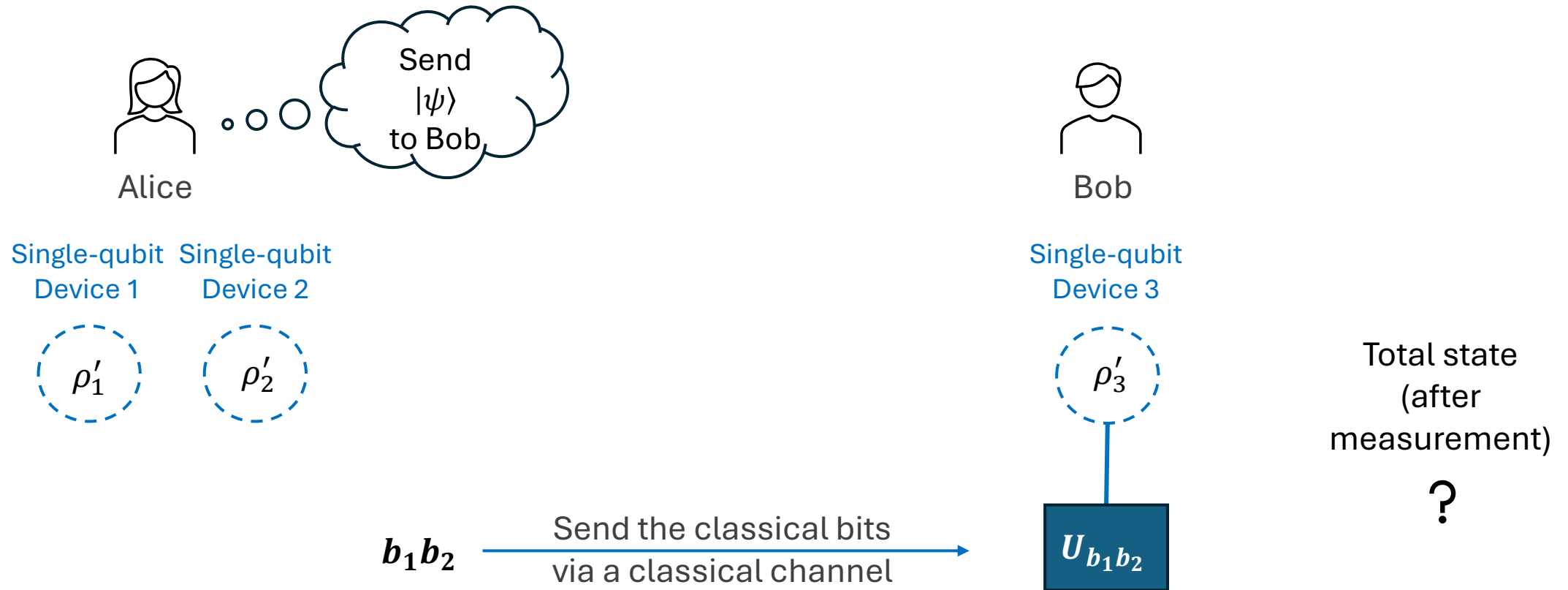




# Quantum Teleportation

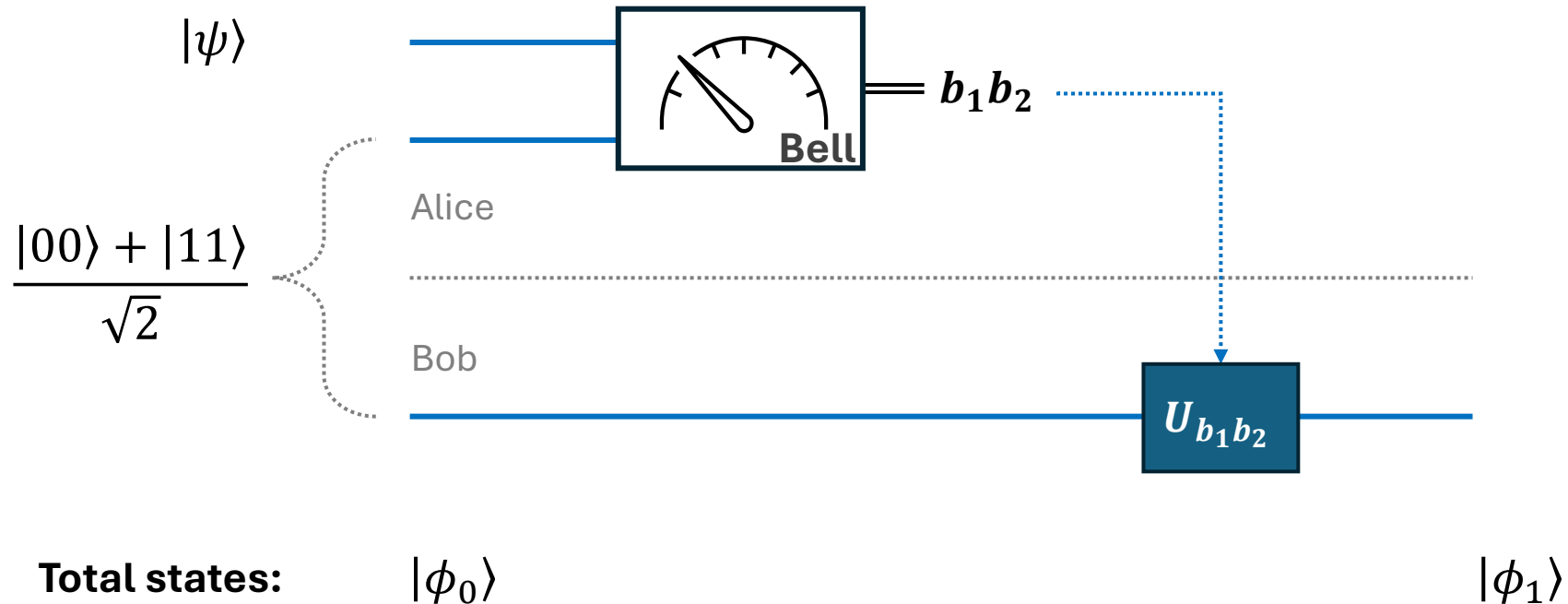


