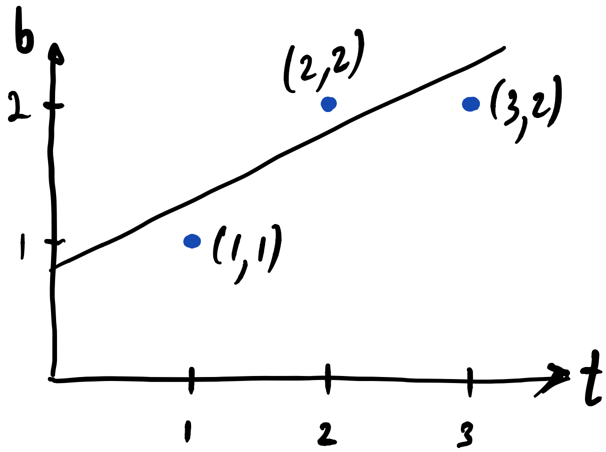
Lectures

Lecture 1 (Tuesday, September 5): Introduction, solving linear equations.

Lecture 2 (Tuesday, September 12): Vector spaces and orthogonality.

Lecture 3 (Tuesday, September 19): Least squares approximation, Gram-Schmidt, determinants, eigenvectors and eigenvalues.

Least squares approximation



$$c + dt = b$$

$$\left.\begin{aligned} c + d\cdot 1 &= 1 \\ c + d\cdot 2 &= 2 \\ c + d\cdot 3 &= 2 \end{aligned}\right\} \Rightarrow \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} c \\ d \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

$Ax = b$

has no solution $\Rightarrow$ multiply by A^T and solve $A^TA\hat{x} = A^Tb$

Project b onto A to find closest solution

$$P = A(A^TA)^{-1}A^T$$

$$p = Pb = A(A^TA)^{-1}A^Tb$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^TAx = A^Tb \qquad x = (A^TA)^{-1}A^Tb$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} \Rightarrow \begin{matrix} 3c + 6d = 5 \\ 6c + 14d = 11 \end{matrix} \Rightarrow \begin{matrix} c = \frac{2}{3} \\ d = \frac{1}{2} \end{matrix}$$

equivalent to

minimize $\|Ax - b\|^2 = \|Ax - p\|^2 + \|e\|^2 \overset{x=\hat{x}}{=} \|e\|^2$

minimize residual sum of squares (RSS)

$$f(c,d) = (c+d-1)^2 + (c+2d-2)^2 + (c+3d-2)^2$$

$$\frac{\partial f}{\partial c} = 2(c+d-1) + 2(c+2d-2) + 2(c+3d-2) = 0$$

$$3c + 6d = 5$$

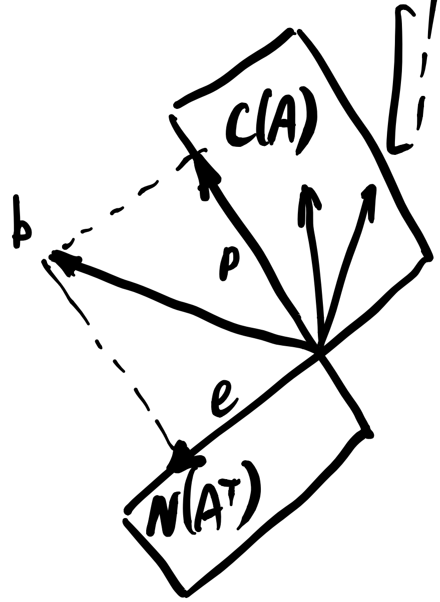
$$\frac{\partial f}{\partial d} = 2(c+d-1) + 2\cdot 2(c+2d-2) + 2\cdot 3(c+3d-2) = 0$$

$$6c + 14d = 11$$

Partial derivatives of $\|Ax-b\|^2$ are 0 when $A^TA\hat{x} = A^Tb$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad x = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{2} \end{bmatrix} \quad Ax = \begin{bmatrix} \frac{7}{6} \\ \frac{10}{6} \\ \frac{13}{6} \end{bmatrix} = p$$

$$e = b - p = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - p = \begin{bmatrix} -\frac{1}{6} \\ \frac{2}{6} \\ -\frac{1}{6} \end{bmatrix}$$



$$A=\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}$$

verify $e \perp p$, $e \perp$ columns of A

Orthonormal bases and Gram-Schmidt

Vectors $q_1, \dots q_n$ are orthonormal if

$$q_i^T q_j = \begin{cases} 0 \text{ when } i \neq j & \text{orthogonal vectors} \\ 1 \text{ when } i = j & \text{unit vectors } \|q_i\| = 1 \end{cases}$$

For Q with orthonormal columns $Q^TQ = I$

$$Q^TQ = \begin{bmatrix} — q_1^T — \\ \vdots \\ — q_n^T — \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \dots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I$$

when Q is square $Q^T = Q^{-1}$

Examples:

Rotation: $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$Q^T = Q^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Permutation: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

If Q has orthonormal columns $Q^TQ = I$

then $\|Qx\| = (Qx)^T(Qx) = x^TQ^TQx = x^Tx = \|x\|$

Lenghts are unchanged (rotation, reflection..)

