7. Let $\mathcal{A} = (A_1, A_2, A_3, A_4, A_5, A_6)$, where

$$\begin{aligned} A_1 &= \{a,b,c\},\ A_2 = \{a,b,c,d,e\},\ A_3 = \{a,b\}, \\ A_4 &= \{b,c\},\ A_5 = \{a\},\ A_6 = \{a,c,e\}. \end{aligned}$$

Does the family $\mathcal{A}$ have an SDR? If not, what is the largest number of sets in the family with an SDR?

7.

$|A_{i_1} \cup A_{i_2} \cup \cdots A_{i_k}| + n - k$

$n = 6$

~~when~~ $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| + 6 - 6 = 5 < 6$

So the family A does not have an SDR.

with $n=6$, $k=1$, $\min_{i=1,2,\dots,6} |A_i| + 6 - 1 = 6$

with $n=6$, $k=2$, $\min_{i=1,2,\dots,6} |A_{i_1} \cup A_{i_2}| + 6 - 2 = 6$

with $n=6$, $k=3$, $\min_{i=1,2,\dots,6} |A_{i_1} \cup A_{i_2} \cup A_{i_3}| + 6 - 3 = 6$

with $n=6$, $k=4$, $\min_{i=1,2,\dots,6} |A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4}| + 6 - 4 = 5$

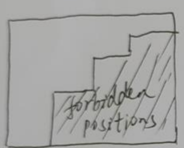
with $n=6$, $k=5$, $\min_{i=1,2,\dots,6} |A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4} \cup A_{i_5}| + 6 - 5 = 5$

~~So~~ Hence, 5 is the largest number of sets in the family with an SDR.

12. Consider a board with forbidden positions which has the property that, if a square is forbidden, so is every square to its right in its row and every square below it in its column. Prove that the chessboard has a tiling by dominoes if and only if the number of allowable white squares equals the number of allowable black squares.

an SDR.

12.



proof:

Let $B = \{b_1, b_2, \dots, b_n\}$ denote all black square;

$W = \{w_1, w_2, \dots, w_m\}$ denote all white square;

define sub-sets of W as $A = (A_1, A_2, \dots, A_n)$

and $A_i = \{w_j \mid w_j \text{ has common edge with } b_i\}$ $i=1,2,\dots,n$

according to the problem, we can know $2 \le |A_i| \le 4$. The chessboard has a tiling by dominoes if and only if the family A has SDR.

when $m = n$, because arbitrary two sub-set of A have maximum overlap square is 2, for any $k = 1, 2, \dots, n$ and ~~for all {1, 2, ..., n}~~ k combination of $\{1, 2, \dots, n\}$ have:

$$|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| \geq k$$

Hence, A has SDR, the chessboard has tiling by dominoes ~~when~~ iff $m = n$.

19. Use the deferred acceptance algorithm to obtain both the women-optimal and men-optimal stable complete marriages for the preferential ranking matrix

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1,3 & 2,3 & 3,2 & 4,3 \\ 1,4 & 4,1 & 3,3 & 2,2 \\ 2,2 & 1,4 & 3,4 & 4,1 \\ 4,1 & 2,2 & 3,1 & 1,4 \end{bmatrix} \end{array}.$$

Conclude that, for the given preferential ranking matrix, there is only one stable complete marriage.

19.

(1) women-optimal

step1. $A \xrightarrow{\text{chooses}} a$, $B \to a$, $C \to b$, $D \to d$; $a \underset{\text{rejects}}{\not\to} B$

step2. $B \to d$; $d \not\to D$

step3. $D \to b$; $b \not\to C$

step4. $C \to a$; $a \not\to A$

step5. $A \to c$: no reject

~~Hence~~. stable complete marriages:

$A \leftrightarrow c$, $B \leftrightarrow d$; $C \leftrightarrow a$, $D \leftrightarrow b$

(2) ~~met~~ men-optimal

step1. $a \to D$, $b \to B$, $c \to D$, $d \to C$; $D \not\to a$

step2. $a \to C$; $C \not\to d$

~~set~~ step3. $d \to B$; $B \not\to b$

step4. $b \to D$; $D \not\to c$

step5. $c \to A$; no reject

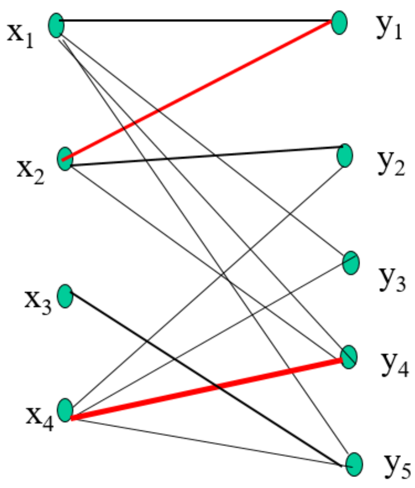
stable compolete marriages:

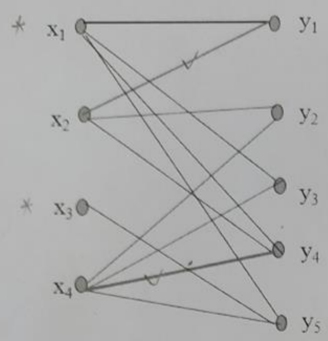
$a \leftrightarrow C$, $b \leftrightarrow D$, $c \leftrightarrow A$, $d \leftrightarrow B$

Hence, there is only one stable complete marriage:

$A \leftrightarrow c$, $B \leftrightarrow d$; $C \leftrightarrow a$, $D \leftrightarrow b$

- **Determine the max-matching and the min-cover of the right graph by applying the matching algorithm. We choose the red edges and obtain a matching M^1.**
- **Find a minimum edge cover for the right graph.**





$M^1 = \{(x_2, y_1), (x_4, y_4)\}$ $U_1 = \{x_1, x_3\}$

step 1. label ~~#~~ x_i in U_1 with *

step 2. scan x_1 label y_3, y_4, y_5 with (x_1)

scan x_3, no y is labeled.

step 3. scan y_3, there is a breakthrough

$r = x_1 y_3$ is a ~~M'-augn~~ M^1-augmenting path.

Hence, $M^2 = \{(x_1, y_3), (x_2, y_1), (x_4, y_4)\}$ $U_2 = \{x_3\}$

step 4. label x_i in U_2 with *

step 5. scan x_3, label y_5 with (x_3)

step 6. scan y_5, there is a breakthough

$M^3 = \{(x_1, y_3), (x_2, y_1), (x_3, y_5), (x_4, y_4\}$. $U_3 = \phi$.

therefore, the ~~#~~ Max-matching is M^3

meanwhile, since $U_3 = \emptyset$, there is no x and y that ~~can~~ be labeled.

So $S = \{x_1, x_2, x_3, x_4\}$ is a Min-cover