# Quantum Teleportation



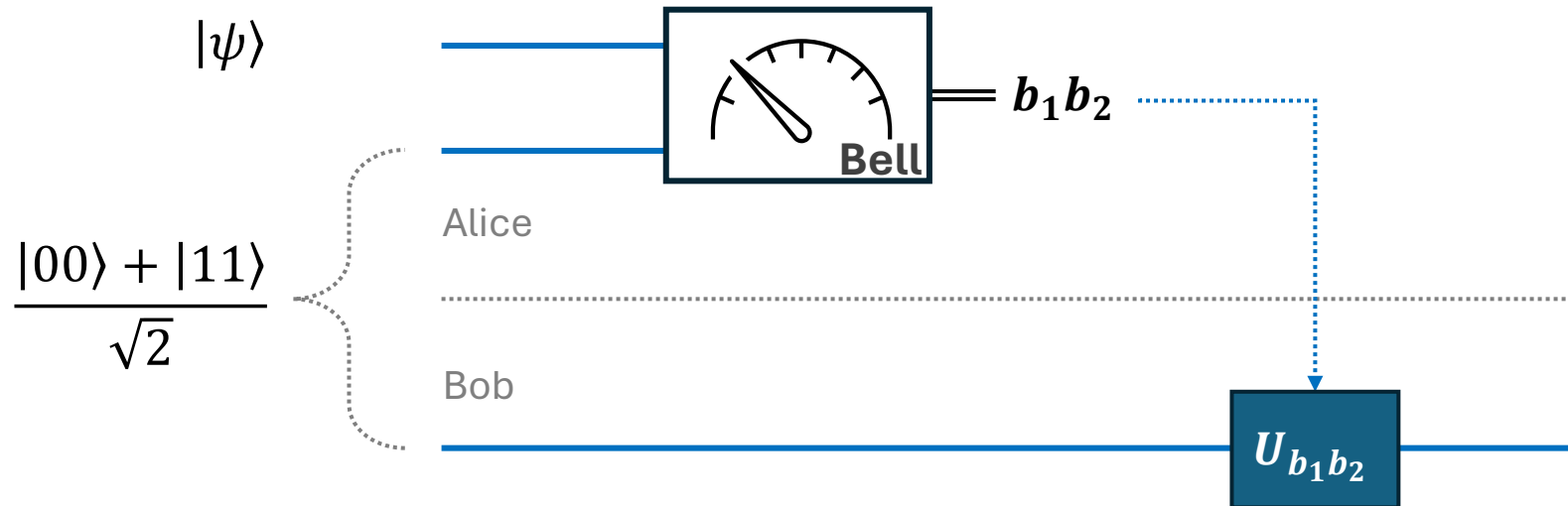
# Quantum Teleportation

- A more compact description (for analyzing states):



