Dynamic Factor Model for Matrix Time Series

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Joint work with Yuefeng Han, Rong Chen and Han Xiao

Introduction: Matrix Time Series

Matrix Time Series

A sequence of observations $\mathbf{X_t}$ in matrix form observed over time

where

$$\boldsymbol{X}_t = \begin{pmatrix} x_{t11} & \cdots & x_{t1n} \\ \vdots & \ddots & \vdots \\ x_{tm1} & \cdots & x_{tmn} \end{pmatrix}$$

Examples:

- A set of economic indicators of a group of countries observed over time;
- A set of financial characteristics of a group of companies observed over time;
- Import-Export among a group of countries observed over time;
- A sequence of time-indexed images.

Review of Matrix Time Series Models

Matrix Autoregressive Model (Chen, Xiao and Yang, 2021+, JoE)

$$\boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_{t-1}\boldsymbol{B}' + \boldsymbol{E}_t.$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $m \times m$ and $n \times n$ coefficient matrices. $\boldsymbol{E}_t$ is a $m \times n$ series of (matrix) white noise.

Matrix Factor Model (Wang, Liu, and Chen (2019, JoE))

$$\boldsymbol{X}_t = \boldsymbol{U}_1\boldsymbol{F}_t\boldsymbol{U}_2' + \boldsymbol{E}_t$$

where $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ are $d_1 \times k_1$ and $d_2 \times k_2$ orthonormal loading matrices. $\boldsymbol{F}_t$ $(k_1 \times k_2)$ unobserved factor matrix. $(k_1 << d_1$ and $k_2 << d_2)$. $\boldsymbol{E}_t$ is a $d_1 \times d_2$ series of (matrix) white noise.

Dynamic Matrix Factor Model

$$
\begin{aligned}
\boldsymbol{X}_t &= \lambda \boldsymbol{U}_1 \boldsymbol{F}_t \boldsymbol{U}_2^\top + \boldsymbol{E}_t \\
\boldsymbol{F}_t &= \boldsymbol{A}_1 \boldsymbol{F}_{t-1} \boldsymbol{A}_2^\top + \boldsymbol{Z}_t
\end{aligned}
$$

where $\boldsymbol{E}_t$ and $\boldsymbol{Z}_t$ are independent matrix white noise processes. $||\boldsymbol{U}_1||_F = ||\boldsymbol{U}_2||_F = 1$ and orthogonal.

Why?

- Adding prediction capability to matrix factor model;
- Reducing dimension for matrix AR model :
  - Matrix AR involves $2d^2$ parameters, here only $2dr + 2r^2$
- More detailed dynamic structure in low dimension;
- Can be extended to allow more flexible dynamic structure, as $\boldsymbol{F}_t$ is of much lower dimension:
  - $vec(\boldsymbol{F}_t) \sim VAR$
  - Each f_{ijt} follows univariate AR model, and independent to each other

Prediction

Let $\mathcal{F}_t$ be the σ-field generated by $\boldsymbol{X}_t, \ldots, \boldsymbol{X}_1$. Under the model:

$$E(\boldsymbol{X}_{t+1} \mid \mathcal{F}_t) = \lambda \boldsymbol{U}_1 \boldsymbol{F}^*_{t+1} \boldsymbol{U}_2^\top.$$

where $\boldsymbol{F}^*_{t+1} = E[\boldsymbol{F}_{t+1} \mid \mathcal{F}_t] = \boldsymbol{A}_1 \boldsymbol{F}_t \boldsymbol{A}_2^\top$.

Hence the least squares predictor is

$$\begin{aligned}
\hat{\boldsymbol{X}}_{t+1} &= \lambda \boldsymbol{U}_1 \hat{\boldsymbol{F}}^*_{t+1} \boldsymbol{U}_2^\top \\
&= \lambda \boldsymbol{U}_1 \boldsymbol{A}_1 \hat{\boldsymbol{F}}_t \boldsymbol{A}_2^\top \boldsymbol{U}_2^\top \\
&= \boldsymbol{U}_1 \boldsymbol{A}_1 \boldsymbol{U}_1^\top \boldsymbol{X}_t \boldsymbol{U}_2 \boldsymbol{A}_2^\top \boldsymbol{U}_2^\top.
\end{aligned}$$

