16. Apply the algorithm for the GCD in Section 10.1 to 15 and 46, and then use the results to determine the multiplicative inverse of 15 in Z_{46}.

16.

A	B	
46	15	$46 = 15 \times 3 + 1$
1	15	$15 = 15 \times 1$
1	0	

Since, $1 = 46 - 3 \times 15 = 46 + 43 \times 15 = 43 \times 15$

Hence, $GCD(46, 15) = 1$, $15^{-1} = 43$ in Z_{46}.

21. Determine the complementary design of the BIBD with parameters $b = v = 7, k = r = 3, \lambda = 1$ in Section 10.2.

21.

$B = \{ B_1 = \{0, 1, 3\}, B_2 = \{1, 2, 4\}, B_3 = \{2, 3, 5\}, B_4 = \{3, 4, 6\}, B_5 = \{0, 4, 5\}, B_6 = \{1, 5, 6\}, B_7 = \{0, 2, 6\} \}$ is a BIBD with parameters $b = v = 7, k = r = 3$, $\lambda = 1$, So $B^c = \{ \bar{B}_1 = \{2, 4, 5, 6\}, \bar{B}_2 = \{0, 3, 5, 6\}, \bar{B}_3 = \{0, 1, 4, 7\}, \bar{B}_4 = \{0, 1, 2, 5\}, \bar{B}_5 = \{1, 2, 3, 6\}, \bar{B}_6 = \{0, 2, 3, 4\} \}$. B^c is the complementary design of the B.

28. Show that $B = \{0, 1, 3, 9\}$ is a difference set in Z_{13}, and use this difference set as a starter block to construct an SBIBD. Identify the parameters of the block design.

28

$-$	0	1	3	9
0	0	12	10	4
1	1	0	11	5
3	3	2	0	7
9	9	8	6	0

From the table, we can see that each of the non-zero integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 in Z_{13} occurs exactly once in the off-diagonal positions, hence B is a difference set in Z_{13}.

Using B as a starter block we obtain the following blocks for a SBIBD with parameters $b = v = 13$, $k = r = 4$ and $\lambda = \frac{k(k-1)}{v-1} = 1$

$B+0 = \{0, 1, 3, 9\}$ $B+1 = \{1, 2, 4, 10\}$ $B+2 = \{2, 3, 5, 11\}$ $B+3 = \{3, 4, 6, 12\}$

$B+4 = \{4, 5, 7, 0\}$ $B+5 = \{5, 6, 8, 1\}$ $B+1 = \{6, 7, 9, 2\}$ $B+7 = \{7, 8, 10, 3\}$

$B+8 = \{8, 9, 11, 4\}$ $B+9 = \{9, 10, 12, 5\}$ $B+10 = \{10, 11, 0, 6\}$ $B+11 = \{11, 12, 1, 7\}$

$B+12 = \{12, 0, 2, 8\}$

32. Use Theorem 10.3.2 to construct a Steiner triple system of index 1 having 21 varieties.

32.

Let $X = \{a_0, a_1, a_2\}$ and $Y = \{b_0, b_1, b_2, b_3, b_4, b_5, b_6\}$ be two sets of varieties. Let $B_1 = \{a_0, a_1, a_2\}$ and $B_2 = \{\{b_0, b_1, b_3\}, \{b_1, b_2, b_4\}, \{b_2, b_3, b_5\}, \{b_3, b_4, b_6\}, \{b_4, b_5, b_0\}, \{b_5, b_6, b_1\}, \{b_6, b_0, b_2\}\}$ be the steiner triple systems of X and Y, respectively. So we can get a 3-by-7 array as following:

$$\begin{array}{c|ccc} & a_0 & a_1 & a_2 \\ \hline b_0 & 0 & 1 & 2 \\ b_1 & 3 & 4 & 5 \\ b_2 & 6 & 7 & 8 \\ b_3 & 9 & 10 & 11 \\ b_4 & 12 & 13 & 14 \\ b_5 & 15 & 16 & 17 \\ b_6 & 18 & 19 & 20 \end{array}$$

(1) The entries in each of the seven rows:

$B_1' = \{\{0,1,2\}, \{3,4,5\}, \{6,7,8\}, \{9,10,11\}, \{12,13,14\}, \{15,16,17\}, \{18,19,20\}\}$

(2) The entries in each of three columns:

$B_2' = \{\{0,3,9\}, \{1,4,10\}, \{2,5,11\}, \{3,6,12\}, \{4,7,13\}, \{5,8,14\}, \{6,9,15\}, \{7,10,16\}, \{8,11,17\}, \{9,12,18\}, \{10,13,19\}, \{11,14,20\}, \{12,15,0\}, \{13,16,1\}, \{14,17,2\}, \{15,18,3\}, \{16,19,4\}, \{17,20,5\}, \{18,0,6\}, \{19,1,7\}, \{20,2,8\}\}$

