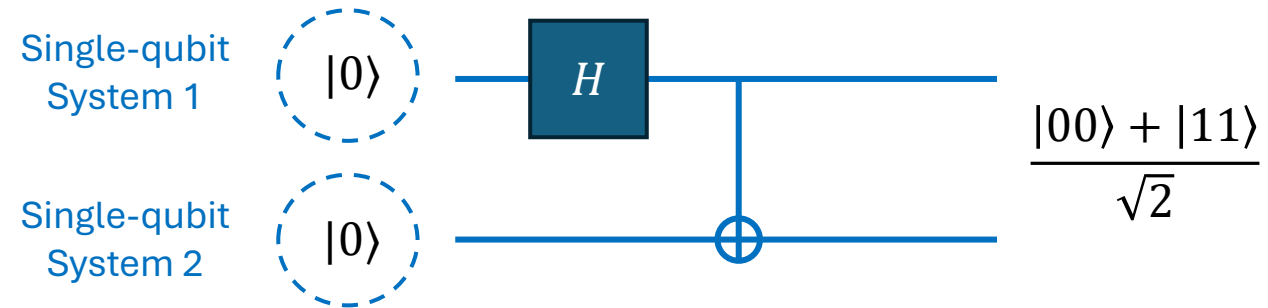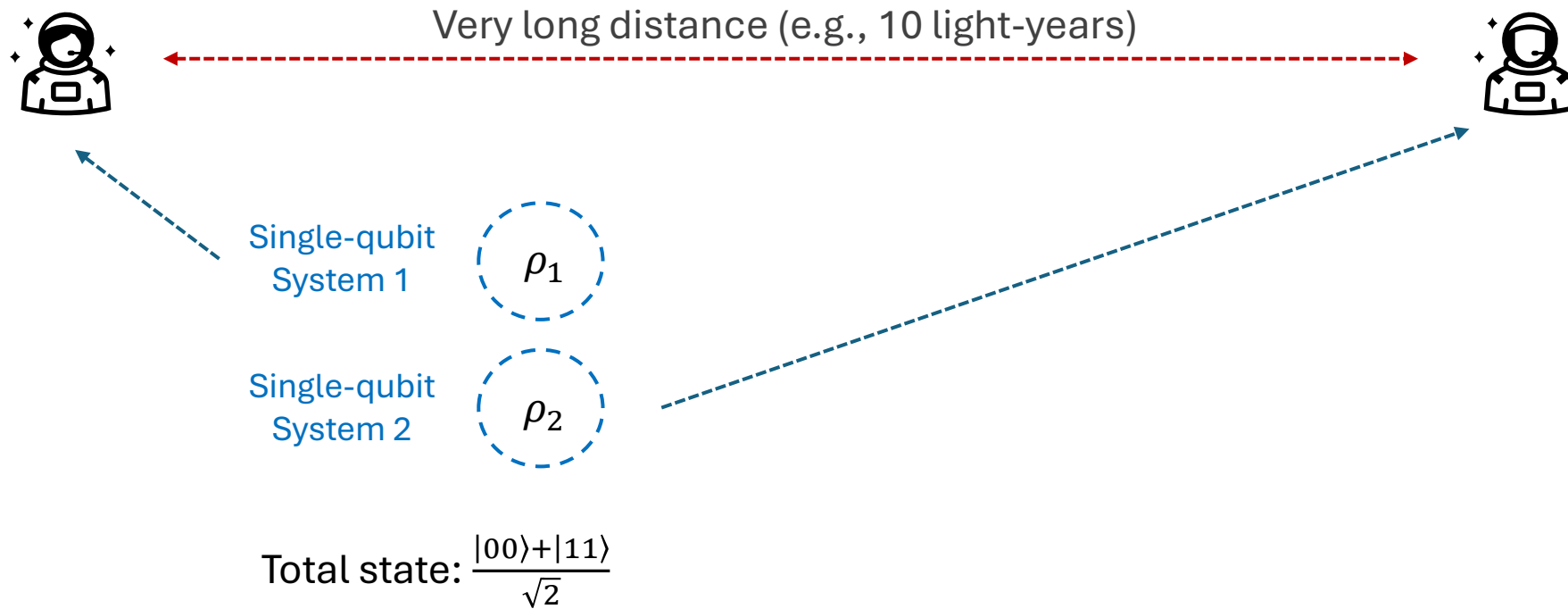# Quantum Computing

- Lectures 9 and 10 (June 4, 2025)

- Today:
  - Superdense coding
  - Quantum teleportation
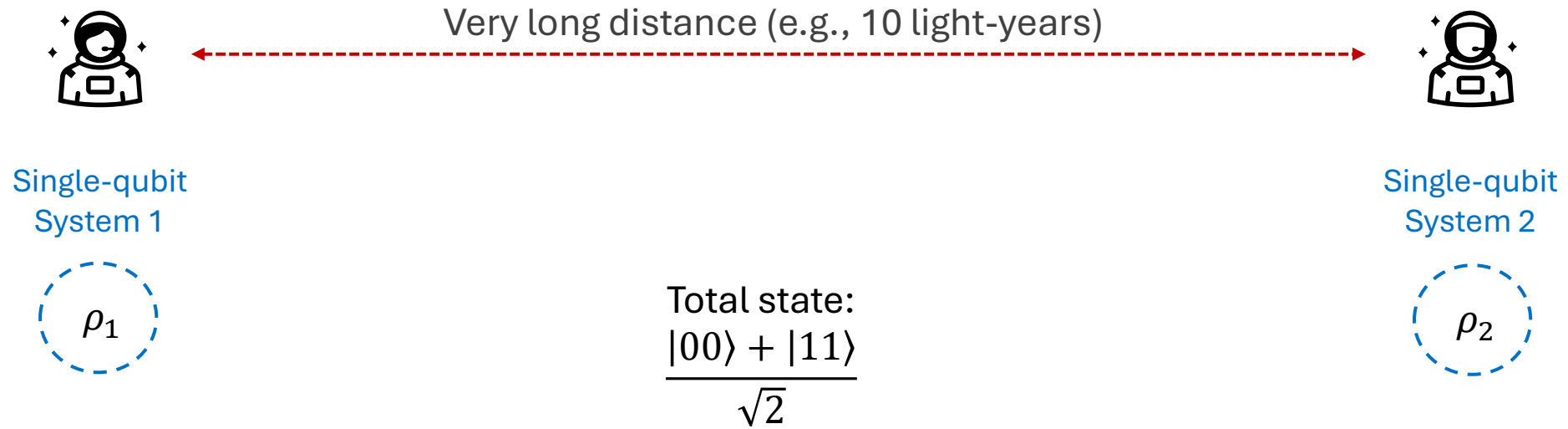
Action at a Distance

Action at a Distance

Very long distance (e.g., 10 light-years)

Single-qubit System 1 — ρ_1

Single-qubit System 2 — ρ_2

Total state: $\dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

Action at a Distance

Very long distance (e.g., 10 light-years)

Single-qubit
System 1

ρ_1

Single-qubit
System 2

ρ_2

Total state:
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Action at a Distance

Very long distance (e.g., 10 light-years)

Single-qubit
System 1

ρ_1

Single-qubit
System 2

ρ_2

Total state:
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Action at a Distance

Very long distance (e.g., 10 light-years)

Single-qubit
System 1

Single-qubit
System 2

$|0\rangle$

$|0\rangle$

Total state:
$|00\rangle$

- Application: Superdense coding

Pauli Matrices

- Pauli matrices:

$$X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\quad$ (Pauli-**X**) $\qquad\qquad\qquad\qquad$ (Pauli-**Y**) $\qquad\qquad\qquad\qquad$ (Pauli-**Z**)

- Some facts:
  - Pauli-X is the qNOT gate (in the computational basis)
  - $\sigma_j^2 = I$ for $j = 1,2,3$

Pauli Matrices

- (Extended) Pauli matrices:

$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(Pauli-**X**)   (Pauli-**Y**)   (Pauli-**Z**)

