1. $\|x\|_\infty = \max_i |x_i|$

① $\cancel{\|x\|_2 = \sqrt{|x_1| + \cdots + |x_n|}}$

$\|x\|_1 = \sum_i |x_i| = |x_1| + \cdots + |x_n|$ $\therefore \cancel{\|x\|_\infty \leq \|x\|_1}$

$\|x\| = \|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$

$(\|x\|_\infty)^2 = \max_i |x_i|^2 < x_1^2 + \cdots + x_n^2 = (\|x\|)^2$

$\|x\|_2^2 = x_1^2 + \cdots + x_n^2 = \sum_{i=1}^n x_i^2 \leq \sum_{i=1}^n x_i^2 + \sum_{i,j} |x_i x_j| = \|x\|_1^2$

$\therefore \|x\|_\infty \leq \|x\| \leq \|x\|_1$

② $\frac{\|x\|}{\|x\|_\infty} = \frac{\sqrt{|x_1|^2 + \cdots + |x_n|^2}}{\max_i |x_i|}$

$|x_1|^2 + \cdots + |x_n|^2 \leq n \cdot \max_i |x_i|^2$

$\therefore \frac{\|x\|}{\|x\|_\infty} \leq \sqrt{n}$.

③ $\frac{\|x_1\|}{\|x\|} = \frac{|x_1| + \cdots + |x_n|}{\sqrt{|x_1|^2 + \cdots + |x_n|^2}}$

According to Cauchy-Schwarz inequity:

$|u_1 v_1 + \cdots + u_n v_n|^2 \leq (u_1^2 + \cdots + u_n^2)(v_1^2 + \cdots + v_n^2)$

we can take $u_i = |x_i|$ $v_i = 1$. into this inequity.

we can obtain: $(|x_1| + \cdots + |x_n|)^2 \leq n \cdot (x_1^2 + \cdots + x_n^2)$

$\Longleftrightarrow \frac{\|x_1\|}{\|x\|} \leq \sqrt{n}$

④. $|x_1| = |x_2| = \cdots = |x_{n-1}| = |x_n|$.

2. ① $\|x+y\|_\infty = \max_i |x_i + y_i| \leq \max_i |x_i| + |y_i| \leq \max_i |x|$

$\leq \max_i |x_i| + \max_j |y_j| = \|x\|_\infty + \|y\|_\infty$

② $\|x+y\|_1 = \sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n |x_i| + |y_i| = \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i|$

$= \|x\|_1 + \|y\|_1$

3. $A = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$. $\lambda = 1 . 0.6$.

$\because u_k = A^k u_0$ converges to $\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$

so $\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$ is the eigenvector of the eigenvalue $\lambda_1 = 1$.

for $\lambda_2 = 0.6$. the eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\because A \cancel{X} = \lambda X$.

$\therefore A^{-1} \cdot A X = A^{-1} \lambda X \iff A^{-1} X = \frac{1}{\lambda} \cdot X$

so the eigenvectors of A^{-1} are $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$

$\underline{u}_K = A^{-k} u_0$ after multiplying by 0.6^k.

will converge to a multiple of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

because $\frac{1}{0.6} \times 0.6 = 1$ so it will converge to the eigenvector of $\lambda_2 = \frac{1}{0.6}$ $\left(\left(\frac{1}{1} \times 0.6 \right)^k = 0.6^k \to 0. \right)$

4. $A q_j = b_{j-1} q_{j-1} + a_j q_j + b_j q_{j+1}$

$q_j^T A q_j = a_j$. ($\because$ q's are orthonormal)

So we can obtain: $A q_1 = b_0 q_0 + a_1 q_1 + b_1 q_2 = a_1 q_1 + b_1 q_2$

$A q_2 = b_1 q_1 + a_2 q_2 + b_2 q_3$

$A q_3 = b_2 q_2 + a_3 q_3 + b_3 q_4$

$$AQ = [Aq_1 \ Aq_2 \cdots Aq_n] = (q_1 \cdots q_n) \begin{pmatrix} a_1 & b_1 & 0 & \\ b_1 & a_2 & b_2 & \\ 0 & b_2 & a_3 & \ddots \\ & & b_3 & \ddots \end{pmatrix}$$

$= QT$ where T is tridiagonal