# Quantum Teleportation

- A more compact description (for analyzing states):

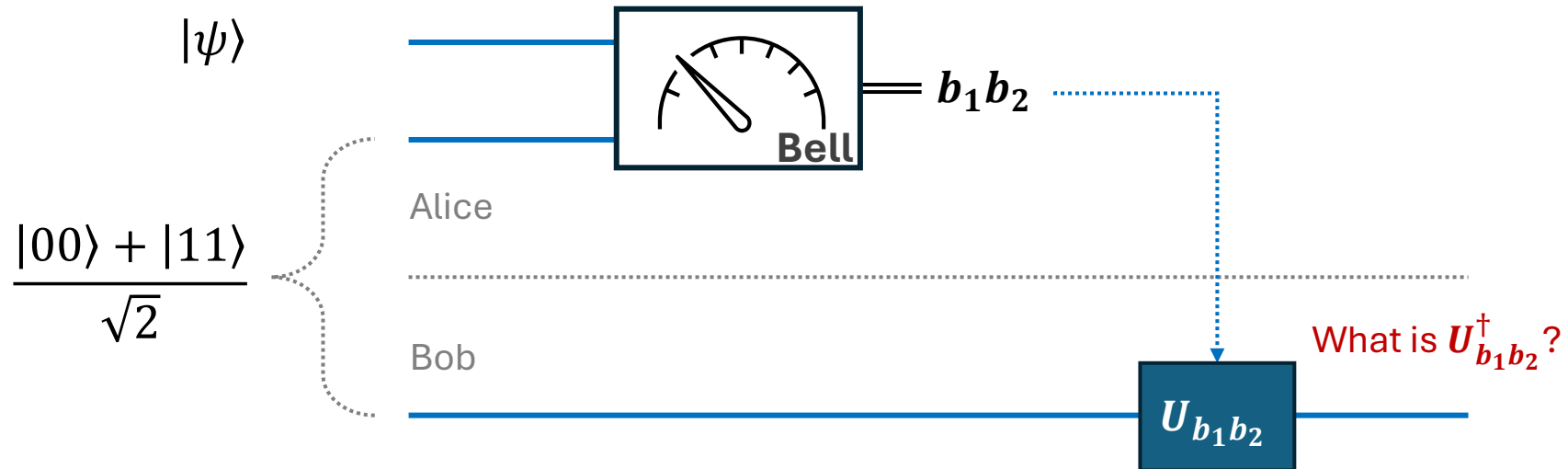


Total states:  $|\phi_0\rangle$

Can we re-write the state of the first two systems using the Bell basis?

# Quantum Teleportation

- A more compact description (for analyzing states):

