# Quantum Computing

- Lecture 4 (May 7, 2025)

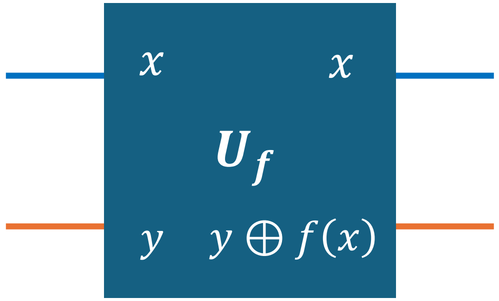
- Today:
    - Unitary operations on multi-qubit systems
    - Some examples (do it on the board)

# Unitary Operations

- A quantum gate is a unitary operator (⇔ A unitary represents some quantum gate)
  - A unitary operator has **linearity:** $\boldsymbol{U}(c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}) = c_1\boldsymbol{U}\boldsymbol{v_1} + c_2\boldsymbol{U}\boldsymbol{v_2}$

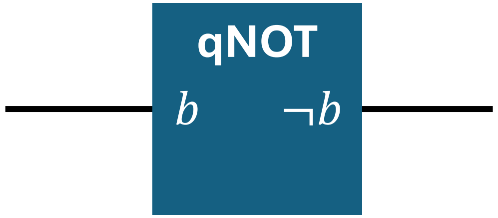
- Quantum gates operate on superposition: **Linearity**
  - View any quantum gate as a unitary linear operator (matrix)
  - Quantum gates act on superpositions according to linearity

- Make a classical computable function unitary $f \rightarrow \boldsymbol{U_f}$
  - Use **input qubits** and **ancilla qubits** to make it invertible
  - Any classical algorithm can be simulated by quantum computers



# Unitary Operations on Single-Qubit States

- Single-qubit unitary:
  - Examples: qNOT, Hadamard transform, …




$$\mathbf{qNOT}(|b\rangle) \rightarrow |\neg b\rangle$$


$$\boldsymbol{H}(|b\rangle) \rightarrow \frac{|0\rangle + (-1)^b|1\rangle}{\sqrt{2}}$$

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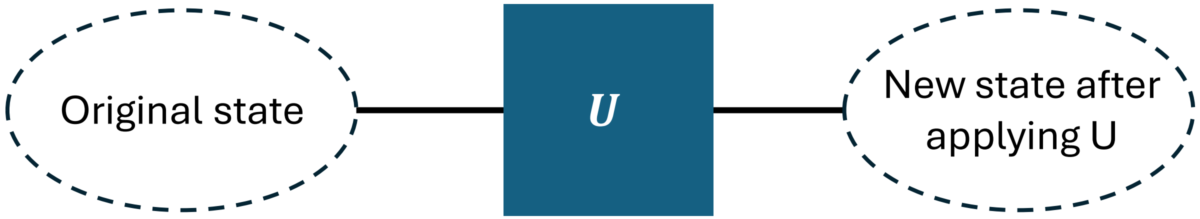
Some **isolated** single-qubit system



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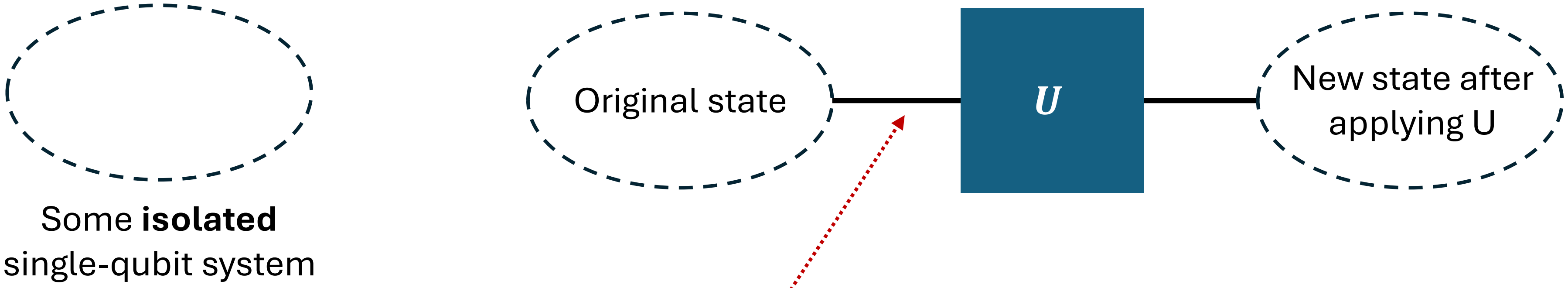


Misunderstand: The “wire” here **does not** represent a real wire!

Instead, it just visually **tracks the state of the system** through time.

