1. (a) False.

$A=\begin{pmatrix}1&2\\1&0\end{pmatrix}$ $\det(A-\lambda I)=\det\begin{vmatrix}1-\lambda&2\\1&-\lambda\end{vmatrix}=\lambda^2-\lambda-2=(\lambda+1)(\lambda-2)$

$\lambda_1=-1.\ \lambda_2=2.$

$(A-\lambda_1 I)=\begin{pmatrix}1&2\\1&0\end{pmatrix}-\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=\begin{pmatrix}2&2\\1&1\end{pmatrix}\begin{pmatrix}x_{11}\\x_{12}\end{pmatrix}=0$ $x_1=(-1,1)^T$

$A-\lambda_2 I=\begin{pmatrix}1&2\\1&0\end{pmatrix}-\begin{pmatrix}2&0\\0&2\end{pmatrix}=\begin{pmatrix}-1&2\\1&-2\end{pmatrix}\begin{pmatrix}x_{21}\\x_{22}\end{pmatrix}=0$ $x_2=(2,1)^T$

$\therefore$ A has real eigenvalues & vectors. but A isn't Symmetric.

(b) True.

$A=S\Lambda S^{-1}$. S is orthonormal. & $S^T=S^{-1}$

$A^T=(S^T)^T\Lambda^T S^T=S\Lambda S^T=A$.

(c) if A is symmetric. A can be decomposed by an orthonormal matrix

$A=Q\Lambda Q^T$ $A^{-1}=Q\Lambda^{-1}Q^{-1}=Q\Lambda^{-1}Q^T$.

$\therefore$ A^{-1} ~~can be~~ is a matrix with real eigenvalues & orthonormal eigenvectors

$\therefore$ from (b). A^{-1} is symmetric.

(d) False.

$A=\begin{pmatrix}1&2\\2&4\end{pmatrix}$. $\lambda_1=0$ $x_1=(2,-1)^T$ $\lambda_2=5$ $x_2=(1,2)^T$

$\therefore S=\begin{pmatrix}2&1\\-1&2\end{pmatrix}$ is not symmetric.

2. For positive definite matrix, $\forall x$. x^TAx must be positive

(1) Assume that $A_{ii}=0$ or <0

let $x=(0\cdots\underset{i}{1}\cdots0)$ (only $x_{ii}=1$)

$x^TAx=A_{ii}<=0$ Contradicted $\therefore$ $a_{ii}>0$. for all i

(2) $(x_1\ x_2\ x_3)\begin{pmatrix}4 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = 4x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3 + 5x_3^2$

$\therefore$ if $x_1=0$, $x_3=0$. the formula $=0$.

$\therefore (x_1, x_2, x_3) = (0, t, 0)$ $t \in R$.

3 (a) $\frac{X^TSX}{X^TX} = \frac{3x_1^2 + 2x_1x_2 + 3x_2^2}{x_1^2+x_2^2} = 3 + \frac{2x_1x_2}{x_1^2+x_2^2} \overset{t=\frac{x_1}{x_2}}{=} 3 + \frac{2t}{t^2+1} = 3 + \frac{2}{t+\frac{1}{t}}$

$(x_1, x_2 \neq 0)$

the max is $3+1=4$ (when $x_1 = x_2 \neq 0$).

if x_1 or $x_2 = 0$. $\frac{X^TSX}{X^TX} = 3 < 4$

$\therefore$ the max is 4

$(x_1\ x_2)S\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 3x_1^2 + 2x_1x_2 + 3x_2^2$ $S = \begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$

(b) $\frac{(x_1+4x_2)^2}{x_1^2+x_2^2} = \frac{x_1^2 + 8x_1x_2 + 16x_2^2}{x_1^2+x_2^2} \overset{\frac{x_2}{x_1}=t}{=} \frac{1+8t+16t^2}{1+t^2} = f(t)$

if $x_1 \neq 0$, $x_2 \neq 0$: $\frac{\partial f(t)}{\partial t} = 0 \Leftrightarrow (8+32t)(t^2+1) - 2t(1+8t+16t^2)$

$= 8t^2 + 8 + \cancel{32t^3} + 32t - 2t - 16t^2 - \cancel{32t^3}$

$= -8t^2 + 30t + 8 = 0$.

$t_1 = -\frac{1}{4}$ $t_2 = 4$.

$f(t)_{max} = f(4) = 17$

if $x_1=0$. $\frac{(x_1+4x_2)^2}{x_1^2+x_2^2} = 16 < 17$ if $x_2=0$. $\frac{(x_1+4x_2)^2}{x_1^2+x_2^2} = 1 < 17$

the max is 17

$Ax = (x_1 + 4x_2)$. $A = (1, 4)$

4. $D = \begin{pmatrix} d_{11} & & \\ & d_{22} & \\ & & d_{33} \end{pmatrix}$.

$$DSD = \begin{pmatrix} d_{11}s_1^2 & d_{11}s_{12} & d_{11}s_{13} \\ d_{22}s_{21} & d_{22}s_2^2 & d_{22}s_{23} \\ d_{33}s_{31} & d_{33}s_{32} & d_{33}s_3^2 \end{pmatrix} \begin{pmatrix} d_{11} & & \\ & d_{22} & \\ & & d_{33} \end{pmatrix}.$$

$$= \begin{pmatrix} d_{11}^2s_1^2 & d_{11}d_{22}s_{12} & d_{11}d_{33}s_{13} \\ d_{11}d_{22}s_{21} & d_{22}^2s_2^2 & d_{22}d_{33}s_{23} \\ d_{11}d_{33}s_{31} & d_{22}d_{33}s_{32} & d_{33}^2s_3^2 \end{pmatrix} = \begin{pmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{pmatrix}$$

$\therefore d_{11} = \pm\frac{1}{s_1}$ $\quad d_{22} = \pm\frac{1}{s_2}$ $\quad d_{33} = \pm\frac{1}{s_3}$

$c_{12} = \frac{s_{12}}{s_1 s_2}$. $\quad c_{13} = \frac{s_{13}}{s_1 s_3}$ $\quad c_{23} = \frac{s_{23}}{s_2 s_3}$