- Some facts:
  - Pauli-X is the qNOT gate (in the computational basis)
  - $\sigma_j^2 = I$ for $j = 0,1,2,3$
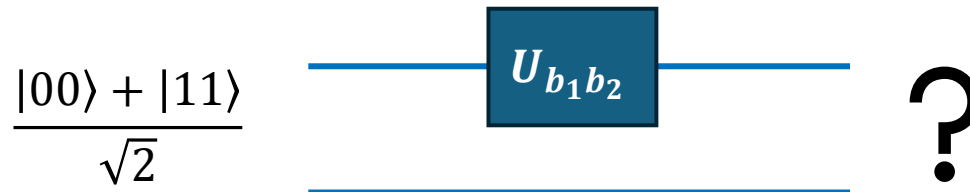
Superdense Coding

- Consider the four matrices

$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let's define:

$$U_{b_1 b_2} := Z^{b_2} X^{b_1} \begin{cases} U_{00} = I \\ U_{01} = X \\ U_{10} = Z \\ U_{11} = Z \cdot X \end{cases}$$

- Small Exercise:
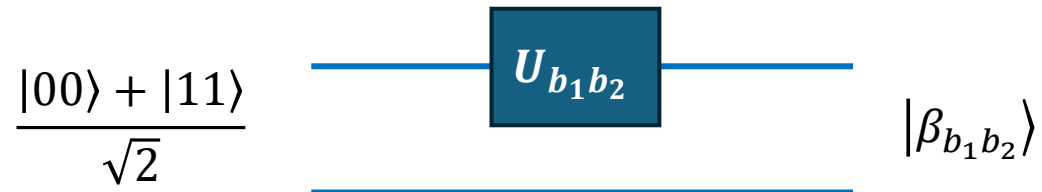
Superdense Coding

- Consider the four matrices

$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let's define:

$$U_{b_1 b_2} := Z^{b_2} X^{b_1} \begin{cases} U_{00} = I \\ U_{01} = X \\ U_{10} = Z \\ U_{11} = Z \cdot X \end{cases}$$

- Small Exercise:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad \boxed{U_{b_1 b_2}} \quad |\beta_{b_1 b_2}\rangle$$
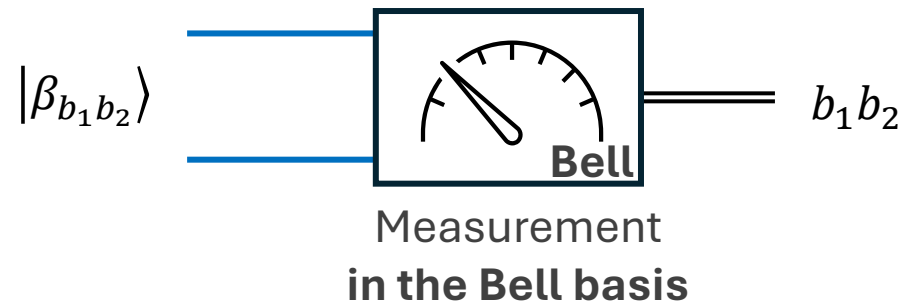
Superdense Coding

- Consider the four matrices

$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let's define:

$$U_{b_1 b_2} := Z^{b_2} X^{b_1} \begin{cases} U_{00} = I \\ U_{01} = X \\ U_{10} = Z \\ U_{11} = Z \cdot X \end{cases}$$
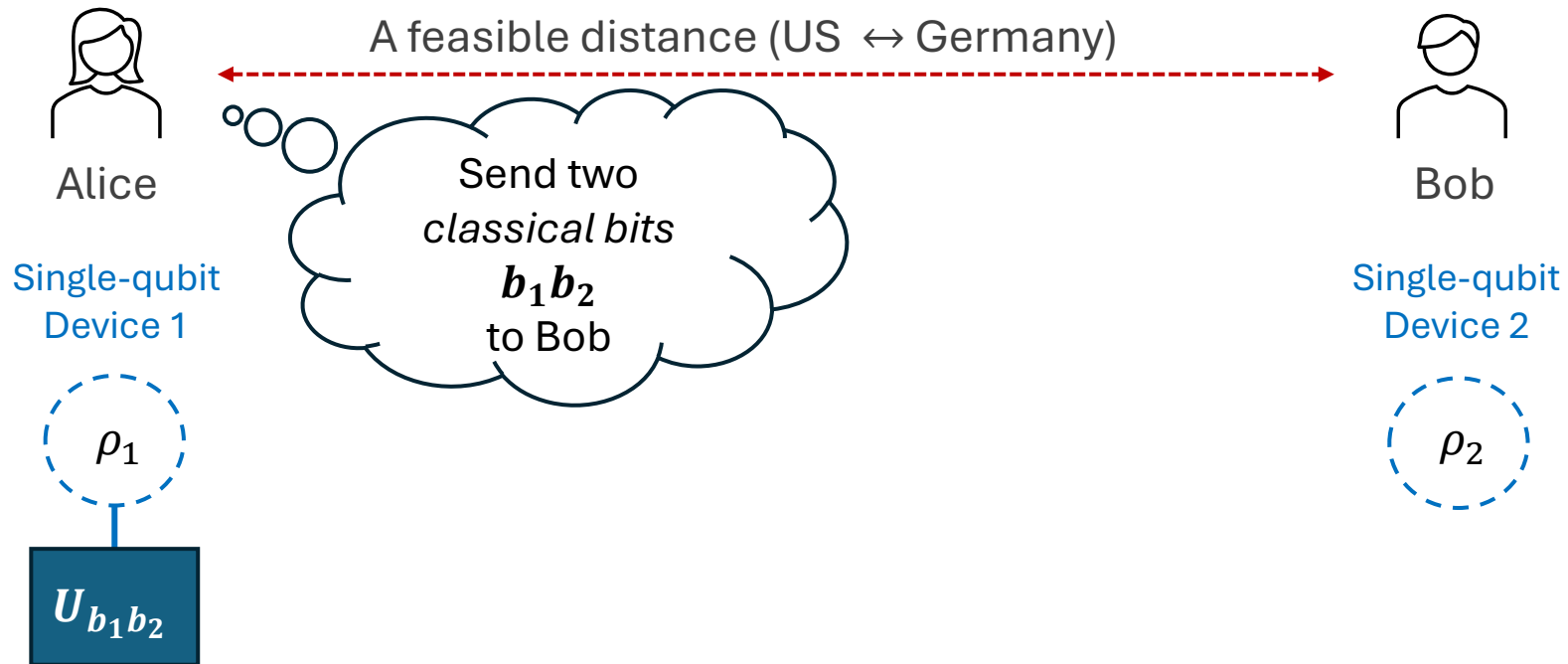
- Small Exercise:

The Bell basis:
$$\begin{cases} M_{00}: |\beta_{00}\rangle\langle\beta_{00}|, \\ M_{01}: |\beta_{01}\rangle\langle\beta_{01}|, \\ M_{10}: |\beta_{10}\rangle\langle\beta_{10}|, \\ M_{11}: |\beta_{11}\rangle\langle\beta_{11}|, \end{cases}$$
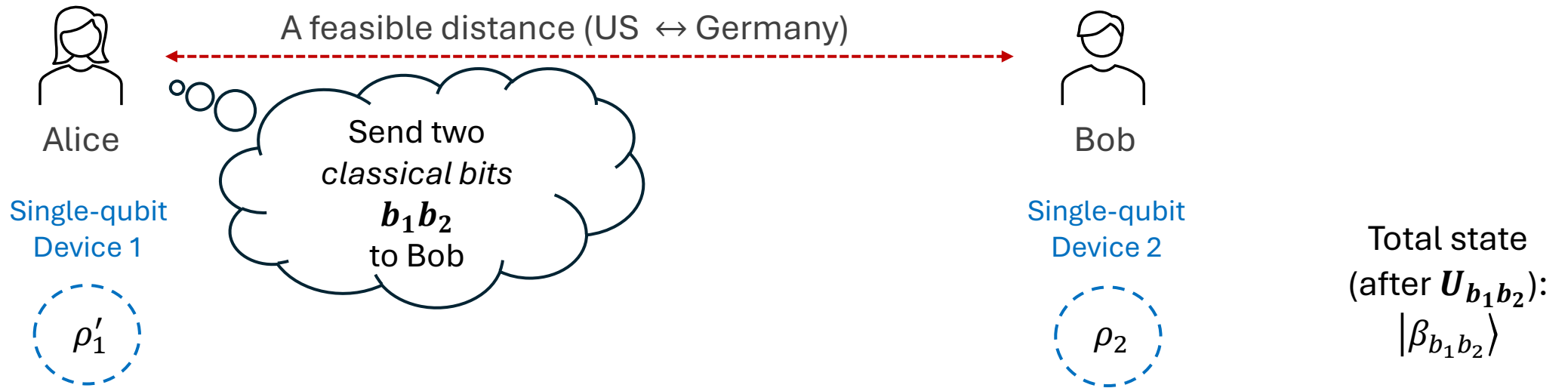
$|\beta_{b_1 b_2}\rangle$ — Bell Measurement in the Bell basis — $b_1 b_2$