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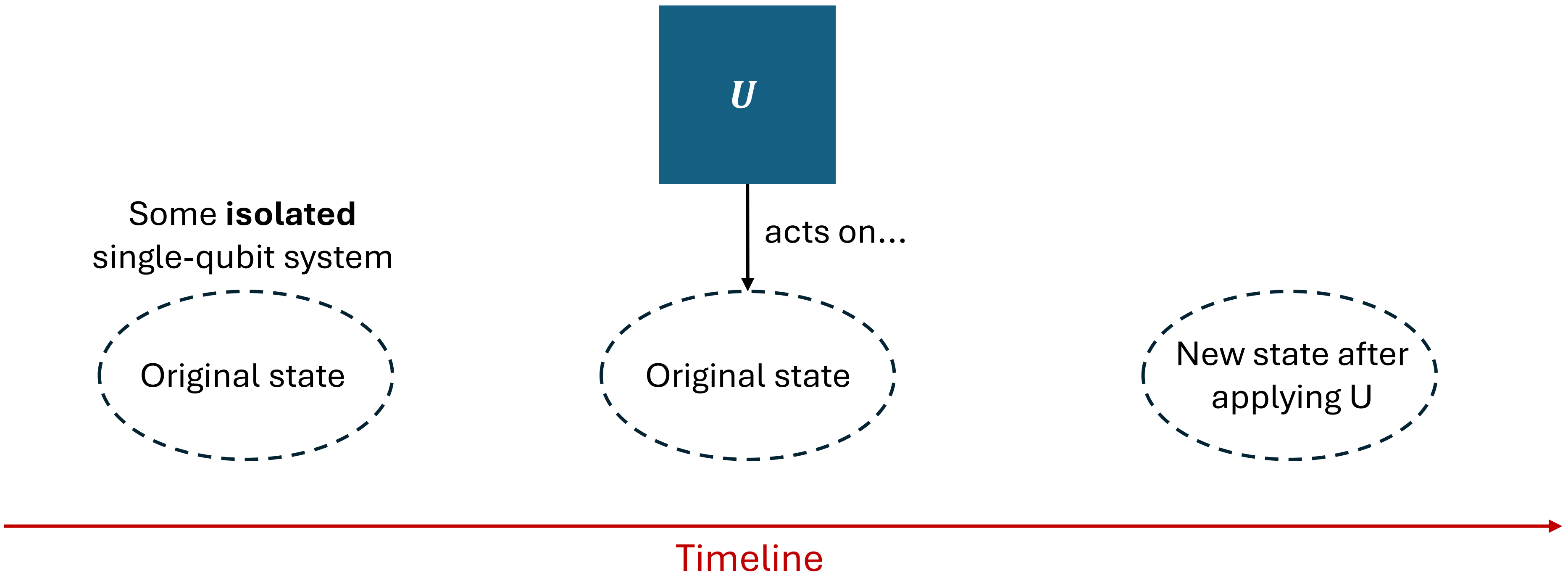
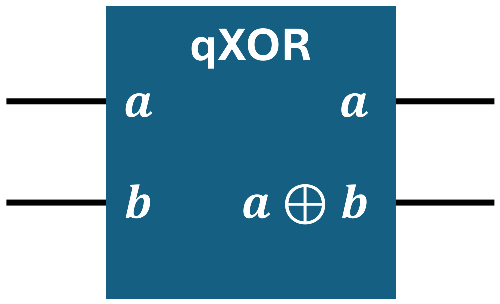
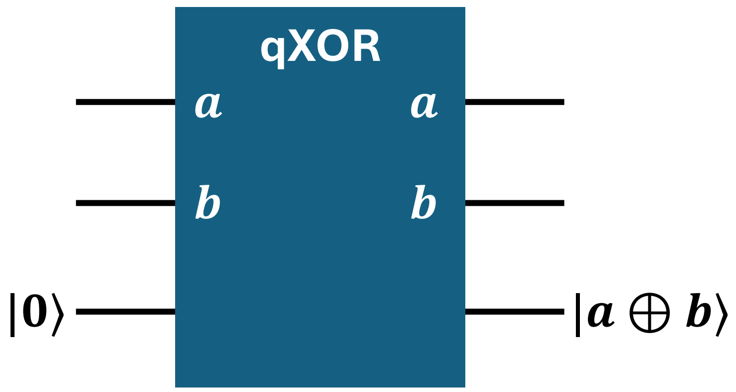
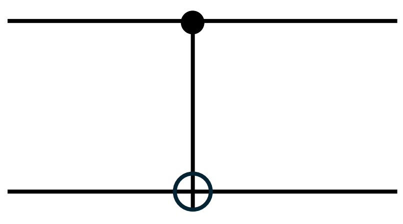


# Unitary Operations on Multi-Qubit States

- Multi-qubit unitary:
  - Examples: qXOR, CNOT, …



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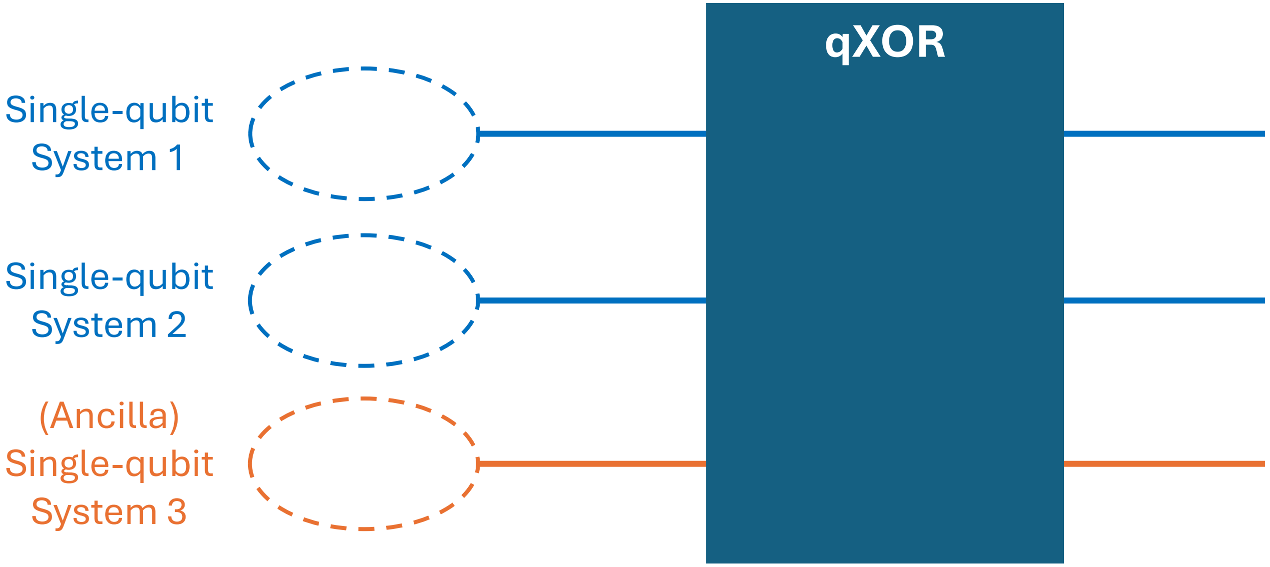

