3. Find the number of integers between 1 and 10,000 that are neither perfect squares nor perfect cubes.

3

Answer:

Let $S = \{1, 2, \ldots, 10000\}$,

$A_1 = \{a^2 \mid a \in S \text{ and } a^2 \in S\}$

$A_2 = \{b^3 \mid b \in S \text{ and } b^3 \in S\}$

because $100^2 = 10000$, $101^2 > 10000 \Rightarrow |A_1| = 100$

Similarly $21^3 = 9261$. $22^3 = 10648 > 10000 \Rightarrow |A_2| = 21$

$A_1 \cap A_2 = \{c^6 \mid c \in S, c^{2\times3} = c^6 \in S\}$

$4^6 = 4096$, $5^6 = 15625 > 1000 \Rightarrow |A_1 \cap A_2| = 4$

$\Rightarrow |\bar{A_1} \cap \bar{A_2}| = |S| - |A_1| - |A_2| + |A_1 \cap A_2| = 10000 - 100 - 21 + 4 = 9883$

Answer: 9883

8. Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 14$ in positive integers x_1, x_2, x_3, x_4 and x_5 not exceeding 5.

8

Answer:

$x_1 + x_2 + x_3 + x_4 + x_5 = 14 \quad 1 \le x_i \le 5$

Let $y_i = x_i - 1$

$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 9 \quad 0 \le y_i \le 4$

Let S is the set of solutions of the equation.

Then, $|S| = \binom{9+5-1}{9} = C_{13}^{9} = C_{13}^{4} = 715$

Let P_i is the property $y_i \geq 6$ $(i = 1, 2, 3, 4, 5)$

Then, A_1 is the solution of the equation $y_1 + y_2 + y_3 + y_4 + y_5 = 9$ in positive integers $y_1 \geq 6$, $y_2 \geq 0$, $y_3 \geq 0$, $y_4 \geq 0$, $y_5 \geq 0$

$|A_1| = \binom{3+5-1}{3} = C_7^3 = \frac{7\times6\times5}{3\times2} = 35$

Similarly, $|A_2| = |A_3| = |A_4| = |A_5| = |A_1| = 35$

But $A_i \cap A_j = \{\phi\} \, (i \neq j)$

Therefore, $|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4} \cap \bar{A_5}| = |S| - 5\times35 = 540$

$\Rightarrow$ Answer: 540

11. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which no even integer is in its natural position.

11. According to the problem, we can get the 8×8 bord with forbidden positions as shown following

	1	2	3	4	5	6	7	8
1								
2		X						
3								
4				X				
5								
6						X		
7								
8								X

$X_1 = \{\phi\}$, $X_2 = \{2\}$, $X_3 = \{\phi\}$, $X_4 = \{4\}$, $X_5 = \{\phi\}$, $X_6 = \{6\}$, $X_7 = \{\phi\}$, $X_8 = \{8\}$

$r_1 = 4$, $r_2 = C_4^2 = 6$, $r_3 = 4$, $r_4 = 1$, $n = 8$

$$|P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)| = n! - r_1(n-1)! + r_2(n-2)! - r_3(n-3)! + r_4(n-4)!$$
$$= 8! - 4\times 7! + 6\times 6! - 4\times 5! + 4!$$

Answer: $8! - 4\times 7! + 6\times 6! - 4\times 5! + 4!$

25. Count the permutations $i_1 i_2 i_3 i_4 i_5 i_6$ of $\{1, 2, 3, 4, 5, 6\}$, where $i_1 \neq 1, 5$; $i_3 \neq 2, 3, 5$; $i_4 \neq 4$; and $i_6 \neq 5, 6$.

25. According to the problem, we can get the following 6×6 bord with forbidden positions as shown

	1	2	3	4	5	6
1	X					
2			X			
3			X			
4				X		
5	X		X			X
6						X

$r_1 = 8$, $r_2 = 6+4+3+3+2+2 = 20$, $r_3 = 20$, $r_4 = 7$

Answer: $6! - 8\times 5! + 20\times 4! - 20\times 3! + 7\times 2!$

31. How many circular permutations are there of the multiset

$$\{2 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\},$$

where, for each type of letter, all letters of that type do not appear consecutively?

A 31. Let S is the set that contains all the circular permutations of the multiset $\{2\cdot a, 3\cdot b, 4\cdot c, 5\cdot d\}$;

A_1 is the set that contains all the circular permutations of the multiset $\{\text{"aa"}, 3b, 4\cdot c, 5\cdot d\}$ that "aa" means that two letters of "a" appear consecutively.

Similarly, A_2 is $\{\text{"bbb"}, 2a, 4c, 5d\}$

A_3 is $\{\text{"cccc"}, 2a, 3b, 5d\}$

A_4 is $\{\text{"ddddd"}, 2a, 3b, 4c\}$

Then, we can get:

$|S| = 12 \times \frac{12!}{3!4!5!}$ $\quad |A_1| = \frac{12!}{3!4!5!}$ $\quad |A_2| = \frac{11!}{2!4!5!}$ $\quad |A_3| = \frac{10!}{2!3!5!}$

$|A_4| = \frac{9!}{2!3!4!}$ $\quad |A_1 \cap A_2| = \frac{10!}{4!5!}$ $\quad |A_1 \cap A_3| = \frac{9!}{3!5!}$ $\quad |A_1 \cap A_4| = \frac{8!}{3!4!}$

$|A_2 \cap A_3| = \frac{8!}{2!5!}$ $\quad |A_2 \cap A_4| = \frac{7!}{2!4!}$ $\quad |A_3 \cap A_4| = \frac{6!}{2!3!}$

$|A_1 \cap A_2 \cap A_3| = \frac{7!}{5!}$ $\quad |A_1 \cap A_2 \cap A_4| = \frac{6!}{4!}$ $\quad |A_1 \cap A_3 \cap A_4| = \frac{5!}{3!}$

$|A_2 \cap A_3 \cap A_4| = \frac{4!}{2!}$ $\quad |A_1 \cap A_2 \cap A_3 \cap A_4| = 3!$

So, $|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4}| = |S| - (|A_1| + |A_2| + |A_3| + |A_4|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4|$

$+ |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|)$

$- (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|)$

$+ |A_1 \cap A_2 \cap A_3 \cap A_4|$