UNI KASSEL
VERSITÄT

Superdense Coding

Alice

Bob

A feasible distance (US ↔ Germany)

Single-qubit
Device 1

ρ_1

Single-qubit
Device 2

ρ_2

Total state:
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Superdense Coding

Alice

A feasible distance (US $\leftrightarrow$ Germany)

Bob

Send two *classical bits* $b_1 b_2$ to Bob

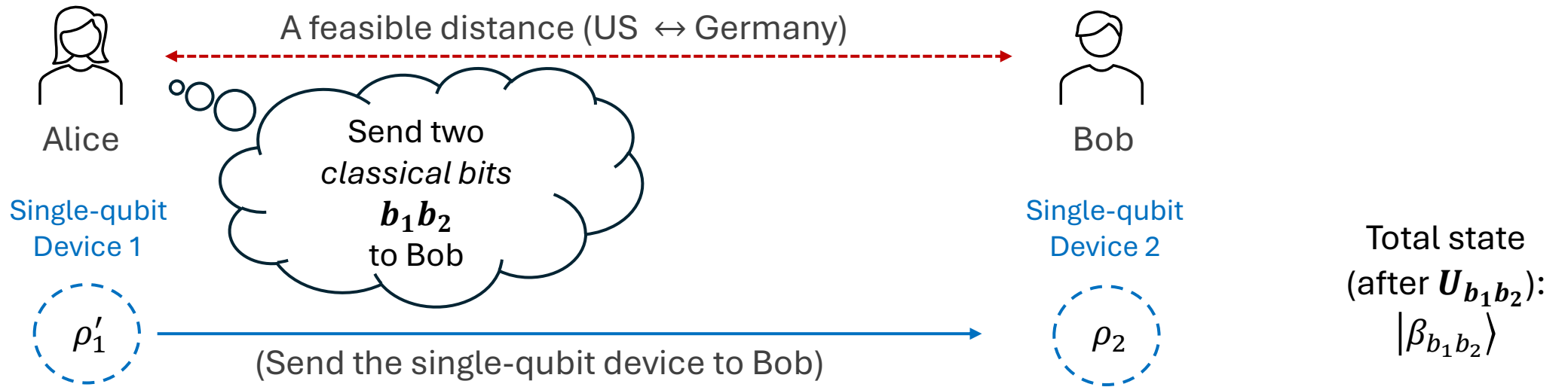Single-qubit
Device 1

ρ_1

$U_{b_1 b_2}$

Single-qubit
Device 2

ρ_2

Total state
(before $U_{b_1 b_2}$):
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Superdense Coding

A feasible distance (US ↔ Germany)

Alice

Single-qubit
Device 1

ρ_1'

Send two
classical bits
$b_1 b_2$
to Bob

Bob

Single-qubit
Device 2

ρ_2

Total state
(after $U_{b_1 b_2}$):
$|\beta_{b_1 b_2}\rangle$

Superdense Coding

Alice

A feasible distance (US ↔ Germany)

Send two
classical bits
$b_1 b_2$
to Bob

Single-qubit
Device 1

ρ_1'

(Send the single-qubit device to Bob)

Bob

Single-qubit
Device 2

ρ_2

Total state
(after $U_{b_1 b_2}$):
$|\beta_{b_1 b_2}\rangle$

Superdense Coding

Alice

Send two
classical bits
$b_1 b_2$
to Bob

Bob

Single-qubit
Device 1

Single-qubit
Device 2

ρ_1'

ρ_2

Total state
(after $U_{b_1 b_2}$):
$|\beta_{b_1 b_2}\rangle$

Superdense Coding

Alice

Send two
classical bits
$b_1 b_2$
to Bob

Bob

Single-qubit
Device 1

Single-qubit
Device 2

ρ_1'

ρ_2

Total state
(after $U_{b_1 b_2}$):
$|\beta_{b_1 b_2}\rangle$

Bell

$b_1 b_2$

Superdense Coding

Alice

Send two
classical bits
$b_1 b_2$
to Bob

Bob

Single-qubit
Device 1

Single-qubit
Device 2

Total state
(after $U_{b_1 b_2}$):
$|\beta_{b_1 b_2}\rangle$

ρ_1'