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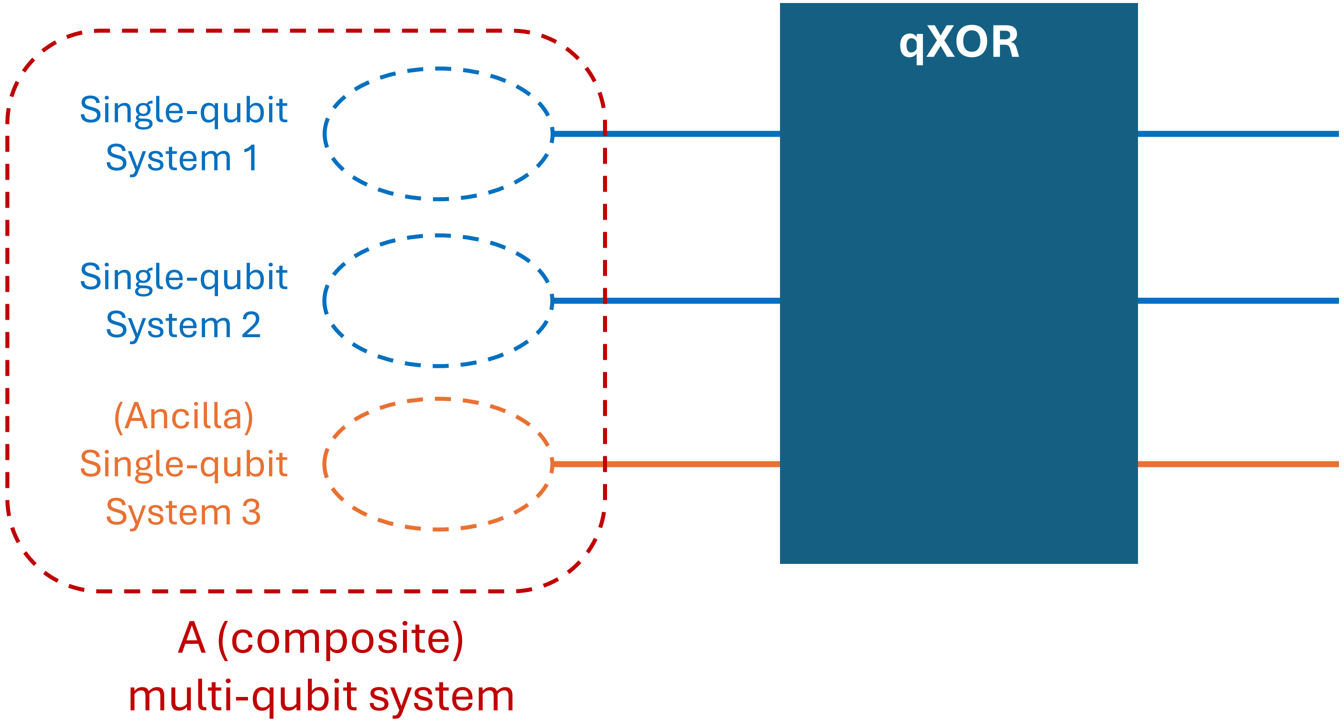

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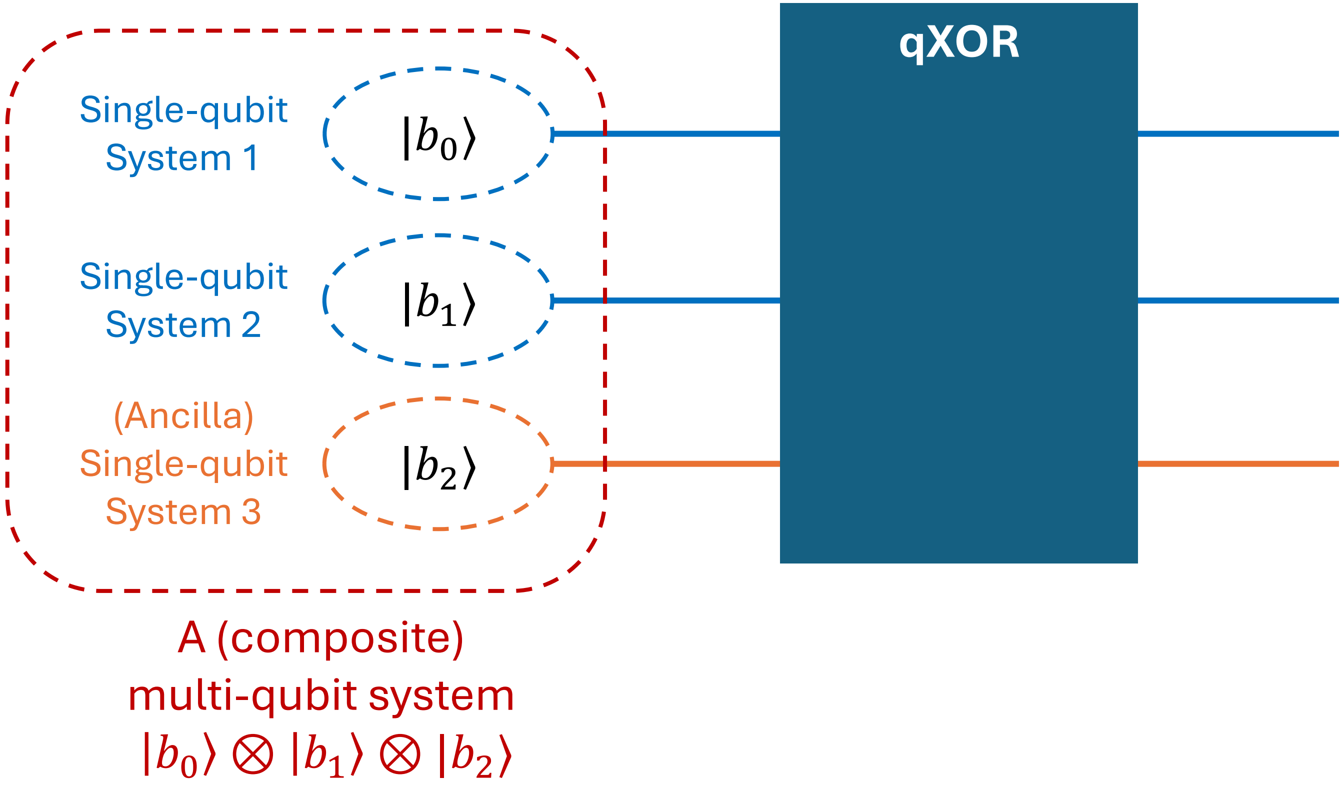
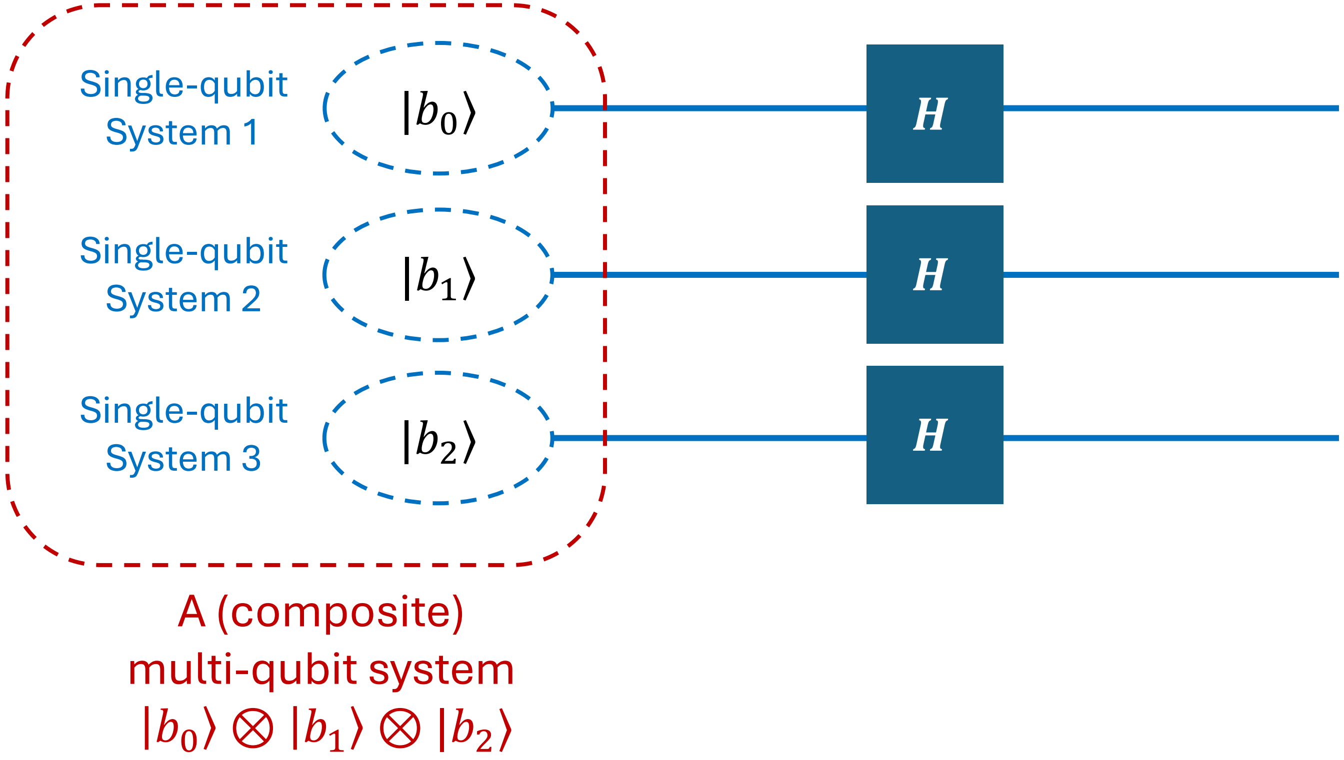