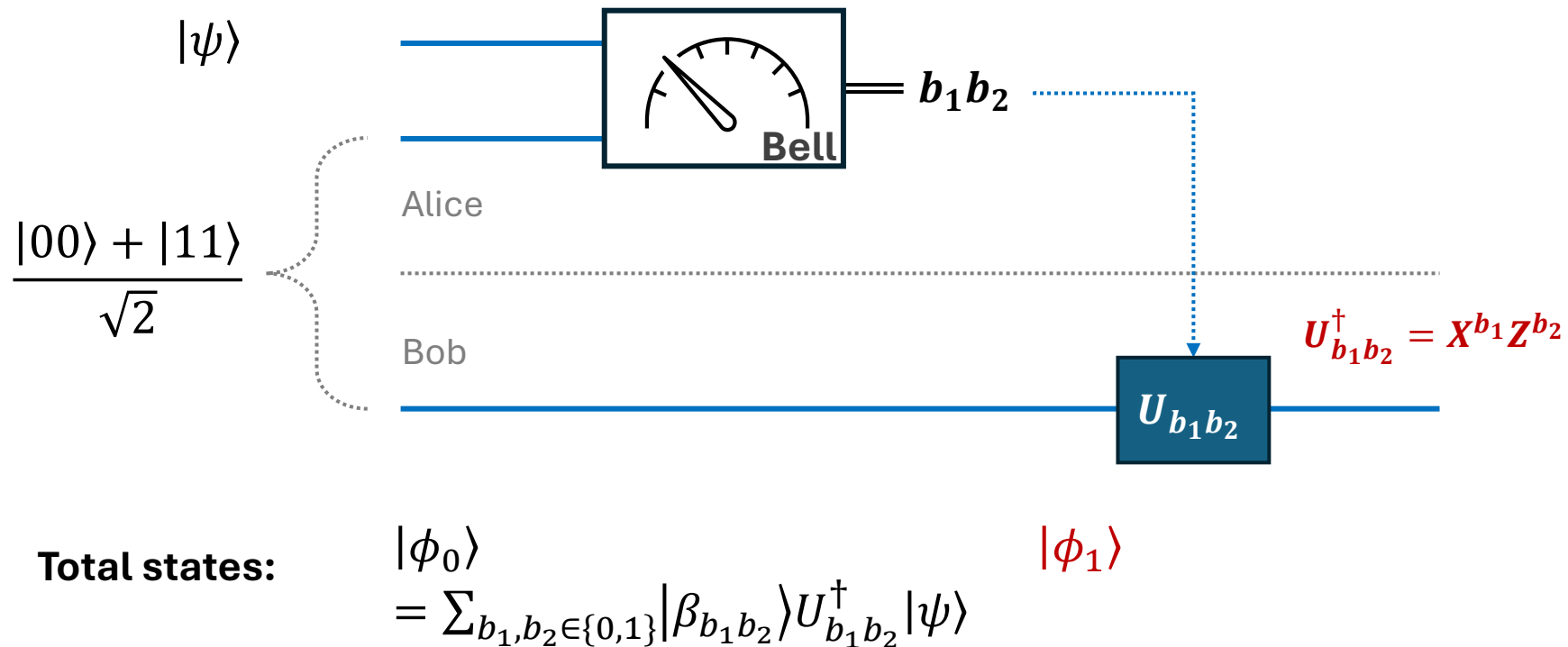


**Total states:**

$$|\phi_0\rangle = \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2}^\dagger |\psi\rangle$$

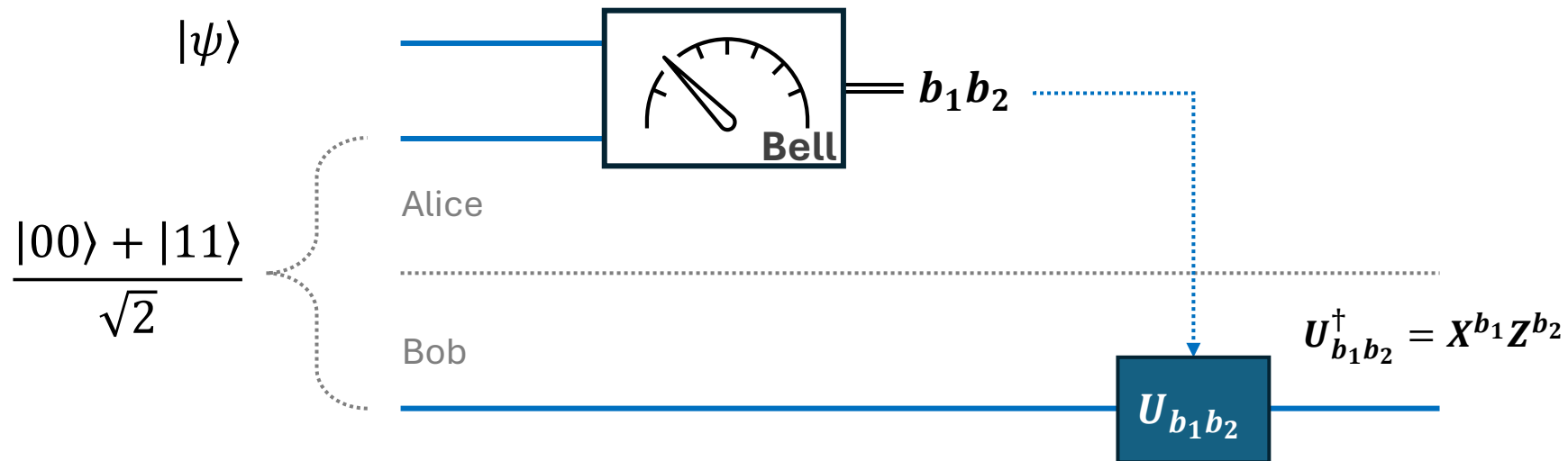
# Quantum Teleportation

- A more compact description (for analyzing states):



# Quantum Teleportation

- A more compact description (for analyzing states):

