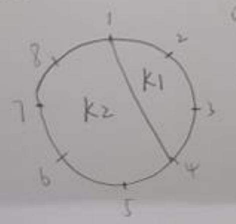
1. Let $2n$ (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number C_n.

1. Assume g_n is the number of ways, and mark the points with $1, 2, \cdots, 2n$ respectively. Get the point marked 1 and any one of the even point $2k$, connect point 1 and point $2k$. This line segment divided the circle two parts K_1 and K_2. In part K_1, there are $(k-1)$ pairs of points, so there are g_{k-1} ways. In part K_2, there are $(n-k)$ pairs of points. so there are g_{n-k} ways.



$n=4$

Hence,

$$
\begin{aligned}
g_n &= \sum_{k=1}^{n} g_{k-1} g_{n-k} \qquad g_0 = 1 \\
&= g_0 g_{n-1} + g_1 g_{n-2} + \cdots + g_{n-1} \cdot g_0 \qquad n \geq 2 \quad g_0 = g_1 = 1 \\
&= \frac{1}{n+1}\binom{2n}{n} \\
&= C_n
\end{aligned}
$$

notes:

Catalan numbers 卡特兰数 $h(n)$

(1) $h(n) = h(0)\cdot h(n-1) + h(1)\cdot h(n-2) + \cdots + h(n-1)h(0)$ $\stackrel{n \geq 2}{\vee}$ with $h(0)=1, h(1)=1$

(2) $h(n) = \frac{4n-2}{n+1} h(n-1)$

(3) $h(n) = \frac{1}{n+1} C_{2n}^{n}$ $(n = 0, 1, 2, \cdots)$

7. The general term h_n of a sequence is a polynomial in n of degree 3. If the first four entries of the 0th row of its difference table are $1, -1, 3, 10$, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.

7. difference table:

$$
\begin{array}{ccccccc}
1 & & -1 & & 3 & & 10 \quad \cdots \\
& -2 & & 4 & & 7 & \cdots \\
& & 6 & & 3 & & \cdots \\
& & & -3 & & & \cdots
\end{array}
$$

$$
\begin{aligned}
h_n &= 1\cdot\binom{n}{0} - 2\binom{n}{1} + 6\binom{n}{2} - 3\binom{n}{3} \\
\sum_{k=0}^{n} h_k &= \sum_{k=0}^{n}\binom{k}{0} - 2\sum_{k=0}^{n}\binom{k}{1} + 6\sum_{k=0}^{n}\binom{k}{2} - 3\sum_{k=0}^{n}\binom{k}{3} \\
&= \binom{n+1}{1} - 2\binom{n+1}{2} + 6\binom{n+1}{3} - 3\binom{n+1}{4}
\end{aligned}
$$

25. Let $t_1, t_2, \ldots, t_m$ be distinct positive integers, and let

$$q_n = q_n(t_1, t_2, \ldots, t_m)$$

equal the number of partitions of n in which all parts are taken from $t_1, t_2, \ldots, t_m$. Define $q_0 = 1$. Show that the generating function for $q_0, q_1, \ldots, q_n, \ldots$ is

$$\prod_{k=1}^{m} (1 - x^{t_k})^{-1}.$$

.

25.

proof:

the generating function for $q_0, q_1, q_2, \cdots, q_n \cdots$ is

$$\sum_{n=0}^{\infty} q_n x^n = \prod_{k=1}^{\infty} (1-x^{t_k})^{-1}$$

The expression on the right equals the product

$$(1+x^{t_1}+\cdots+x^{a_1t_1}+\cdots)(1+x^{t_2}+\cdots+x^{a_2t_2}+\cdots)\cdots(1+x^{t_m}+\cdots+x^{a_mt_m}+\cdots)$$

A term x^n arise in this product by choosing a term $x^{a_1t_1}$ from the first factor, $x^{a_2t_2}$ from the second, $x^{a_3t_3}$ from the third, and so on, with

$$a_1t_1 + a_2t_2 + \cdots + a_mt_m = n$$

Thus each partition of n contributes 1 to the coefficient of x^n equals the number q_n of partitions of n.