# Unitary Operations on Multi-Qubit States

- Multi-qubit unitary:
  - Examples: qXOR, CNOT, **Parallel action of Hadamard gates**…



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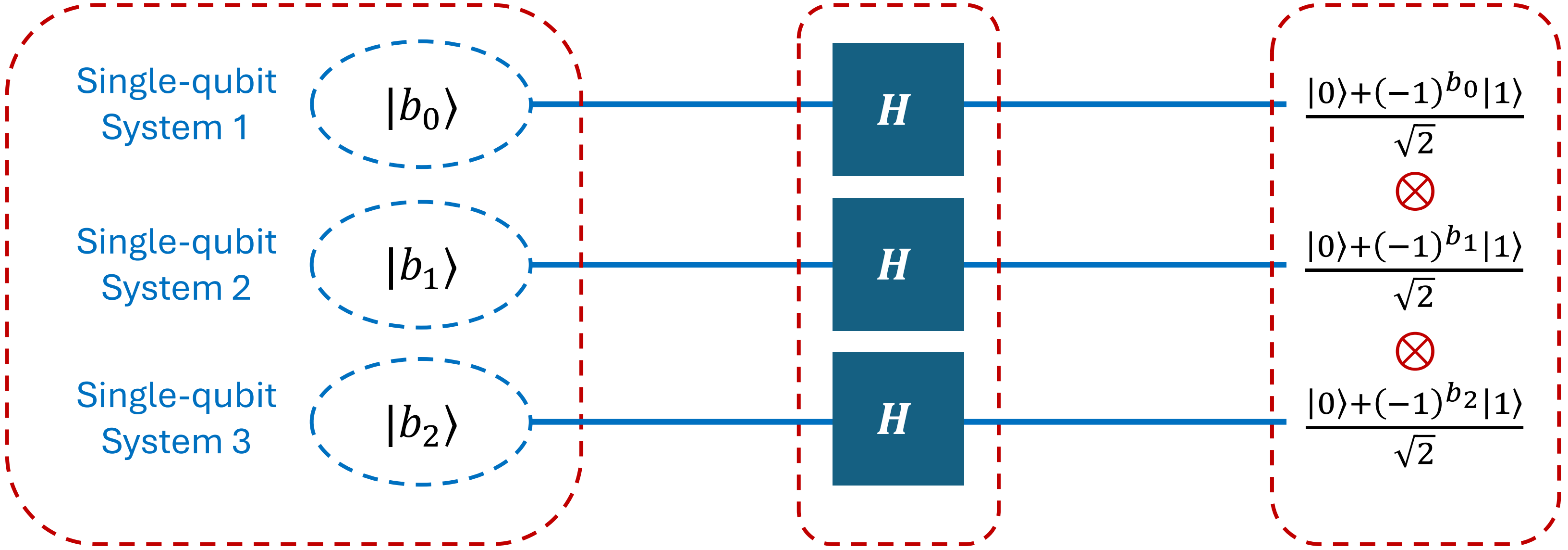



A (composite) multi-qubit system $|b_0\rangle \otimes |b_1\rangle \otimes |b_2\rangle$

$\boldsymbol{H}^{\otimes 3}$ (Parallel H acts on 3-qubit system)

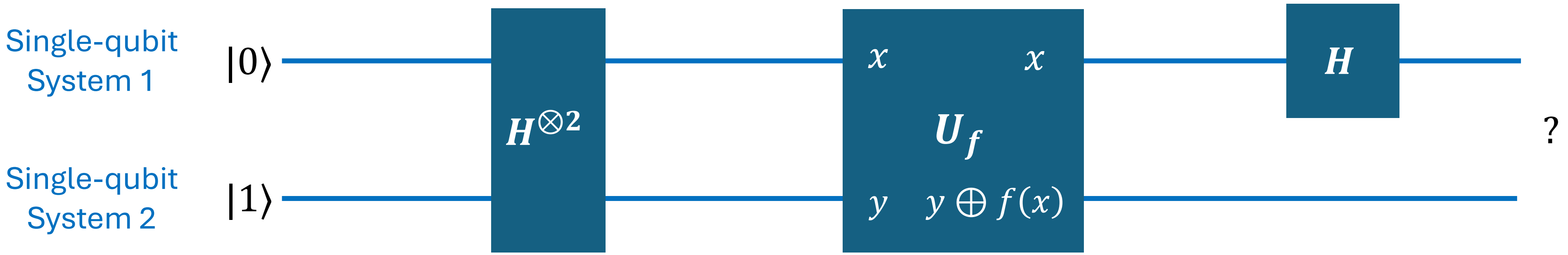
The state of the composite system after applying $\boldsymbol{H}^{\otimes 3}$

# Unitary Operations on Multi-Qubit States

- Multi-qubit unitary:
  - **Parallel action of Hadamard gates…**

# Deutsch's Algorithm

- Let $f$ be a bit function...
- (Do it on the board)