Q has orthonormal columns

Projection onto $C(Q)$

$$P = Q(Q^TQ)^{-1}Q^T = QQ^T = I$$

for square Q

Multiple projections $(Q\underbrace{Q^T)(Q}_{I}Q^T) = I$

Normal equations $A^TA\hat{x} = A^Tb$

for Q $\quad Q^TQ\hat{x} = Q^Tb$

$$\hat{x} = Q^Tb \qquad \hat{x}_i = q_i^Tb$$

the projection of b onto the whole space is b

$$p = Q\hat{x} = QQ^Tb = b$$

for square Q

$$p = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} q_1^Tb \\ \vdots \\ q_n^Tb \end{bmatrix} = q_1(q_1^Tb) + \cdots + q_n(q_n^Tb) = b$$

Gram-Schmidt

Create orthonormal vectors

Given independent vectors a, b, c create

orthogonal vectors A, B, C divide by length

to get orthanormal vectors $q_1 = \frac{A}{\|A\|}$ $q_2 = \frac{B}{\|B\|}$ $q_3 = \frac{C}{\|C\|}$

first vector $A = a$ $\qquad q_1 = \frac{A}{\|A\|}$

second vector b, subtract its projection onto A

$$B = b - \frac{A^T b}{A^T A} A \qquad q_2 = \frac{B}{\|B\|}$$

$$A \perp B \qquad A^T B = A^T b - A^T b = 0$$

third vector c, subtract its projection onto A and onto B

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B \qquad \begin{matrix} C \perp A \\ C \perp B \end{matrix} \qquad q_3 = \frac{C}{\|C\|}$$

subtract from every new vector its projections in the directions already set

Finally, divide vectors by their lengths

Example:

$$a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad c = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

$$A = a$$

$$B = b - \frac{A^T b}{A^T A} A = b - \frac{2}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$A^T B = 0$$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B = c - \frac{6}{2} A - \frac{6}{6} B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_1 = \frac{A}{\|A\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$q_3 = \frac{C}{\|C\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} \qquad A = QR$$

$$Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$$

$R = Q^T A$ is upper triangular

later q's are orthogonal to earlier a's

$$\underset{A}{\begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix}} = \underset{\substack{Q \\ \text{orthogonal}}}{\begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}} \underset{\substack{R \\ \text{triangular}}}{\begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}}$$

Useful for least squares

$$A^T A = (QR)^T QR = R^T Q^T QR = R^T R$$

least squares equation $A^T A \hat{x} = A^T b$

simplifies to $R^T R \hat{x} = R^T Q^T b$ which is

$$R\hat{x} = Q^T b$$

$$\hat{x} = R^{-1} Q^T b$$

```python
A = [[1,2,3],[-1,0,-3],[0,-2,3]]
```

```python
Q, R = np.linalg.qr(A)
```

```python
print("Q = ", Q)
print("R = ", R)
```

```
Q =  [[-0.70710678 -0.40824829  0.57735027]
 [ 0.70710678 -0.40824829  0.57735027]
 [-0.          0.81649658  0.57735027]]
R =  [[-1.41421356 -1.41421356 -4.24264069]
 [ 0.         -2.44948974  2.44948974]
 [ 0.          0.          1.73205081]]
```

```python
b = [3,4,5]
```

```python
x = np.linalg.solve(A,b)
```

```python
print("x =", x)
```

```
x = [-16.    3.5   4. ]
```

```python
x = np.dot(np.dot(np.linalg.inv(R), Q.T), b)
```

```python
print("x = ", x)
```

```
x =  [-16.    3.5   4. ]
```

Determinants

$\det(A) = |A|$ of square matrix A is a number s.t.