ρ_2

Bell

$b_1 b_2$
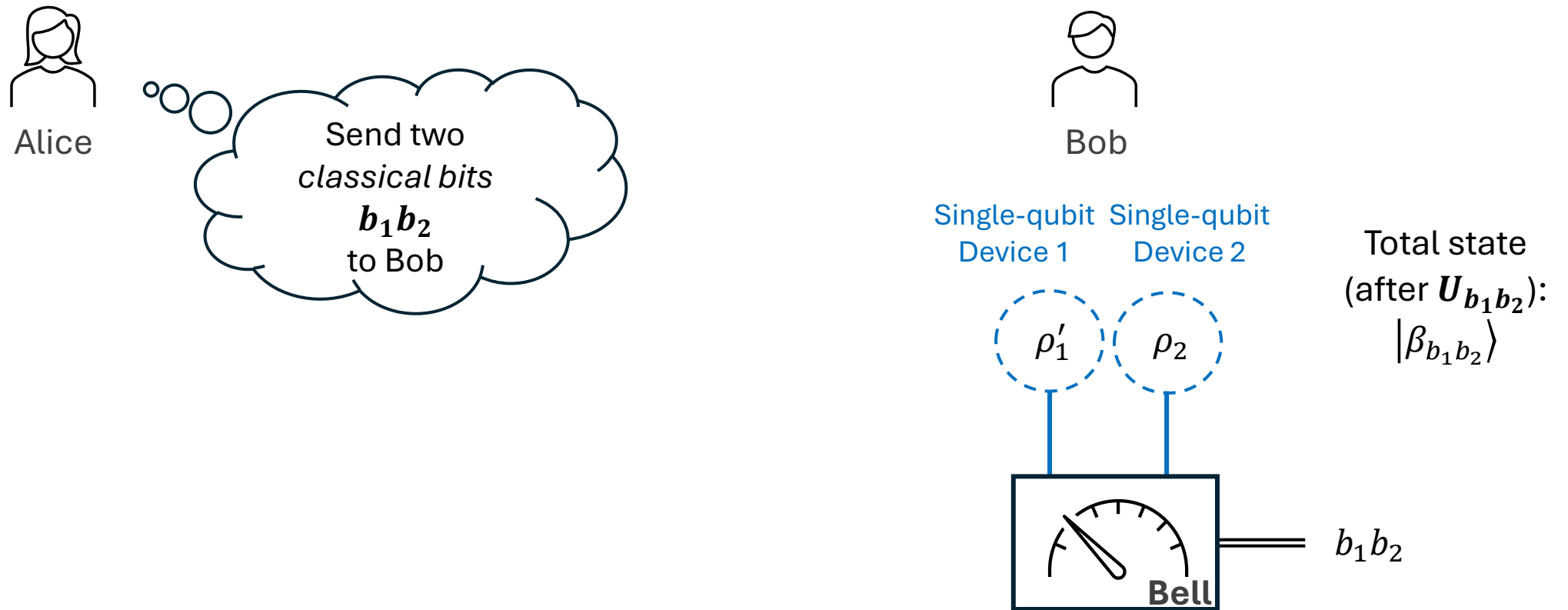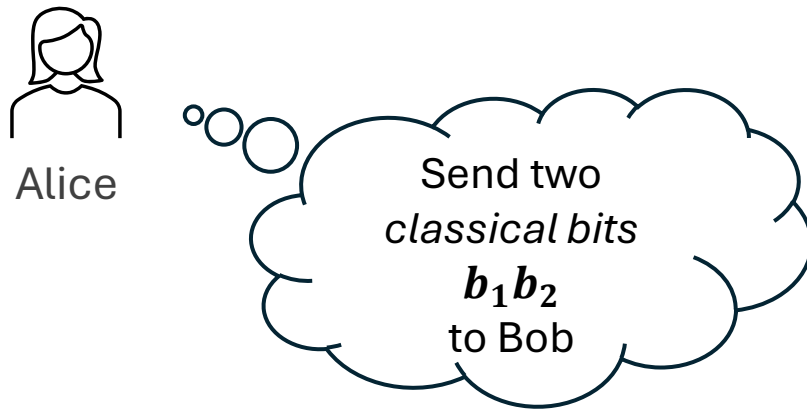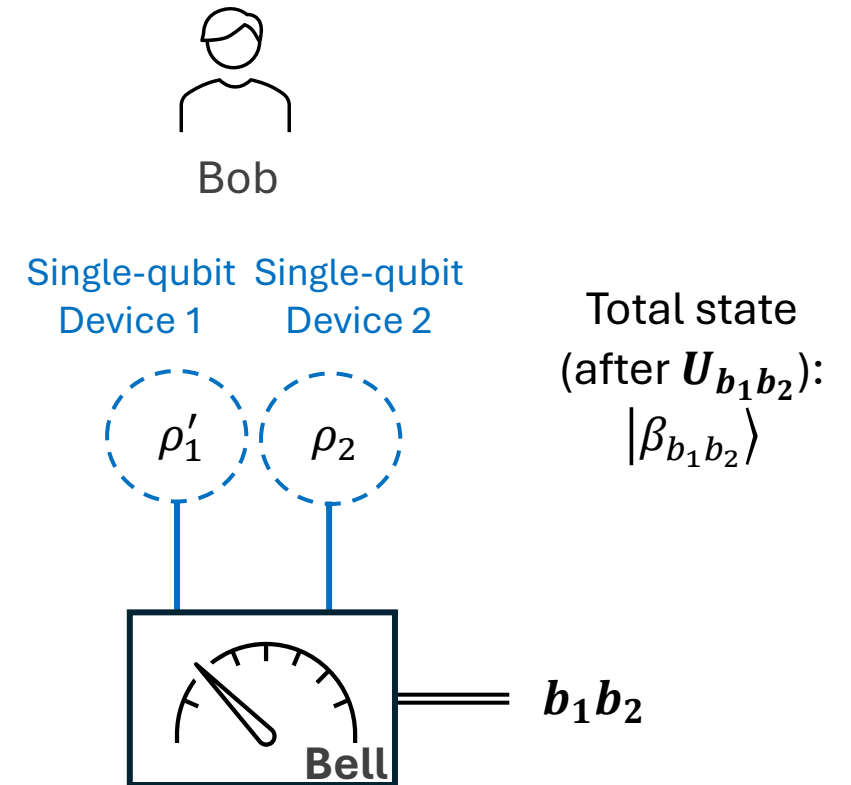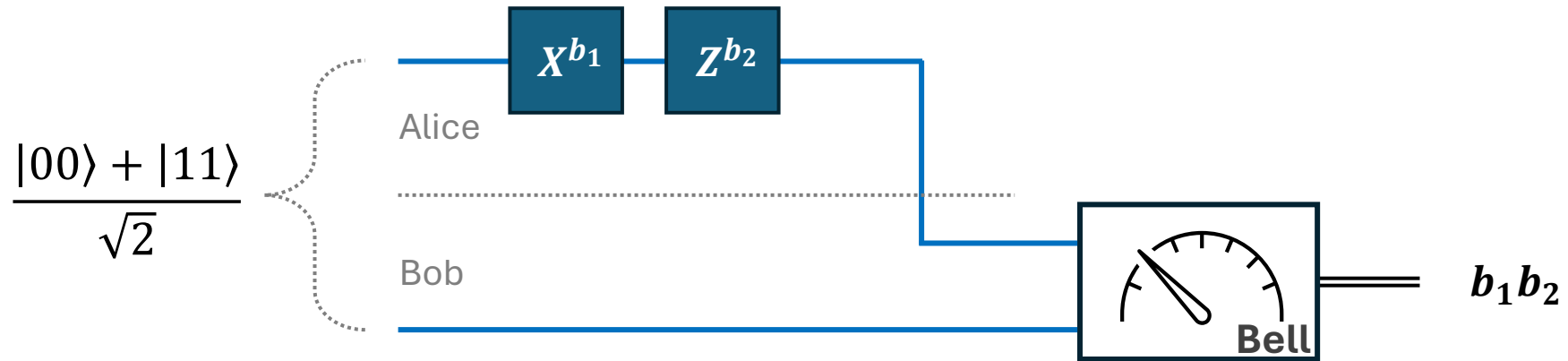
- **One** physical qubit can transmit **two** classical bits of information
  - (Require prior entanglement)
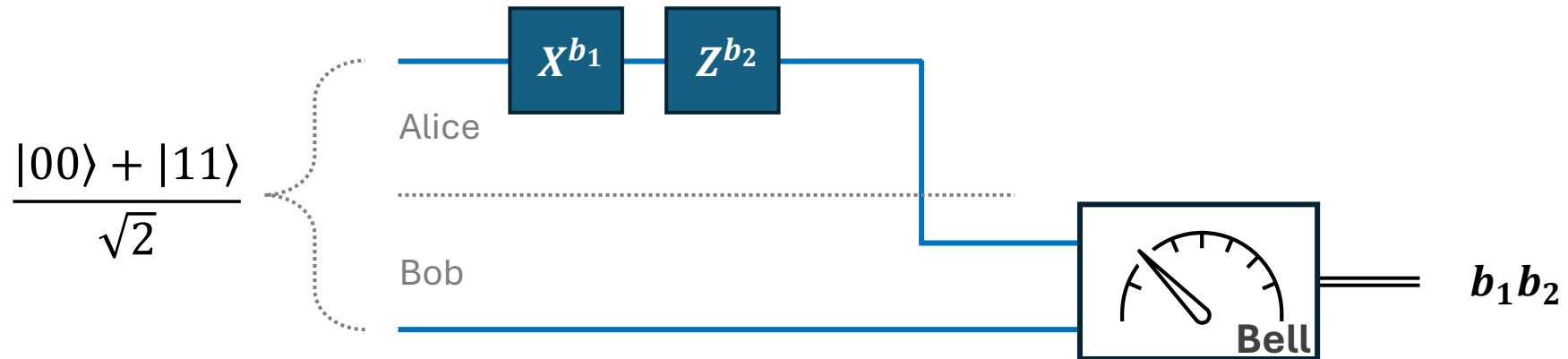- **Superdense coding:** One qubit "encodes" two classical bits...

Superdense Coding

- A more compact description of the experiment:

Superdense Coding

- A more compact description of the experiment:

- Quick questions:
- (1) We wrote $U_{b_1 b_2} := Z^{b_2} X^{b_1}$ before, but why does X come first here?
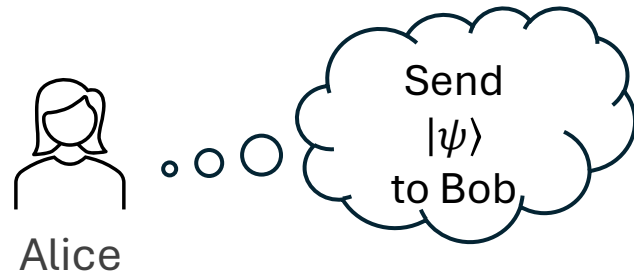- (2) How can we **transmit $2n$ classical bits using only n qubits?**