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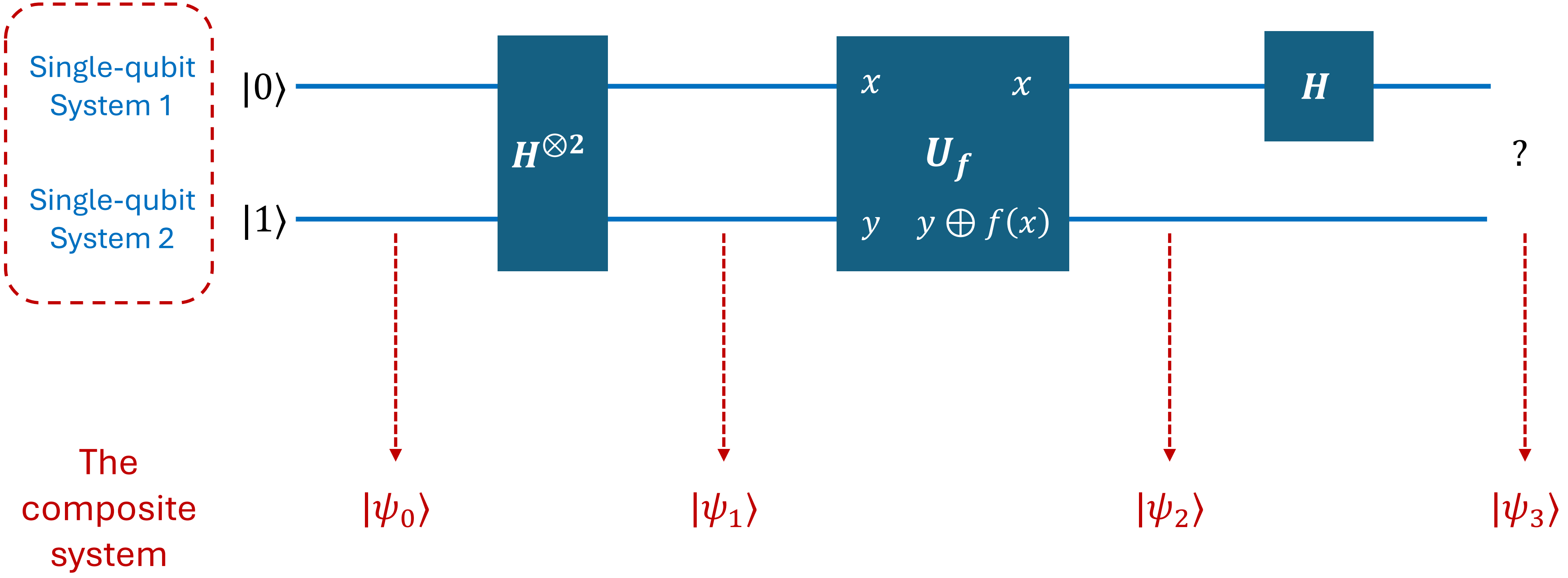

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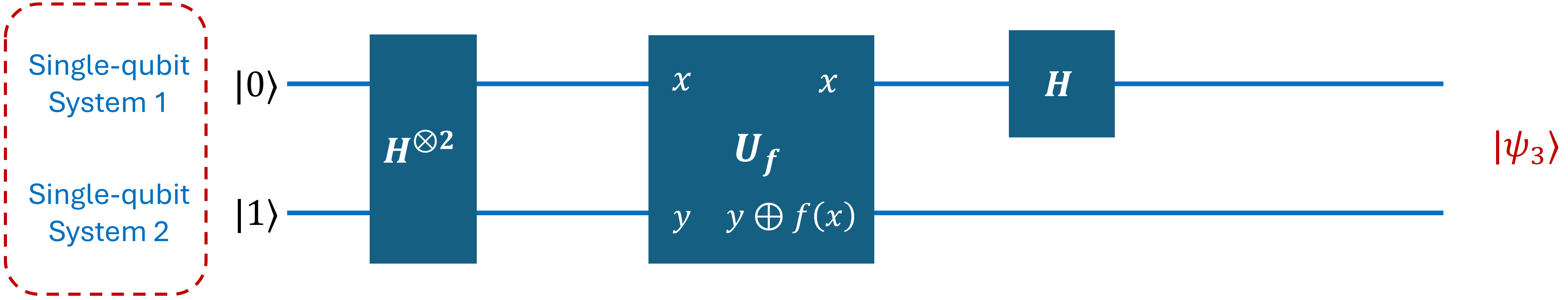


The composite system

$$|\psi_3\rangle = \pm|f(0) \oplus f(1)\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

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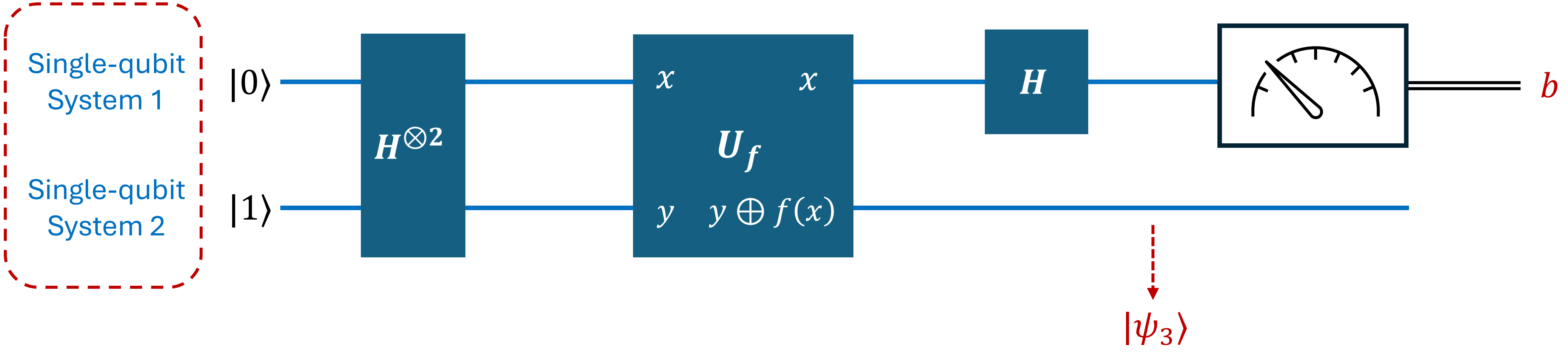


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$$|\psi_3\rangle = \pm|f(0) \oplus f(1)\rangle \left[\frac{0 - |1\rangle}{\sqrt{2}}\right] \xrightarrow{\text{Measure the first system}} b = f(0) \oplus f(1)$$