1. $\det I = 1$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} = 1$$

2. Determinant changes sign when two rows exchanged

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

$\det P = 1$ for even number of row exchanges

$= -1$ for odd number of row exchanges

3. Determinant is a linear function of each row

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2$$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

```python
I = np.identity(3)
```

```python
detI = np.linalg.det(I)
```

```python
print("detI = ", detI)
```

```
detI =  1.0
```

4. If two rows are equal then $\det A = 0$

Exchange rows to get the same matrix

$\det A = -\det A$ so $\det A = 0$

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = -\begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

5. Subtracting a multiple of one row from another leaves $\det A$ unchanged.

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \underset{(3)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{\overset{\shortparallel}{0}\ (4)} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

determinant is not changed by elimination (except exchanges)

6. Matrix with row of zeros has $\det A = 0$

for $t = 0$

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

7. If U is triangular then $\det U = d_{11} \cdot d_{22} \cdots d_{nn}$

a product of diagonal entries

$$\det U = \begin{vmatrix} d_{11} & & * \\ & d_{22} & \\ 0 & & \ddots \\ & & & d_{nn} \end{vmatrix} \underset{(5)}{=} \begin{vmatrix} d_{11} & & 0 \\ & d_{22} & \\ 0 & & \ddots \\ & & & d_{nn} \end{vmatrix} =$$

$$\underset{(3)}{=} d_{11} d_{22} \cdots d_{nn} \underbrace{\begin{vmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{vmatrix}}_{I} =$$

$$\underset{(1)}{=} d_{11} \cdot d_{22} \cdots d_{nn} \quad \text{product of pivots}$$

8. If A is singular then $\det A = 0$

I A is invertible then $\det A \neq 0$

$$A \to U \to D$$

if A singular then U has zero row $\det A = \det U = 0$

(6)

if A invertible then U has pivots along diagonal

(7)

$\det A = \pm \det U = \pm$ product of pivots

and the product of non-zero pivots gives $\det A \neq 0$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix} = ad - bc$$

9. $\det(AB) = \det A \cdot \det B$

$\det(AA^{-1}) = \det A \cdot \det A^{-1} = 1$

(1) $\det I = 1$

$\det A^{-1} = \dfrac{1}{\det(A)}$

$\det A^2 = (\det A)^2$

$$\det 2A = \begin{vmatrix} 2 \text{ row } 1 \\ \\ 2 \text{ row } n \end{vmatrix} = 2^n \det A$$

(3)

show that $\det(AB)$ has properties 1, 2, 3

10. $\det A^T = \det A$

if A is singular $\det A = 0$

otherwise $PA = LU$

transpose both sides $A^T P^T = U^T L^T$

$$|P||A| = |L||U|$$

$$|P^T||A^T| = |L^T||U^T|$$

$$\left.\begin{array}{ll} \text{same diagonal so} & |L| = |L^T| \\ & |U| = |U^T| \\ P^T P = I \text{ so} & |P| = |P^T| \end{array}\right\} \Rightarrow |A| = |A^T|$$

Every rule for rows applies to columns

Permutations and cofactors

Pivot formula

$\det P \cdot \det A = \det L \cdot \det U$ gives $\det A = \pm(d_1, \ldots, d_n)$

k^{th} pivot $\quad d_k = \dfrac{d_1 d_2 \cdots d_k}{d_1 d_2 \cdots d_{k-1}} = \dfrac{\det A_k}{\det A_{k-1}}$

Big formula

2×2 case:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \underset{(3)}{=} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} =$$

$$\underset{(3)}{=} \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} =$$

$$\underset{(6,10,\text{ pivot formula})}{=} 0 + ad - bc + 0$$

3×3 case:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{31} & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{23} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} =$$

all other terms have zero row or column hence 0

$$= a_{11}a_{22}a_{33} - \underbrace{a_{11}a_{23}a_{31}}_{\text{row exchange}} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{23}a_{31}$$

n×n case:

$\det A$ = sum over all $n!$ column permutations $P = (\alpha, \beta, \ldots \omega)$

Big formula $\quad \sum \underbrace{(\det P)}_{\pm 1} a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$

Determinant by cofactors

3×3 case

$$a_{11}(a_{22}a_{33}-a_{23}a_{31}) + a_{12}(-a_{21}a_{33}+a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{23}a_{31})$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

Cofactor of a_{ij} is $C_{ij} = (-1)^{i+j} \det M_{ij}$

where M_{ij} is the submatrix without row i and column j

$+$ if $i+j$ is even, $-$ if $i+j$ is odd

Cofactor formula

The determinant is the dot product of any row i with its cofactors using other rows:

3×3
$$\det A = a_{11}C_{11} + a_{12}C_{22} + a_{13}C_{33}$$

n×n
$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

Example:

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 2\underbrace{\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}}_{C_{11}} - (-1)\underbrace{\begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}}_{C_{12}}$$

$a_{13} = 0 \quad a_{14} = 0$

Cramer's rule

Solve $Ax = b$ using determinants

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{21} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1$$

A with b as col 1

$$\det A \cdot x_1 = \det B_1 \qquad x_1 = \frac{\det B_1}{\det A}$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} = B_1$$

A with b as col 1

$$\det A \cdot x_2 = \det B_2 \qquad x_2 = \frac{\det B_2}{\det A}$$

$$\det A \cdot x_j = \det B_j \qquad x_j = \frac{\det B_j}{\det A}$$

B_j has j^{th} column of A replaced by b

Solving $n \times n$ system requires evaluating $n+1$ determinants
very inefficient

Inverse

$$A^{-1} = \frac{1}{\det A} C^T \qquad (A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$

multiply both sides by A

$$AA^{-1} = \frac{1}{\det A} AC^T$$

$$\det A \cdot I = AC^T$$

for 3x3

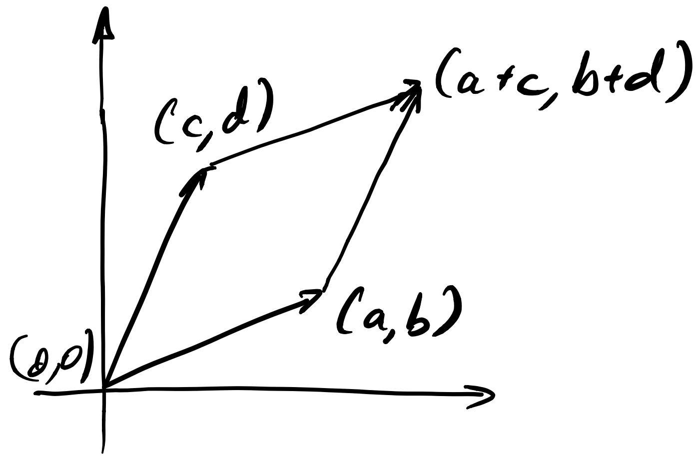
$$\begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \det A \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

row 1 of A · col 1 of $C^T = \det A$

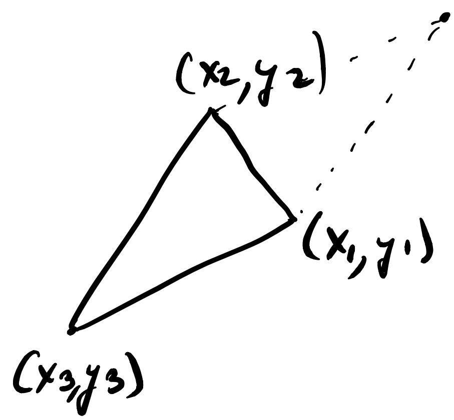
$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \det A$$

Volume

2×2 case



$$\det\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{area of parallelogram}$$



$$\frac{1}{2}\det\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \text{area of triangle}$$

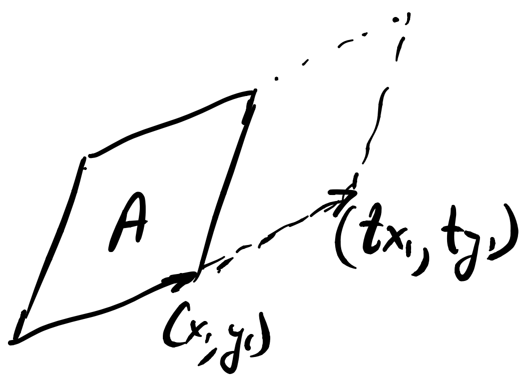
for $(x_3,y_3)=(0,0)$ area $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

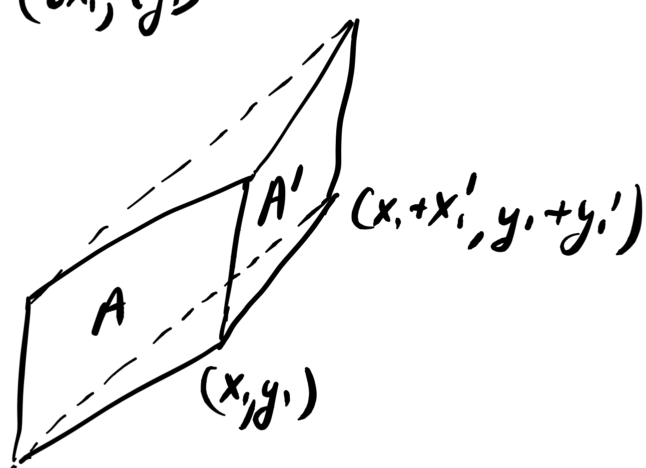
1. When $A = I$ parallelogram = unit square