Quantum Teleportation

- Superdense coding: Transmit classical bits via qubits

- Quantum teleportation: **Transmit qubits via classical bits**

Alice

$|\psi\rangle$
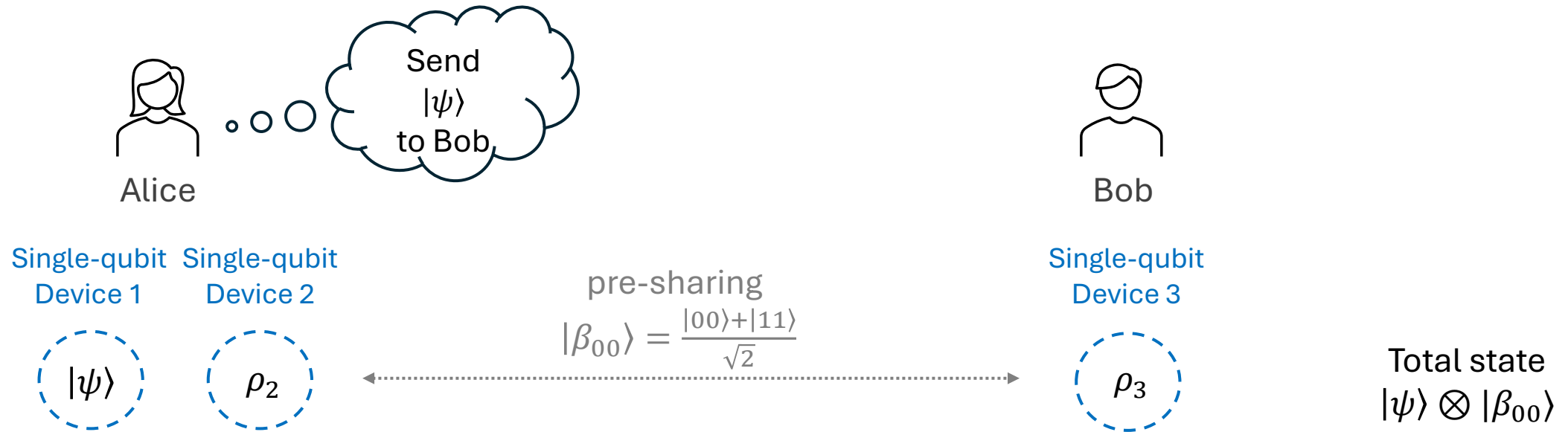
Send $|\psi\rangle$ to Bob

Bob

- A trivial approach (transportation): Physically deliver the device carrying $|\psi\rangle$ to Bob

- But here we consider **Teleportation**: Transmit via (quantum) channel
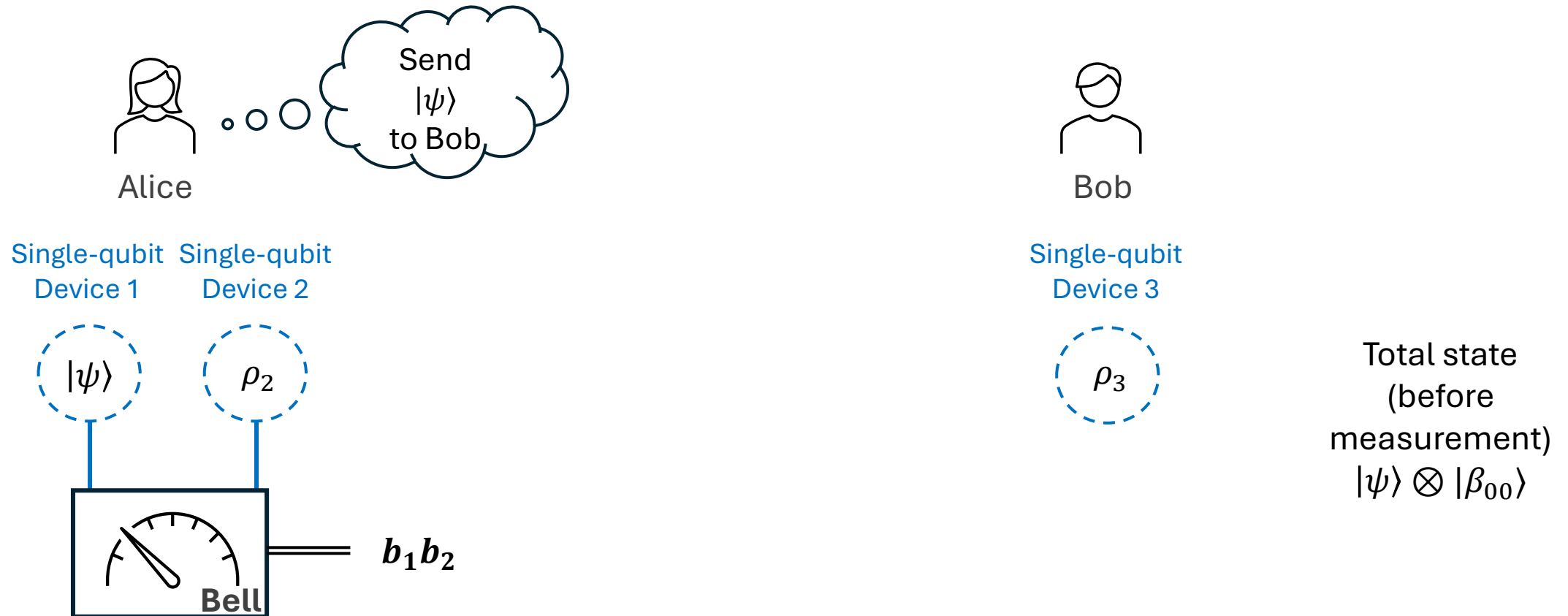
Quantum Teleportation