(3) Three entries, no two from the same row or column:

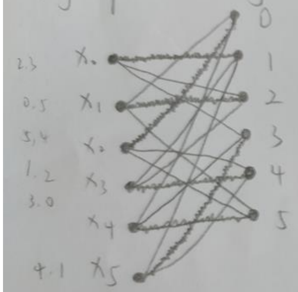
$B_3' = \{\{0,4,11\}, \{0,5,10\}, \{1,3,11\}, \{1,5,9\}, \{2,3,10\}, \{2,4,9\}, \{3,7,14\}, \{3,8,13\}, \{4,6,14\}, \{4,8,12\}, \{5,6,13\}, \{5,7,12\}, \{6,10,17\}, \{6,11,16\}, \{7,9,17\}, \{7,11,15\}, \{8,10,15\}, \{8,9,16\}, \{9,13,20\}, \{9,14,19\}, \{10,14,18\}, \{10,12,20\}, \{11,12,19\}, \{11,13,18\}, \{12,16,2\}, \{12,17,1\}, \{13,17,0\}, \{13,15,2\}, \{14,16,0\}, \{14,15,1\}, \{15,19,5\}, \{15,20,4\}, \{16,20,3\}, \{16,18,5\}, \{17,19,3\}, \{17,18,4\}, \{0,7,20\}, \{0,8,19\}, \{1,6,20\}, \{1,8,18\}, \{2,6,19\}, \{2,7,18\}\}$

Hence, $B_1' \cup B_2' \cup B_3' = B$ is a Steiner triple system of index 1 having 21 varieties.

52. Construct a completion of the 3-by-6 Latin rectangle

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 & 0 \\ 5 & 4 & 3 & 0 & 1 & 2 \end{bmatrix}.$$

52. According to the 3-by-6 Latin rectangle, we can construct a bigraph as following:



we can find three prefect matching are

$\{(x_0,1), (x_1,2), (x_2,0), (x_3,4), (x_4,5), (x_5,3)\}$

$\{(x_0,2), (x_1,0), (x_2,5), (x_3,1), (x_4,3), (x_5,4)\}$

$\{(x_0,3), (x_1,5), (x_2,4), (x_3,2), (x_4,0), (x_5,1)\}$

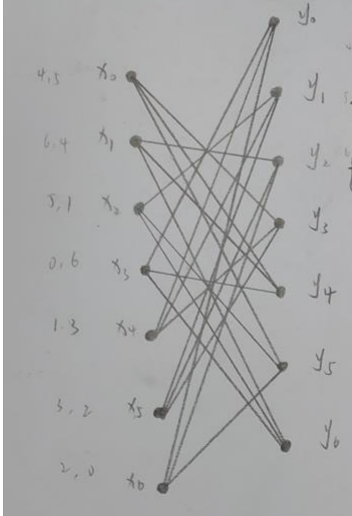
So we can get:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 & 0 \\ 5 & 4 & 3 & 0 & 1 & 2 \\ 1 & 2 & 0 & 4 & 5 & 3 \\ 2 & 0 & 5 & 1 & 3 & 4 \\ 3 & 5 & 4 & 2 & 0 & 1 \end{bmatrix}$$

56. Construct a completion of the semi-Latin square

$$\begin{bmatrix} 0 & 2 & 1 & & & & 3 \\ 2 & 0 & & 1 & & 3 & \\ 3 & & 0 & 2 & 1 & & \\ & 3 & 2 & 0 & & 1 & \\ & & 3 & & 0 & 2 & 1 \\ 1 & & & & 3 & 0 & 2 \\ & 1 & & 3 & 2 & & 0 \end{bmatrix}.$$

56. According to the 7-by-7 semi-Latin square, we can construct a bigraph as following:



we can find three prefect matching are

$\{(x_0,y_3), (x_1,y_2), (x_2,y_6), (x_3,y_4), (x_4,y_0), (x_5,y_1), (x_6,y_5)\}$

$\{(x_0,y_4), (x_1,y_6), (x_2,y_5), (x_3,y_0), (x_4,y_1), (x_5,y_3), (x_6,y_2)\}$

$\{(x_0,y_5), (x_1,y_4), (x_2,y_1), (x_3,y_6), (x_4,y_3), (x_5,y_2), (x_6,y_0)\}$

So we can get:

$$\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 4 & 5 & 6 & 3 \\ 2 & 0 & 4 & 1 & 6 & 3 & 5 \\ 3 & 6 & 0 & 2 & 1 & 5 & 4 \\ 5 & 3 & 2 & 0 & 4 & 1 & 6 \\ 4 & 5 & 3 & 6 & 0 & 2 & 1 \\ 1 & 4 & 6 & 5 & 3 & 0 & 2 \\ 6 & 1 & 5 & 3 & 2 & 4 & 0 \end{bmatrix} \end{array}$$