$\det A = 1$ area

2. Exchanging rows, determinants reverse signs
area stay the same

3. row multiplied by t, area multiplied by t





3x3 case:

$\det A$ = volume of box

Eigenvalues and eigenvectors

Eigenvectors Ax parallel to x

$$A x = \underbrace{\lambda}_{\text{eigenvalue}} \underbrace{x}_{\text{eigenvector}}$$

two unkowns

Examples:

projection matrices

Any x in plane $Px = x$ $\lambda_1 = 1$

Any $x \perp$ plane $Px = 0$ $\lambda_2 = 0$

permutation matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad Ax = x \quad \lambda_1 = 1$$

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad Ax = -x \quad \lambda_2 = -1$$

$\text{trace}(A) = a_{11} + \dots + a_{nn} = \text{sum of eigenvalues } \lambda$

Solve $Ax = \lambda x$

$$(A - \lambda I)x = 0$$

λ is eigenvalue of A iff $A - \lambda I$ is singular

so $\det(A - \lambda I) = 0$ characteristic polynomial

Find λ's, then x's by elimination

Example:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0 \quad (\lambda - 4)(\lambda - 2) = 0$$

$$\lambda_1 = 4 \quad \lambda_2 = 2$$

$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Find λ's

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - 4 = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = 5$$

Find x's

$$(A - \lambda_1 I)x_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \end{bmatrix} = 0 \qquad x_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$(A - \lambda_2 I)x_2 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_2 \end{bmatrix} = 0 \qquad x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

x_1, x_2 are in the nullspaces: $(A - \lambda I)x = 0$

is $Ax = \lambda x$

Determinant and trace

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \qquad \lambda_1 = 0 \quad \lambda_2 = 7$$

$$\det = \lambda_1 \cdot \lambda_2 = 1$$

$$\text{trace} = \lambda_1 + \lambda_2 = 0$$

product of n eigenvalues = determinant

sum of n eigenvalues = trace

$$\lambda_1 + \cdots + \lambda_n = \text{trace} = a_{11} + a_{22} + \cdots + a_{nn}$$

Example:

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{orthogonal } 90^\circ \text{ rotation}$$

No real eigenvectors, since no vector is parallel to itself after rotation

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_1 = i \qquad \lambda_2 = -i$$

```python
import numpy as np
```

```python
D = np.diag((1,2,3))
```

```python
eigenvalues, eigenvectors = np.linalg.eig(D)
```

```python
print("lambda = ", eigenvalues)
print("x = ", eigenvectors)
```

```
lambda =  [ 1.  2.  3.]
x =  [[ 1.  0.  0.]
 [ 0.  1.  0.]
 [ 0.  0.  1.]]
```

```python
A = [[1,3],[2,6]]
```

```python
eigenvalues, eigenvectors = np.linalg.eig(A)
```

```python
print("lambda = ", eigenvalues)
print("x = ", eigenvectors)
```

```
lambda =  [ 0.  7.]
x =  [[-0.9486833  -0.4472136 ]
 [ 0.31622777 -0.89442719]]
```

```python
Q = [[0,-1],[1,0]]
```

```python
eigenvalues, eigenvectors = np.linalg.eig(Q)
```

```python
print("lambda = ", eigenvalues)
print("x = ", eigenvectors)
```

```
lambda =  [ 0.+1.j  0.-1.j]
x =  [[ 0.70710678+0.j          0.70710678-0.j          ]
 [ 0.00000000-0.70710678j  0.00000000+0.70710678j]]
```

Matrix diagonalization

$A_{n\times n}$ has n independent eigenvectors $x_1 \dots x_n$

$$AS = A\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ | & & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{\text{eigenvector matrix}} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\text{eigenvalue matrix}} = S\Lambda$$

$$AS = S\Lambda$$

$$S^{-1}AS = \Lambda \qquad A = S\Lambda S^{-1}$$

if $Ax = \lambda x$

$$A^2 x = \lambda A x = \lambda^2 x$$

same eigenvectors, squared eigenvalues

$$A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1} \qquad A^k = S\Lambda^k S^{-1}$$

$A^k \to 0$ as $k \to \infty$ if all $|\lambda_i| < 1$