Alice

Bob

Send $|\psi\rangle$ to Bob

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Quantum Teleportation

Alice

Send $|\psi\rangle$ to Bob

Bob

Single-qubit Device 1

Single-qubit Device 2

Single-qubit Device 3

pre-sharing

$|\beta_{00}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$

$|\psi\rangle$

ρ_2

ρ_3

Total state
$|\psi\rangle \otimes |\beta_{00}\rangle$

UNIKASSEL
VERSITÄT

Quantum Teleportation

Alice

Send
$|\psi\rangle$
to Bob

Bob

Single-qubit
Device 1

Single-qubit
Device 2

Single-qubit
Device 3

$|\psi\rangle$

ρ_2

ρ_3

Total state
(before
measurement)
$|\psi\rangle \otimes |\beta_{00}\rangle$

$b_1 b_2$

Bell

UNIKASSEL
VERSITÄT

Quantum Teleportation

Quantum Teleportation
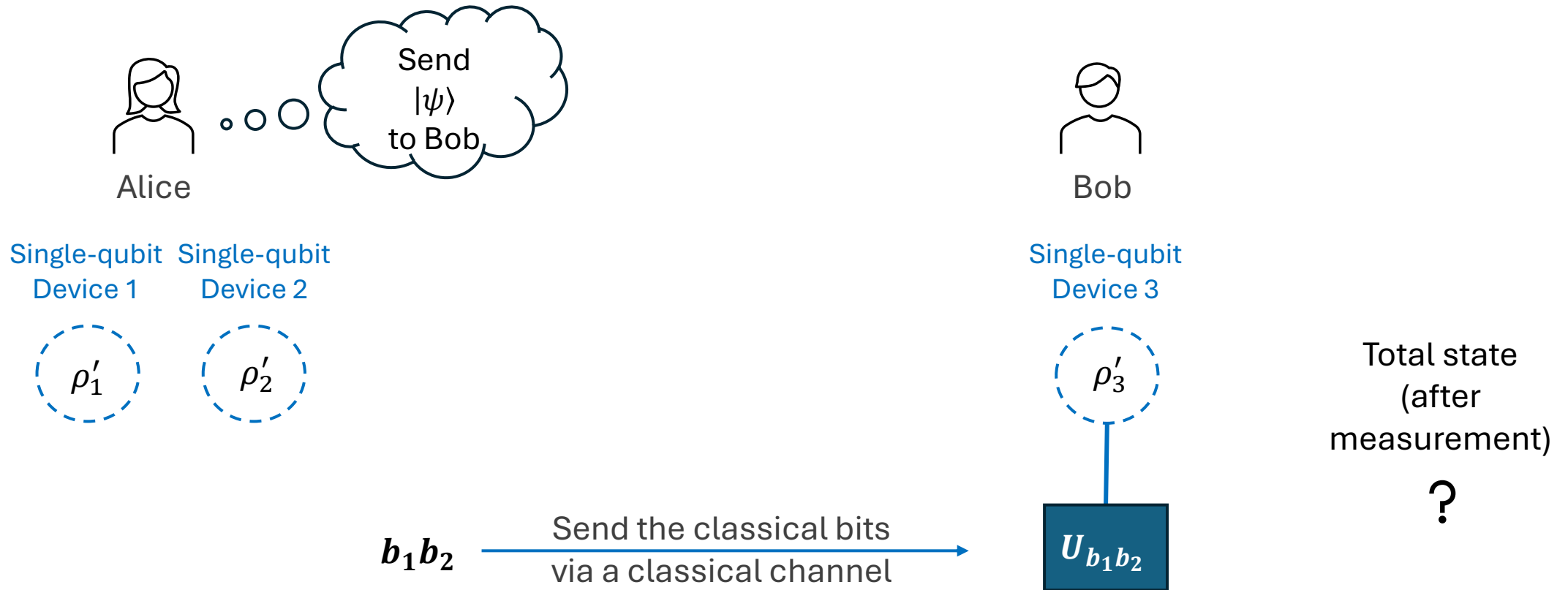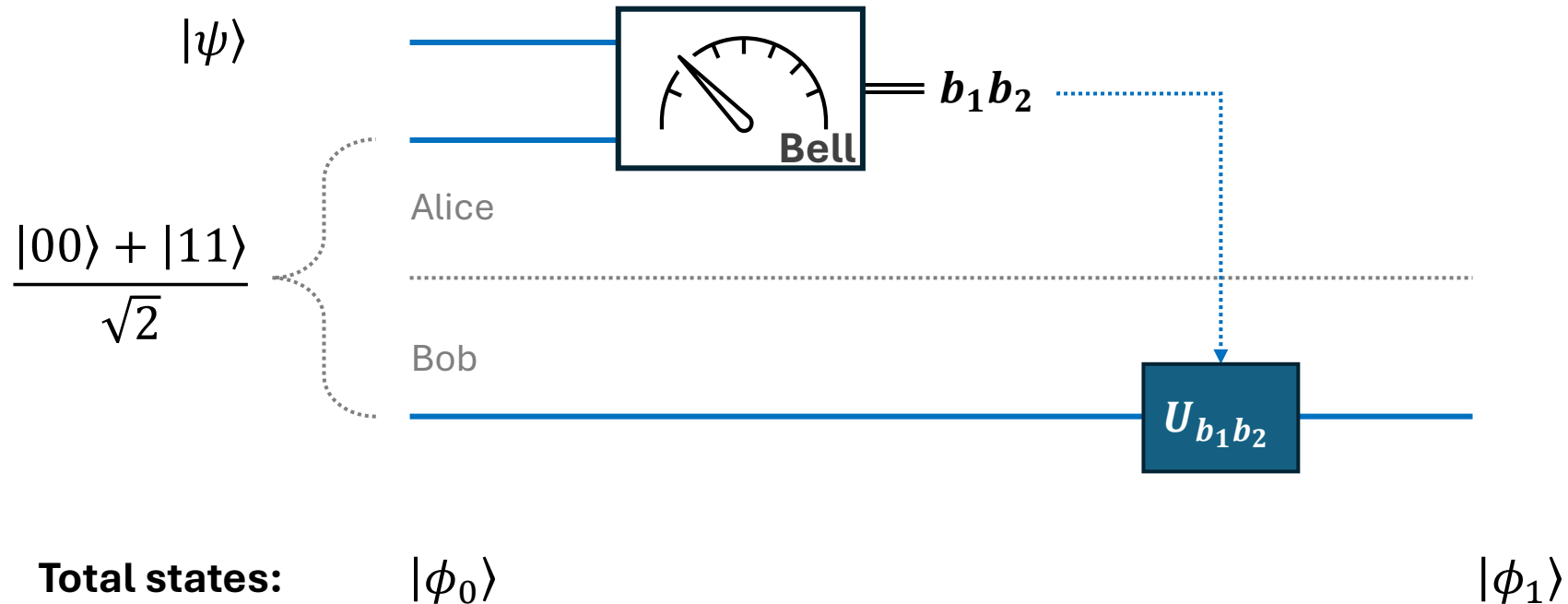
- **A more compact description (for analyzing states):**

$|\psi\rangle$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$b_1 b_2$

Bell

Alice

Bob

$U_{b_1 b_2}$

Total states: $|\phi_0\rangle$ $|\phi_1\rangle$

Quantum Teleportation

- **A more compact description (for analyzing states):**

$$|\psi\rangle$$

$$b_1 b_2$$

Bell

Alice

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
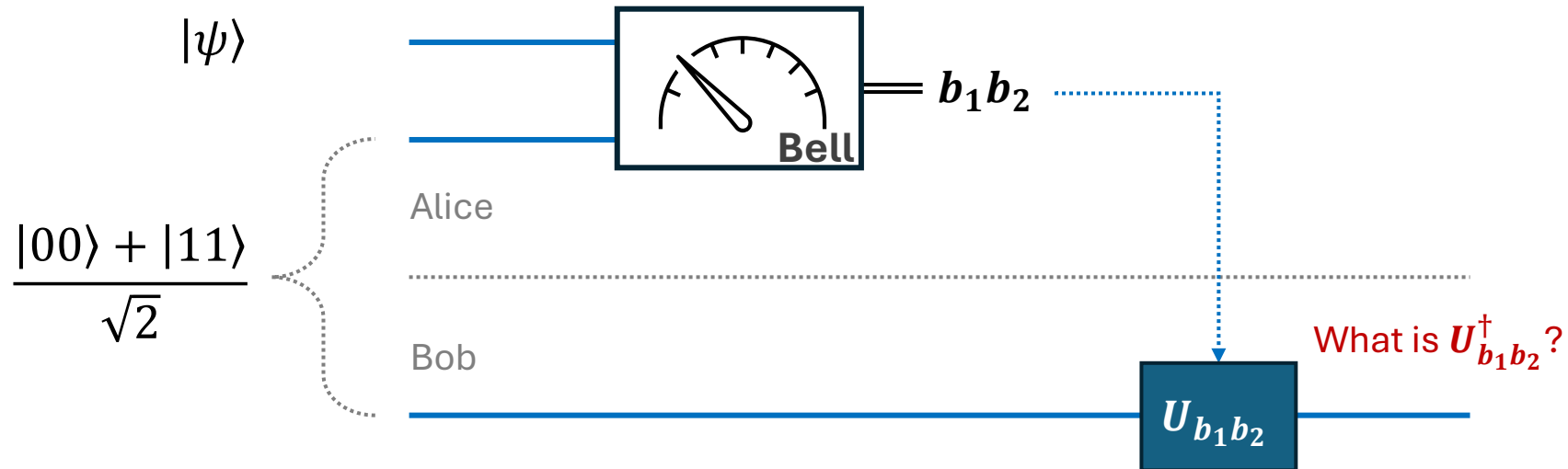
Bob
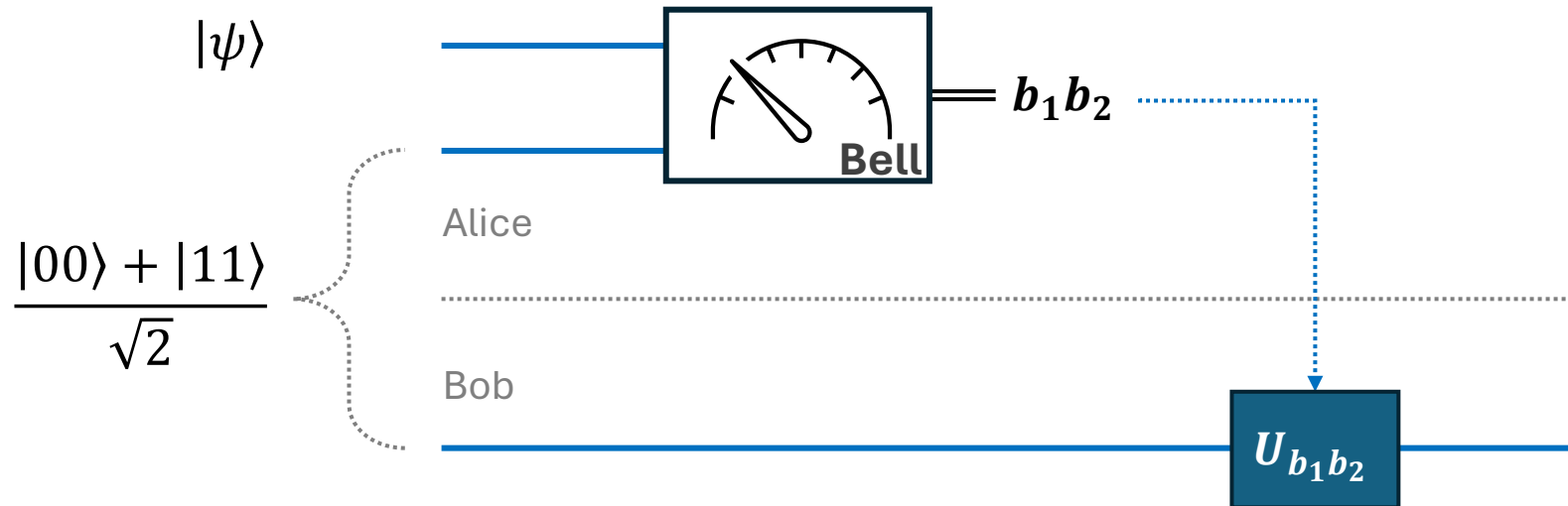
$$U_{b_1 b_2}$$

Total states: $\quad |\phi_0\rangle$

Can we re-write the state of the first two systems using the Bell basis?

