13. Let f and g be the permutations in Exercise 1. Consider the coloring $\mathbf{c} = (R, B, B, R, R, R)$ of $1, 2, 3, 4, 5, 6$ with the colors R and B. Determine the following actions on $\mathbf{c}$:

(b) $f^{-1} * \mathbf{c}$

13.(b)

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$$

$$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 2 & 5 & 1 \end{pmatrix}$$

$$f^{-1} * c = (R, B, R, B, R, R)$$

(d) $(g \circ f) * \mathbf{c}$ and $(f \circ g) * \mathbf{c}$

d)

$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}$$

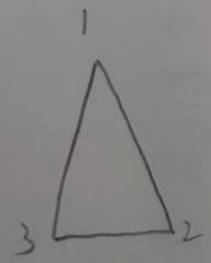
$$(g \circ f) * c = (B\ R\ B\ R\ R\ R)$$

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 3 & 4 & 6 \end{pmatrix}$$

$$(f \circ g) * c = (R\ B\ R\ B\ R\ R)$$

20. Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of a triangle that is isoceles, but not equilateral, with the colors red and blue. Do the same with p colors (cf. Exercise 4).

20



$$P_3^0 = [1] \circ [2] \circ [3] \Rightarrow (3, 0, 0) \Rightarrow z_1^3$$

$$r_1 = [1] \circ [2, 3] \Rightarrow (1, 1, 0) \Rightarrow z_1 \cdot z_2$$

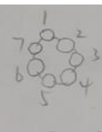
$$P(z_1, z_2, z_3) = \frac{z_1^3 + z_1 z_2}{2}$$

$$P(k, k, k) = \frac{k^3 + k^2}{2}$$

because $k = 2 \Rightarrow P(2, 2, 2) = 6$

26. How many different necklaces are there that contain four red and three blue beads?

26.



$$\rho_7^0 = [1][2][3][4][5][6][7] \Rightarrow z_1^7$$

$$\rho_7^1 = [1, 2, 3, 4, 5, 6, 7]$$

$$\rho_7^2 = [1, 3, 5, 7, 2, 4, 6]$$

$$\rho_7^3 = [1, 4, 7, 3, 6, 2, 5]$$

$$\cdots$$

$$\rho_7^6 = [1, 7, 6, 5, 4, 3, 2]$$

$$6 \Rightarrow 6 z_7$$

$$r_1 = [1][2,7][3,6][4,5]$$

$$r_2 = [2][1,3][4,7][5,6]$$

$$\cdots$$

$$r_6 = [6][5,7][1,4][2,3]$$

$$r_7 = [7][1,6][2,5][3,4]$$

$$7 \Rightarrow 7 z_1 \cdot z_2^3$$

$$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + \cdots + C_n^i a^{n-i} b^i + \cdots + C_n^n b^n = \sum_{i=0}^{n} C_n^i a^{n-i} b^i$$

$$\Rightarrow P_{D_7}(z_1, z_2, z_3, z_4, z_5, z_6) = \frac{(z_1^7 + 6z_7 + 7z_1 \cdot z_2^3)}{14}$$

$$P_{D_7}(r+b, r^2+b^2, r^3+b^3, r^4+b^4, r^5+b^5, r^6+b^6, r^7+b^7)$$

$$= \frac{[(r+b)^7 + 6(r^7+b^7) + 7(r+b)(r^2+b^2)^3]}{14}$$

$$= \frac{\sum_{i=0}^{7} C_7^i r^{7-i} b^i + 7r\sum_{i=0}^{3} C_3^i r^{2(3-i)} b^{2i} + 7b\sum_{i=0}^{3} C_3^i r^{2(3-i)} b^{2i} + 6(r^7+b^7)}{14}$$

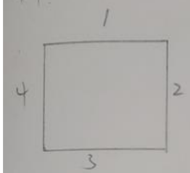
$$= \frac{(C_7^3 + 7C_3^1) r^4 b^3 + f(r,b)}{14}$$

$$= 4r^4b^3 + f(r,b)$$

$\Rightarrow$ Therefore, three are 4 kinds of inequivalent colorings.

44. Determine the generating function for nonequivalent colorings of the edges of a square with the colors red and blue. How many nonequivalent colorings are there with k colors (cf. Exercise 43)?

44.



$$\rho_4^0 = [1][2][3][4] \Rightarrow z_1^4$$

$$\rho_4^1 = [1, 2, 3, 4] \Rightarrow z_4^1$$

$$\rho_4^2 = [1, 3][2, 4] \Rightarrow z_2^2$$

$$\rho_4^3 = [1, 4, 3, 2] \Rightarrow z_4^1$$

$$r_1 = [1][3][2,4] \Rightarrow z_1^2 z_2$$

$$r_2 = [2][4][1,3] \Rightarrow z_1^2 z_2$$

$$r_3 = [1,4][2,3] \Rightarrow z_2^2$$

$$r_4 = [1,2][3,4] \Rightarrow z_2^2$$

$$\Rightarrow P_{D_4}(z_1, z_2, z_3, z_4) = \frac{z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4^1}{8}$$

$$P_{D_4}(k, k, k, k) = \frac{k^4 + 2k^3 + 3k^2 + 2k}{8} = \frac{2k + 3k^2 + 2k^3 + k^4}{8}$$

$$P_{D_4}(r+b, r^2+b^2, r^3+b^3, r^4+b^4) = \frac{(r+b)^4 + 2(r+b)^2(r^2+b^2) + 3(r^2+b^2)^2 + 2(r^4+b^4)}{8}$$

$$= \frac{C_4^0 r^4 + C_4^1 r^3 b + C_4^2 r^2 b^2 + C_4^3 r b^3 + C_4^4 b^4 + 2(r^2+2rb+b^2)(r^2+b^2) + 3(r^4+2r^2b^2+b^4) + 2(r^4+b^4)}{8}$$

$$= \frac{r^4 + 4r^3b + 6r^2b^2 + 4rb^3 + b^4 + 2r^4 + 4r^3b + 4r^2b^2 + 4rb^3 + 2b^4 + 3r^4 + 6r^2b^2 + 3b^4 + 2r^4 + 2b^4}{8}$$

$$= \frac{8r^4 + 8r^3b + 16r^2b^2 + 8rb^3 + 8b^4}{8}$$

$$= r^4 + r^3b + 2r^2b^2 + rb^3 + b^4$$

therefore:

$$P_{D_4}(r+b, r^2+b^2, r^3+b^3, r^4+b^4) = r^4 + r^3b + 2r^2b^2 + rb^3 + b^4$$

$$P_{D_4}(k, k, k, k) = \frac{2k + 3k^2 + 2k^3 + k^4}{8}$$