9. Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.

Assignment 5_ Combinatorics

9.

Solve: If the first chessboard is colored red, then the second chessboard only can be colored blue or white and the $n-2$ chessboards of remain have h_{n-2} ways to be colored. This is case 1.

If the first chessboard is colored blue or white, then the $n-1$ chessboards of remain have h_{n-1} ways to be colored. This is case 2.

So, $h_n = 2h_{n-2} + 2h_{n-1}$

$\Rightarrow$ $h_n - 2h_{n-1} - 2h_{n-2} = 0 \cdots\cdots ①$

The characteristic equation of the recurrence relation ① is

$x^2 - 2x - 2 = 0$

$x_1 = 1+\sqrt{3}$, $x_2 = 1-\sqrt{3}$

$\Rightarrow h_n = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$ $\quad h_1 = 3,\ h_2 = 3^2 - 1 = 8$

$$\begin{cases} c_1(1+\sqrt{3}) + c_2(1-\sqrt{3}) = 3 \\ c_1(1+\sqrt{3})^2 + c_2(1-\sqrt{3})^2 = 8 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{\sqrt{3}+2}{2\sqrt{3}} \\ c_2 = \frac{\sqrt{3}-2}{2\sqrt{3}} \end{cases}$$

Answer: $h_n = \frac{\sqrt{3}+2}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{\sqrt{3}-2}{2\sqrt{3}}(1-\sqrt{3})^n \quad n = 1, 2, \cdots$

16. Formulate a combinatorial problem for which the generating function is

$$(1+x+x^2)(1+x^2+x^4+x^6)(1+x^2+x^4+\cdots)(x+x^2+x^3+\cdots).$$

16:

Let h_n denote the number of solutions of the equation $x_1+x_2+x_3+x_4 = n$ in nonnegative integer x_1, x_2, x_3 and x_4 with $0 \le x_1 \le 2$, x_2 is even number and $0 \le x_2 \le 6$, x_3 is even number, $x_4 \ge 1$

25. Let h_n denote the number of ways to color the squares of a 1-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence $h_0, h_1, \ldots, h_n, \ldots$, and then find a simple formula for h_n.

25.

solve.

$$g^{(e)}(x) = (1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots)(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots)(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots)$$

$$= \frac{e^x+e^{-x}}{2}\cdot\frac{e^x-e^{-x}}{2}\cdot e^{2x}$$

$$= \frac{e^{2x}(e^{2x}-e^{-2x})}{4}$$

$$= \frac{1}{4}(e^{4x}-1)$$

$$= -\frac{1}{4}+\sum_{n=0}^{\infty}\frac{4^n}{4}\cdot\frac{x^n}{n!}$$

$$= -\frac{1}{4}+\sum_{n=0}^{\infty}4^{n-1}\cdot\frac{x^n}{n!}$$

Hence, $h_n = 4^{n-1}$ $h_0=0$

48. Solve the following recurrence relations by using the method of generating functions as described in Section 7.4:

(b) $h_n = h_{n-1} + h_{n-2}$, $(n \geq 2)$; $h_0 = 1, h_1 = 3$

48 (b)

Solve:

$$h_n - h_{n-1} - h_{n-2} = 0$$

$$x^2 - x - 1 = 0$$

$$\Rightarrow x_1 = \frac{1+\sqrt{5}}{2}, \quad x_2 = \frac{1-\sqrt{5}}{2}$$

$$h_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\begin{cases} h_0 = c_1+c_2 = 1 \\ h_1 = c_1\cdot\frac{1+\sqrt{5}}{2}+c_2\cdot\frac{1-\sqrt{5}}{2} = \frac{1}{2}(c_1+c_2)+\frac{\sqrt{5}}{2}(c_1-c_2) = 3 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{1+\sqrt{5}}{2} \\ c_2 = \frac{1-\sqrt{5}}{2} \end{cases}$$

$$\Rightarrow h_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$