Quantum Teleportation
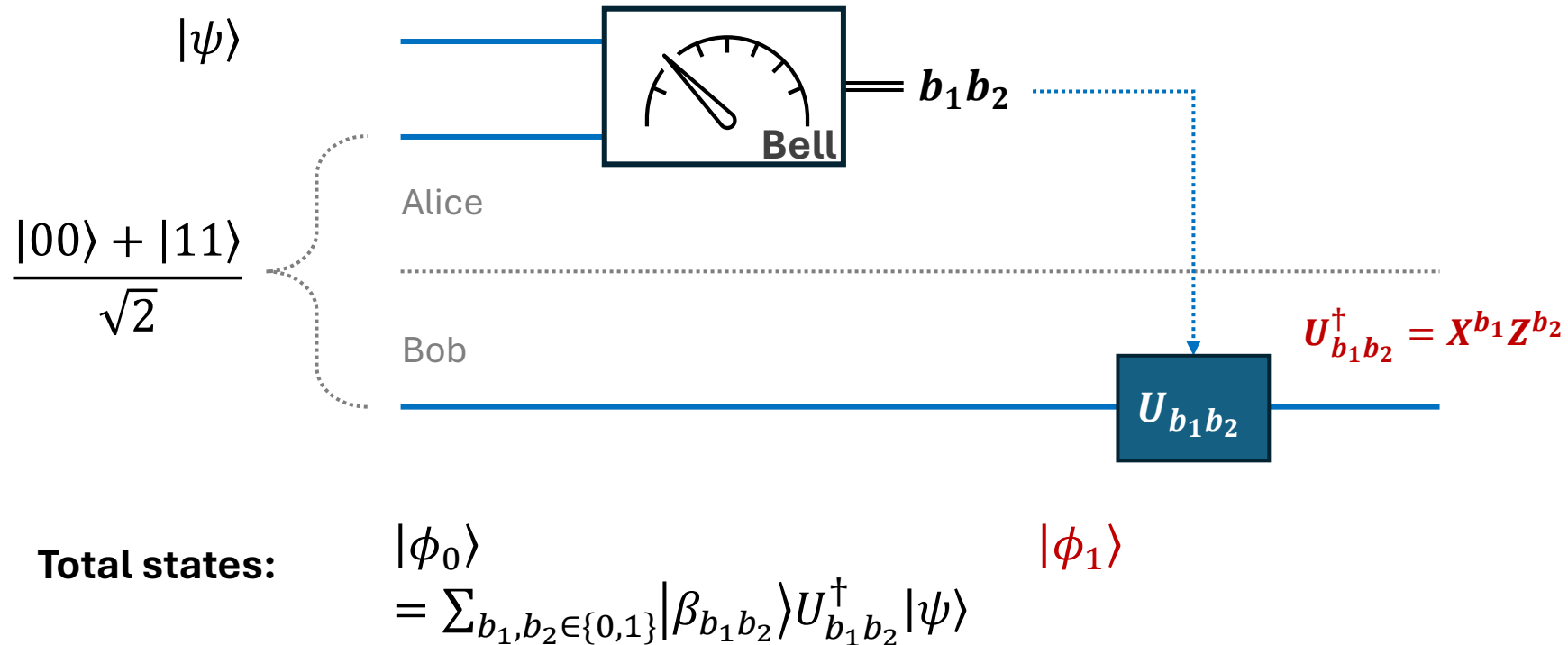
- **A more compact description (for analyzing states):**

$$|\psi\rangle$$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Alice

$$b_1 b_2$$

Bell

Bob

What is $U_{b_1 b_2}^{\dagger}$?

$$U_{b_1 b_2}$$

Total states:

$$|\phi_0\rangle$$
$$= \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2}^{\dagger} |\psi\rangle$$

UNI KASSEL
VERSITÄT

Quantum Teleportation

- **A more compact description (for analyzing states):**

$|\psi\rangle$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Bell

$b_1 b_2$

Alice

Bob

$U_{b_1 b_2}$

$U_{b_1 b_2}^\dagger = X^{b_1} Z^{b_2}$

Total states:

$|\phi_0\rangle$

$|\phi_1\rangle$

$$= \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2}^\dagger |\psi\rangle$$

Quantum Teleportation

- **A more compact description (for analyzing states):**

$|\psi\rangle$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Alice

Bob

Bell $= b_1 b_2$

$U_{b_1 b_2}^\dagger = X^{b_1} Z^{b_2}$
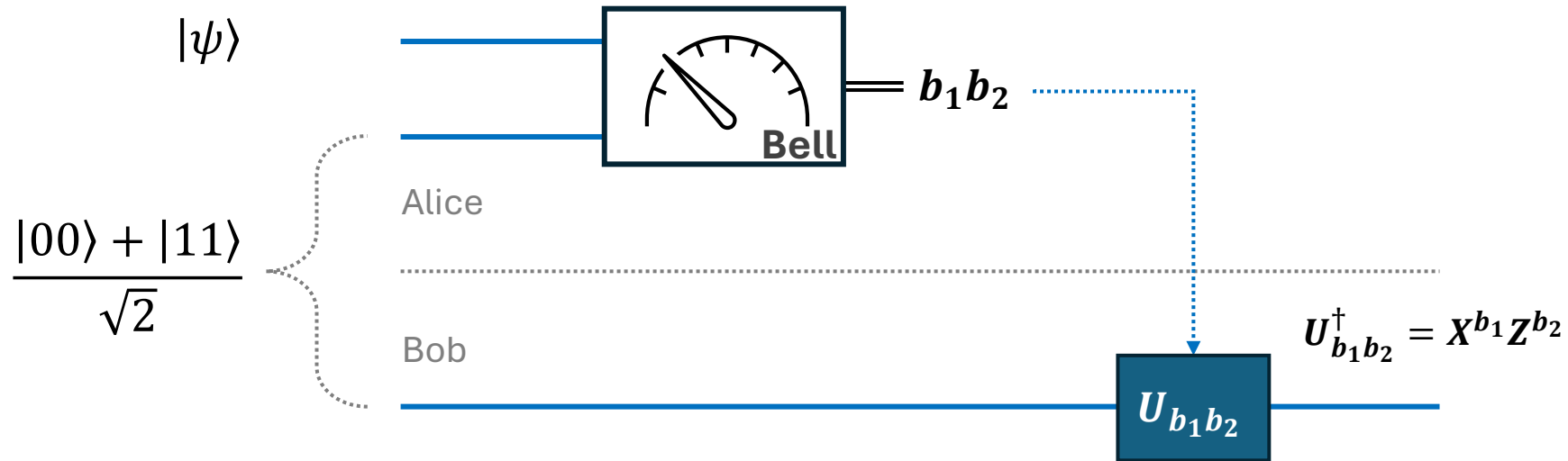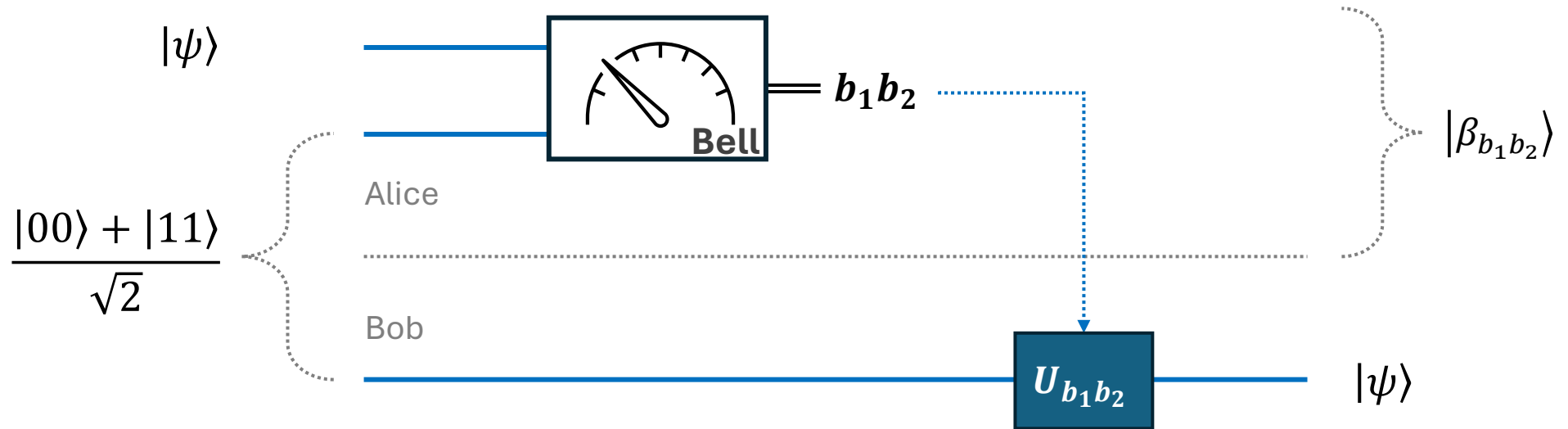
$U_{b_1 b_2}$