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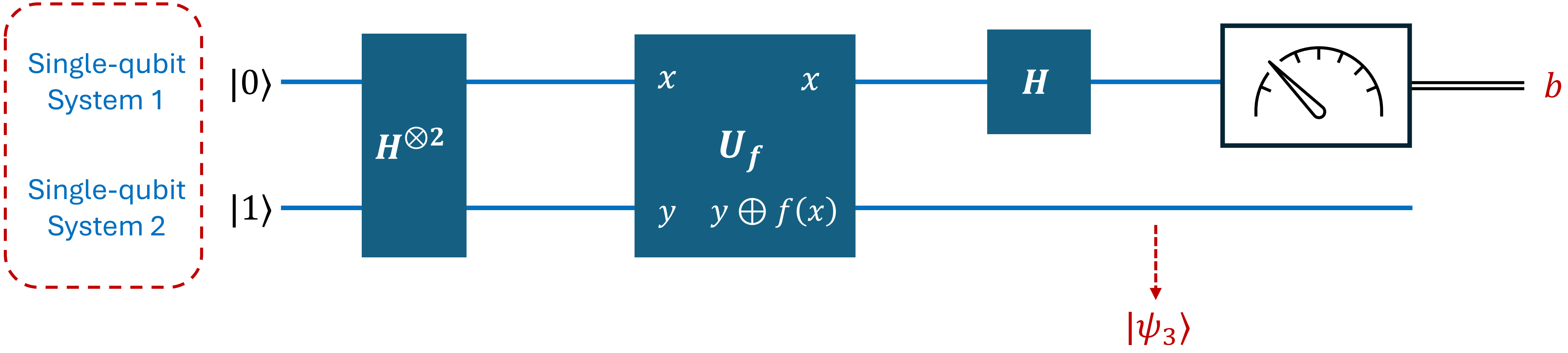
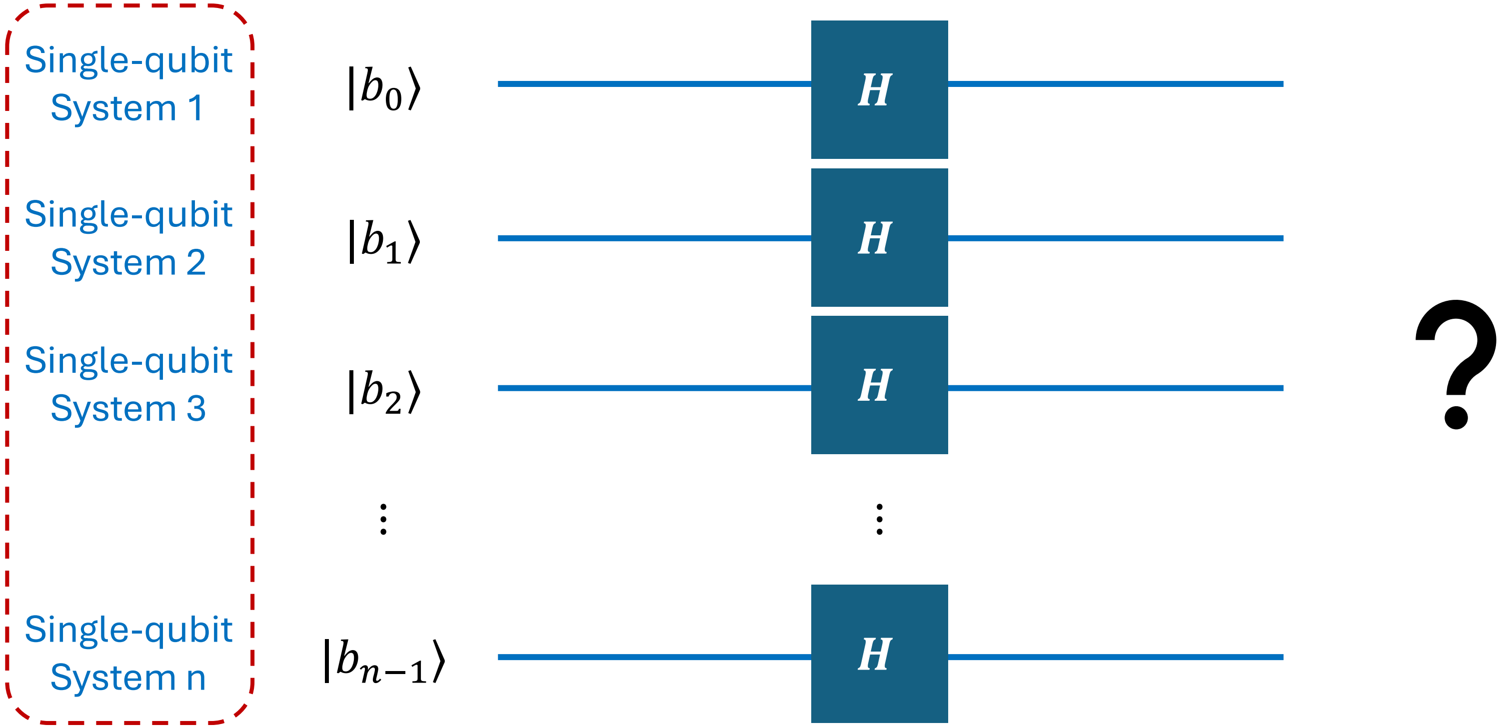



The composite system

$$|\psi_3\rangle \xRightarrow{\text{Measure the first system}} b = f(0) \oplus f(1)$$