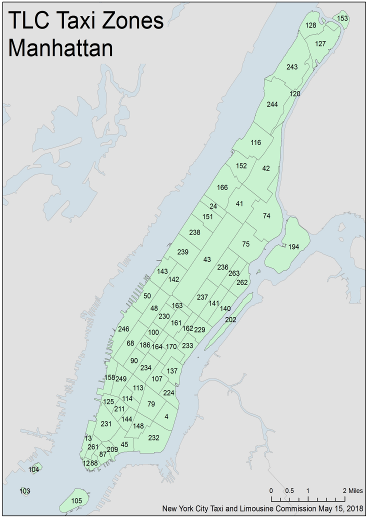
- It is in a Matrix AR(1) form – though the model is not a MAR(1).
- The coefficient matrices $\boldsymbol{U}_1 \boldsymbol{A}_1 \boldsymbol{U}_1^\top$ and $\boldsymbol{U}_2 \boldsymbol{A}_2^\top \boldsymbol{U}_2^\top$ are of low rank.

Estimation

A two-step estimation procedure through a combination of (an extended version of) PCA and (lower dimensional) LS:

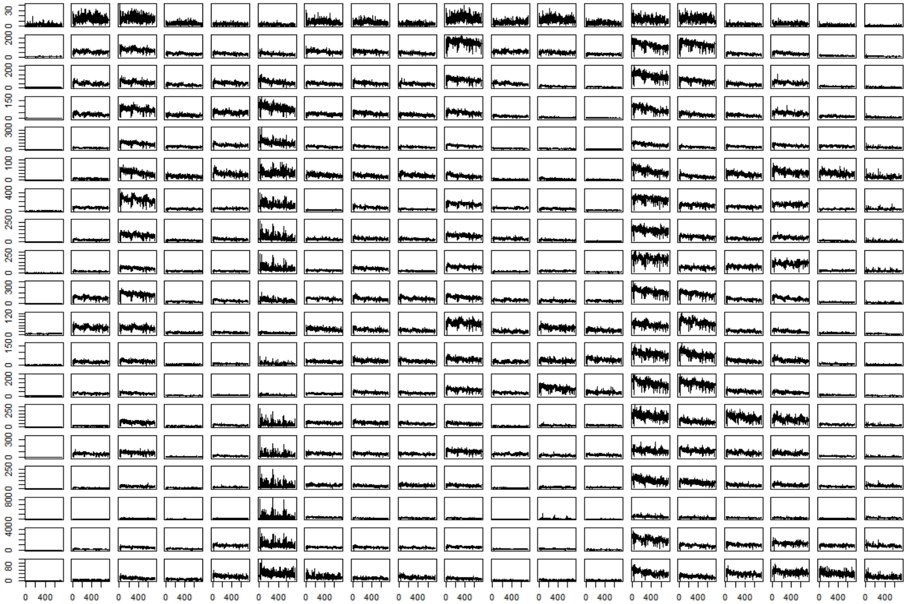
1. Estimate the matrix factor model to obtain $\hat{\boldsymbol{U}}_1, \hat{\boldsymbol{U}}_2, \hat{\boldsymbol{F}}_t$,
2. Estimate a matrix AR model (or VAR, or individual AR), treating $\hat{\boldsymbol{F}}_t$ as true

- Two stage estimation is commonly used to estimate the dynamic factor model in vector case.
- Joint likelihood estimation is possible, through a state-space model approach, with a modified Kalman filter.
    - Poor performance as the number of parameters (the loading matrices and the MAR coefficient matrices and the noise variances).



- The NYC taxi data on Manhattan is tensor data of dimensions $69 \times 69 \times 24 \times T$, which is the taxi counts from one region to another in a certain hour in a day
- Use a subset of the data: Sum of morning rush hours (7-10am) in 19 regions in middle town for years 2015 to 2017. The dimension is $19 \times 19 \times 750$.

Real example : NYC taxi data



Real example : NYC taxi data

Table: Use morning rush hours data to do rolling forecasts

Dynamic Factor		Vector factor		Autoregressive Models	
Method	RMSE	Method	RMSE	Method	RMSE
F.MAR1	**12.68**			X.MAR1	12.71
F.VAR1	12.75	VF.VAR1	12.71	X.VAR1	19.05
F.iAR1	12.81	VF.iAR1	12.74	X.iAR1	12.86
F.rw	14.69	VF.rw	14.9	X.rw	16.12
F.mean	13.34	VF.mean	13.33	X.mean(ES)	13.35

Note: ES means we're using exponential smoothing to do prediction. rw is the random walk prediction, where we use the data at last time point to predict next.

THANK YOU!