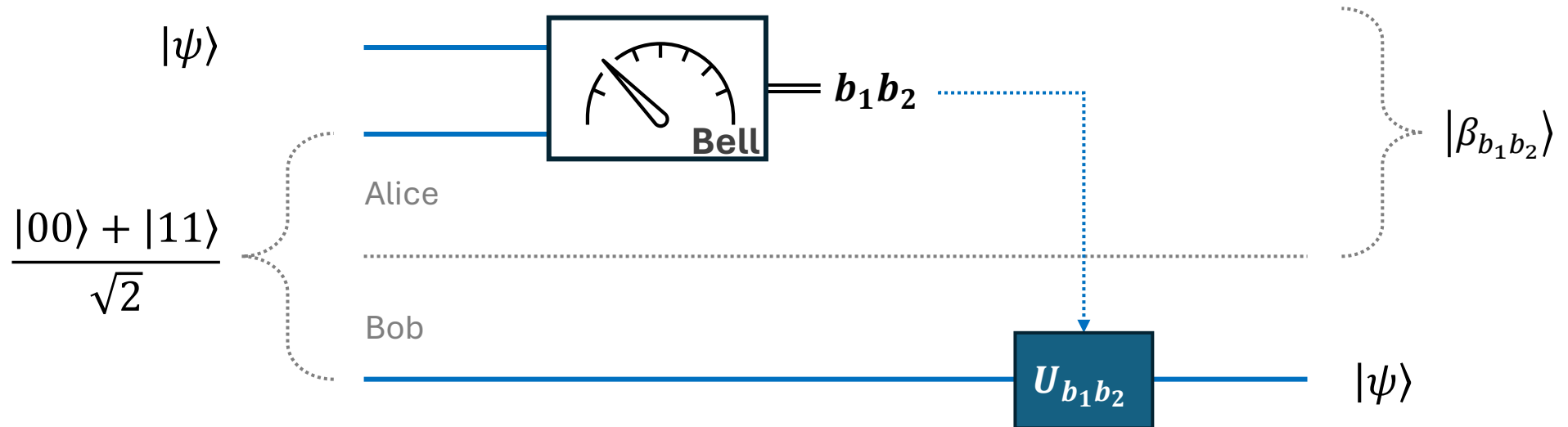


**Total states:**

$$|\phi_0\rangle = \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1b_2}\rangle U_{b_1b_2}^\dagger |\psi\rangle \quad |\phi_1\rangle = |\beta_{b_1b_2}\rangle U_{b_1b_2}^\dagger |\psi\rangle \quad |\phi_2\rangle$$

# Quantum Teleportation

- A more compact description (for analyzing states):