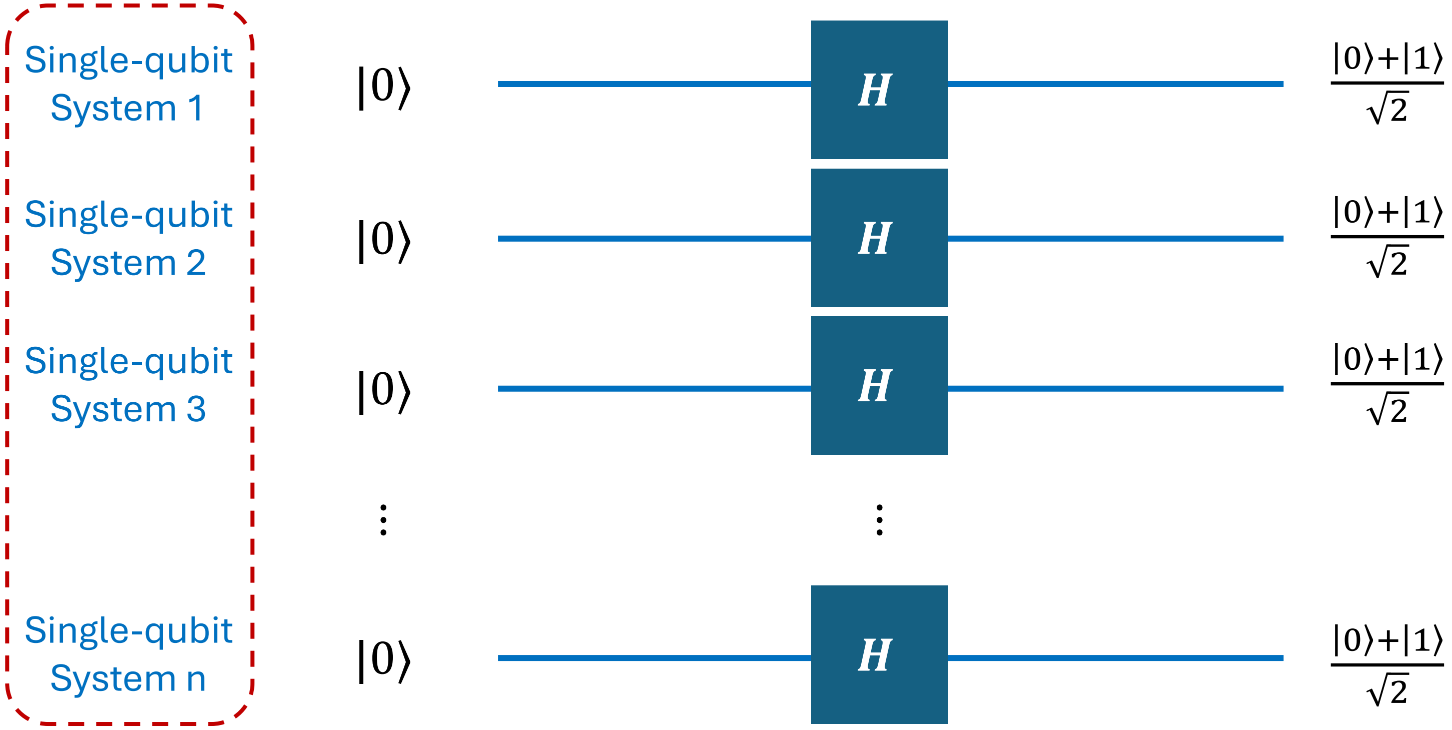
One (quantum) evaluation on $f$ and get $f(0) \oplus f(1)$

# Parallel Hadamard Gates




$|b_0\rangle \otimes |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_{n-1}\rangle$

# Parallel Hadamard Gates




$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |\boldsymbol{x}\rangle$$

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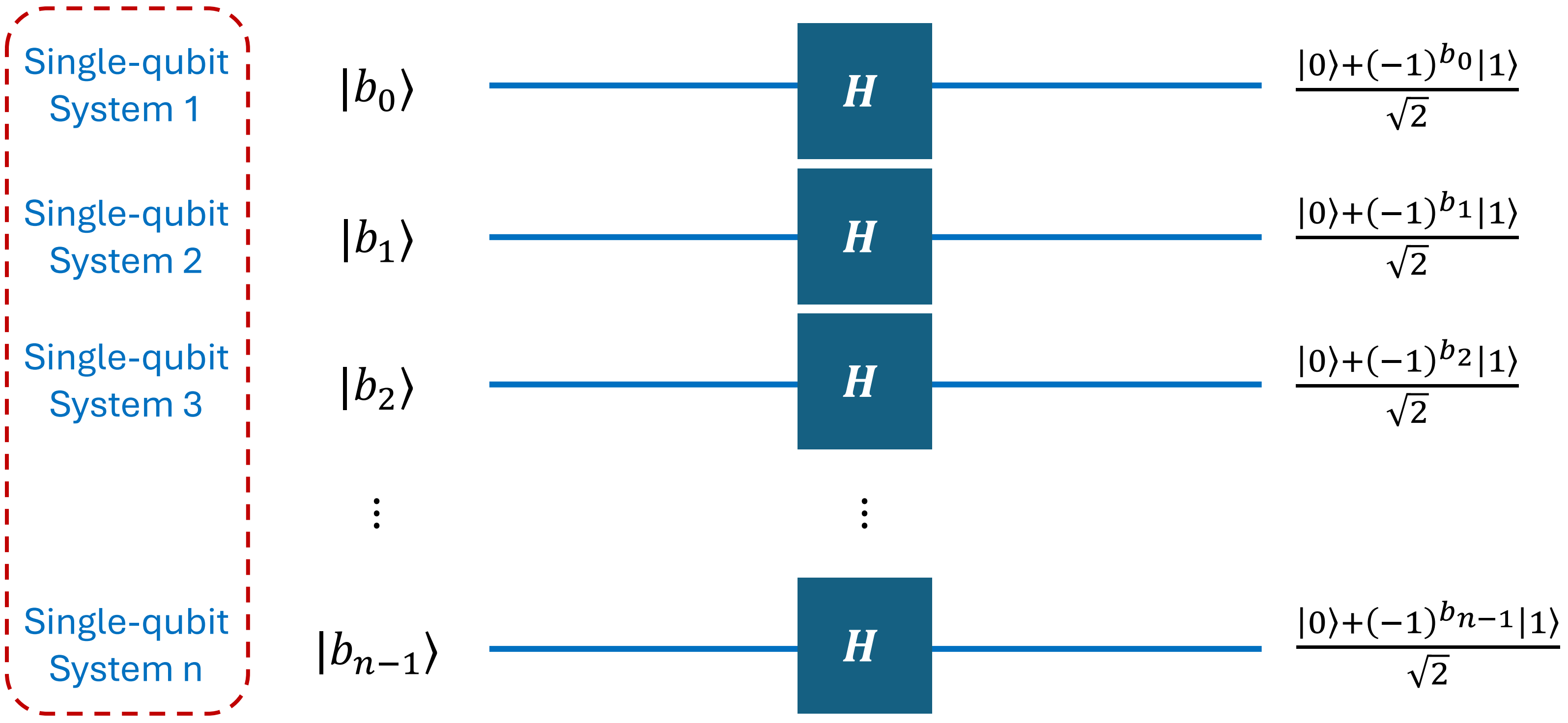


$|b_0\rangle \otimes |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_{n-1}\rangle$

Let $\boldsymbol{b} := b_{n-1}b_{n-2}\ldots b_0$
be the classical bit string

$$\boldsymbol{H}^{\otimes n}|\boldsymbol{b}\rangle = \sum_{x\in\{0,1\}^n} \frac{(-1)^{\boldsymbol{x}^T\boldsymbol{b}}}{\sqrt{2^n}}|\boldsymbol{x}\rangle$$