Total states:

$|\phi_0\rangle$
$= \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2}^\dagger |\psi\rangle$

$|\phi_1\rangle$
$= |\beta_{b_1 b_2}\rangle U_{b_1 b_2}^\dagger |\psi\rangle$
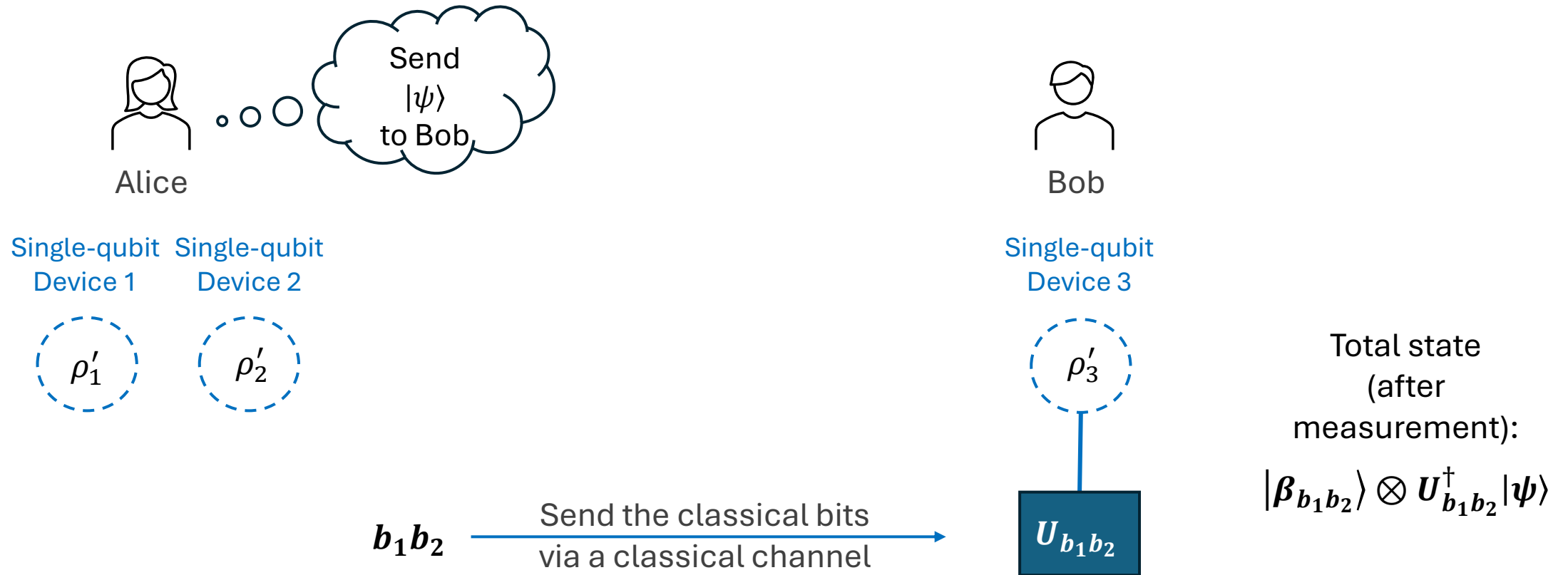
$|\phi_2\rangle$

Quantum Teleportation
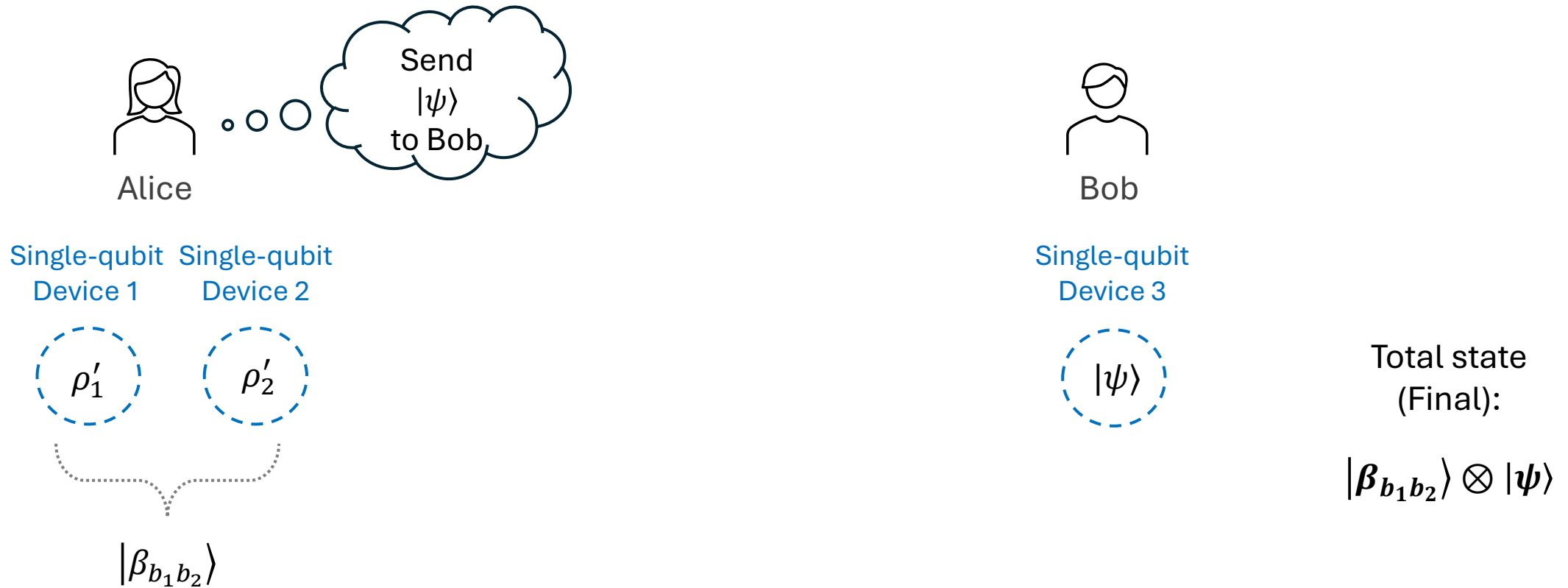
- **A more compact description (for analyzing states):**

$|\psi\rangle$

$\dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

Bell $= b_1 b_2$

Alice

Bob

$U_{b_1 b_2}$

$|\beta_{b_1 b_2}\rangle$

$|\psi\rangle$

Total states:

$|\phi_0\rangle$
$= \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2}^\dagger |\psi\rangle$

$|\phi_1\rangle$
$= |\beta_{b_1 b_2}\rangle U_{b_1 b_2}^\dagger |\psi\rangle$

$|\phi_2\rangle$
$= |\beta_{b_1 b_2}\rangle \otimes |\psi\rangle$

UNI KASSEL
VERSITÄT

Quantum Teleportation

Reference

- **[NC00]:** Sections 1.3.6 and 1.3.7
- **[KLM07]:** Chapter 5

Next topic

- **Quantum Circuits**
  - **Controlled operations and Measurement**