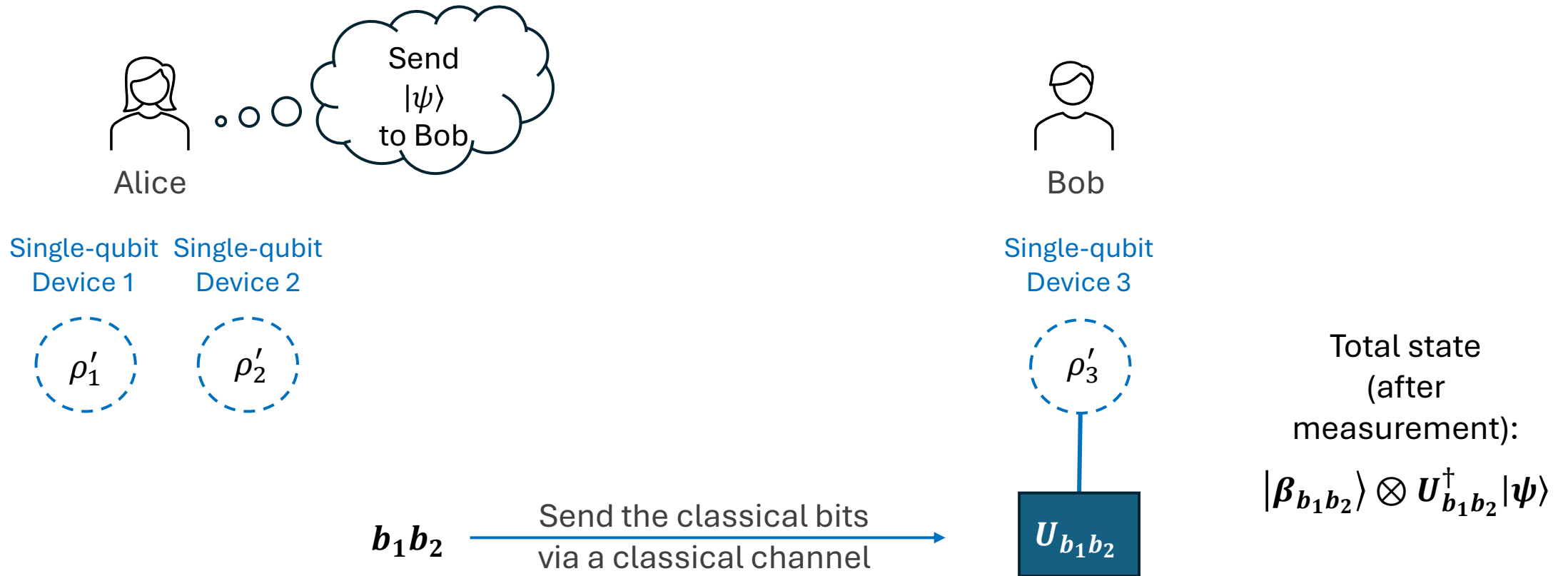


**Total states:**

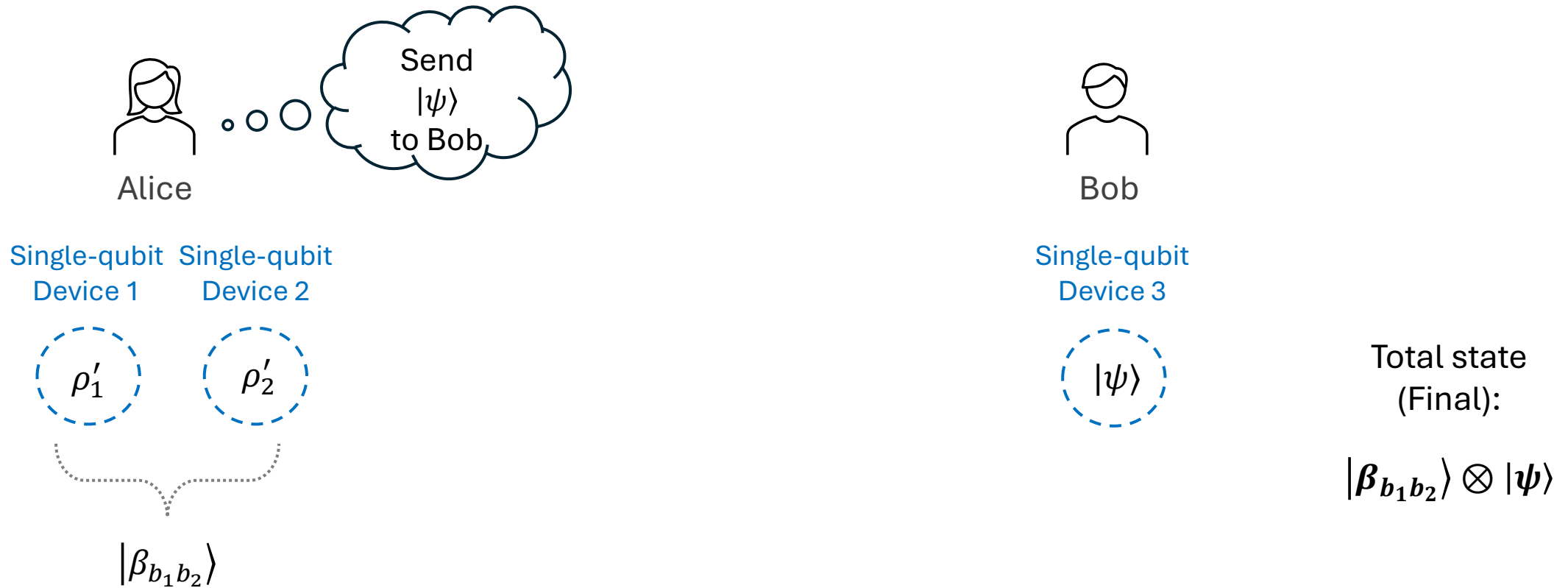
$$\begin{aligned}
 |\phi_0\rangle &= \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1b_2}\rangle U_{b_1b_2}^\dagger |\psi\rangle & |\phi_1\rangle &= |\beta_{b_1b_2}\rangle U_{b_1b_2}^\dagger |\psi\rangle & |\phi_2\rangle &= |\beta_{b_1b_2}\rangle \otimes |\psi\rangle
 \end{aligned}$$



# Quantum Teleportation



# Quantum Teleportation



# Reference

- **[NC00]**: Sections 1.3.6 and 1.3.7
- **[KLM07]**: Chapter 5

# Next topic

- **Quantum Circuits**
  - **Controlled operations and Measurement**