# The Deutsch-Jozsa Algorithm

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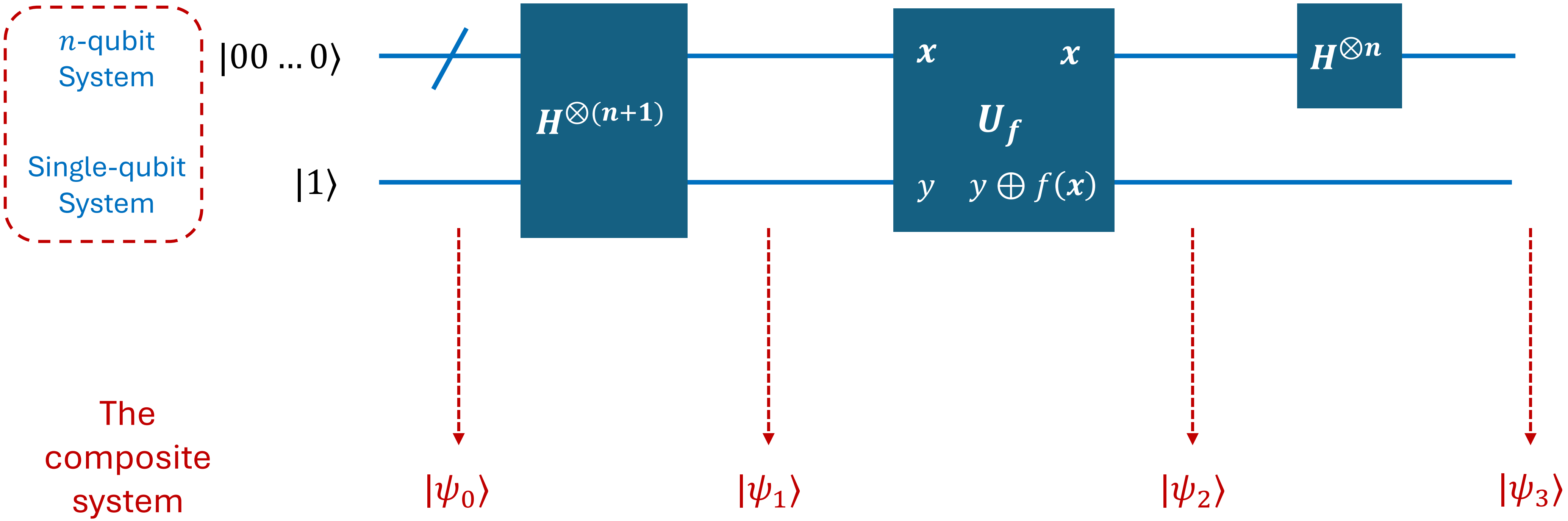

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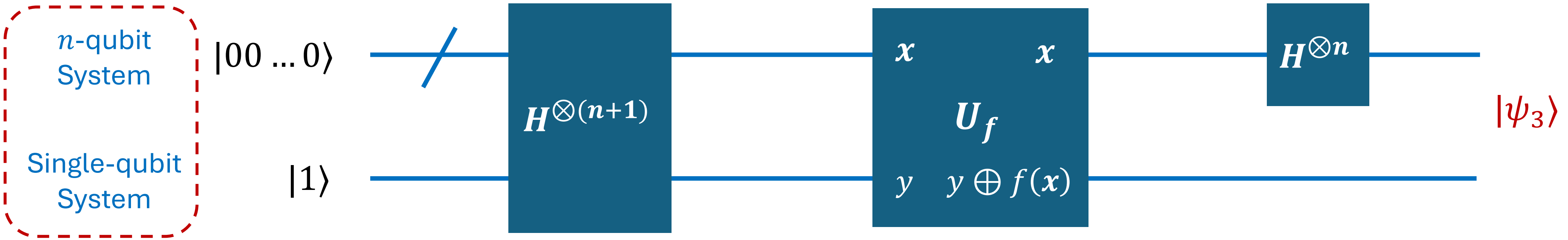


The composite system

$$|\psi_3\rangle = \sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} \frac{(-1)^{x^T z + f(x)} |z\rangle}{\sqrt{2^n}} \left[ \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right]$$

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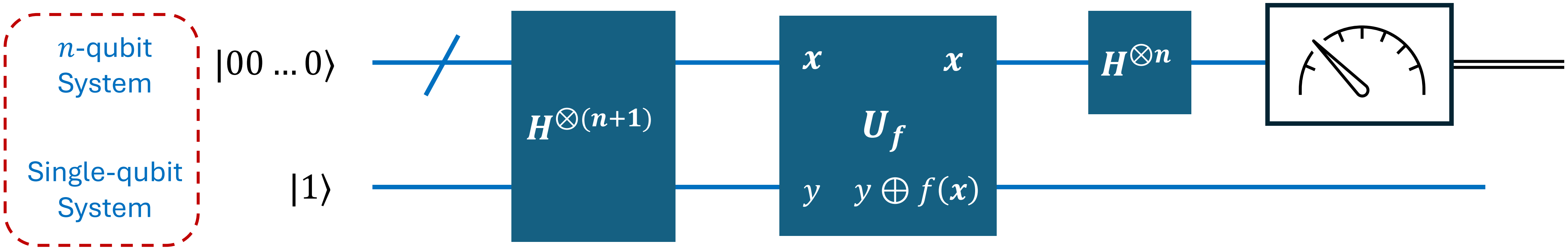


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- What's the probability of $\boldsymbol{z} = 00 \ldots 0$ ? What about $\boldsymbol{z} \neq 00 \ldots 0$ ?

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- What's the probability of $\boldsymbol{z} = 00 \ldots 0$ ? What about $\boldsymbol{z} \neq 00 \ldots 0$ ?
- What if $f$ is a *zero* function: $\forall \boldsymbol{x} \in \{0,1\}^n, f(\boldsymbol{x}) = 0$? Or $f(\boldsymbol{x}) = 1$?
- What if $f$ is a *non-zero balanced* function: $\sum_{\boldsymbol{x} \in \{0,1\}^n} f(\boldsymbol{x}) = 0$

# Deutsch-Jozsa Problem

- Constant-vs-balanced problem
- Let $f: \{0,1\}^n \to \{0,1\}$ be a bit function such that it is in either two cases:
  - $f$ is a *constant* function: $\forall \boldsymbol{x} \in \{0,1\}^n, f(\boldsymbol{x})$ is always a constant (0 or 1)
  - $f$ is a *balanced* function: $\sum_{x \in \{0,1\}^n} f(\boldsymbol{x}) = 0$ (i.e., outputs 0 for half the inputs, and 1 for the other half)

- To decide whether $f$ is constant or balanced, **how many times** must we evaluate $f$?

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**Classical Computer**

Worst-case: $2^n$

*Probabilistic algorithm:*

$l$ times,

with a failure rate of $1 - \frac{1}{2^l}$

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**Quantum Computers:**

Evaluate **once**,

with failure rate $0$

# References

- **[NC00]:** Sections 1.4.3