Numerrical Analysis

欧阳尚可 3190102458

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1 Assignments

Problem 1

首先我们证明 $L_\rho^2[a,b]$ 关于数乘和加法是封闭的。对于 $\forall f,g \in L_\rho^2[a,b]$，有 $f,g \in L[a,b]$ 且 $\rho(x)|f(x)|^2 \in L[a,b]$ 且 $\rho(x)|f(x)|^2 \in L[a,b]$，因此有 $f+g \in L[a,b]$ 且 $\rho(x)|f(x)+g(x)|^2 \in L[a,b]$，因此 $f+g \in L_\rho^2[a,b]$。同理可知 $\forall a \in \mathbb{R}, af(x) \in L_\rho^2[a,b]$。由函数的一些性质很容易验证这些线性空间性质：

$$(VSA-1) \quad \forall f,g \in L_\rho^2[a,b], f+g = g+f$$

$$(VSA-2) \quad \forall f,g,h \in L_\rho^2[a,b], f+(g+h) = f+g+h$$

$$(VSA-3) \quad \forall f \in L_\rho^2[a,b], \forall a,b \in \mathbb{R}, a(bf) = abf$$

$$(VSA-4) \quad 0 \in L[a,b],\ \ 0 \in L_\rho^2[a,b]; \forall f \in L_\rho^2[a,b], \exists g = 0 \in L_\rho^2[a,b], f+g = f$$

$$(VSA-5) \quad \forall f \in L_\rho^2[a,b], f \in L_\rho^2[a,b],\ -f \in L_\rho^2[a,b], \rho|f|^2 = \rho|-f|^2 \in L_\rho^2[a,b],$$
$$-f \in L_\rho^2[a,b]$$

$$(VSA-6) \quad \forall f \in L_\rho^2[a,b], \exists a = 1, af = f$$

$$(VSA-7) \quad \forall f,g \in L_\rho^2[a,b], \forall a,b \in \mathbb{R}, (a+b)f = af+bf, a(f+g) = af+ag$$

下面证明在该内积的定义下它是一个内积空间。

$$(IP-1) \forall v \in L_\rho^2[a,b], \rho(x) > 0, |v(x)|^2 \geq 0, \quad < v,v > \geq 0$$

$$(IP-2) \ \ \rho(x) > 0, \int_a^b \rho(x)|v(x)|^2 dx = 0 \quad iff |v| = 0 iff v = 0$$

$$(IP-3) \forall f,g,h \in L_\rho^2[a,b], < f+g,h >= \int_a^b \rho(t)[f(t)+g(t)]\overline{h(t)}dt$$
$$= \int_a^b \rho(t)f(t)\overline{h(t)}dt + \int_a^b \rho(t)g(t)\overline{h(t)}dt =< f,h > + < g,h >$$

$$(IP-4) \quad \forall f,g \in L_\rho^2[a,b], \forall a \in \mathbb{R}, < af,g >= \int_a^b \rho(t)af(t)\overline{g(t)}dt$$
$$= a\int_a^b \rho(t)f(t)g\hat{(t)}dt = a < f,g >$$

$$(IP-5) \quad \forall f,g \in L_\rho^2[a,b], \overline{< g,t >} = \overline{\int_a^b \rho(t)g(t)\overline{f(t)}dt}$$
$$= \int_a^b \overline{\rho(t)}f(t)\overline{g(t)}dt = \int_a^b \rho(t)f(t)\overline{g(t)}dt =< f,g >$$

Problem 2

对积分进行简单的计算

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} cos(narccost)cos(marccost)dt = \int_{0}^{\pi} cosn\theta cosm\theta d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi} cos((m+n)\theta)d\theta + \frac{1}{2}\int_{0}^{\pi} cos((m-n)\theta)d\theta,$$

若 $m \neq n$，有 $\int_0^\pi cos((m+n)\theta)d\theta = \int_0^\pi cos((m-n)\theta)d\theta = 0$；若 $m = n$，有 $\int_0^\pi cos((m+n)\theta)d\theta = 0, \frac{1}{2}\int_0^\pi cos((m-n)\theta)d\theta = \frac{\pi}{2} > 0$；若 $m = n = 0$，有 $\int_0^\pi cos((m+n)\theta)d\theta + \frac{1}{2}\int_0^\pi cos((m-n)\theta)d\theta = \pi > 0$。由此可得正交性。

仿照 Example5.20 有，若从 1 开始

$$u_1 = x, ||u_1||^2 = \frac{\pi}{2}, u_1^* = \sqrt{\frac{2}{\pi}}x$$

$$u_2 = cos(2arccosx) = 2x^2 - 1, ||u_2||^2 = \frac{\pi}{2}, u_2^* = \sqrt{\frac{2}{\pi}}(2x^2 - 1)$$

$$u_3 = cos(3arccosx) = 4x^3 - 3x, ||u_3||^2 = \frac{\pi}{2}, u_3^* = \sqrt{\frac{2}{\pi}}(4x^3 - 3x)$$

若从 0 开始

$$u_1 = 1, ||u_1||^2 = \pi, u_2^* = \sqrt{\frac{1}{\pi}}$$

$$u_2 = x, ||u_2||^2 = \frac{\pi}{2}, u_2^* = \sqrt{\frac{2}{\pi}}x$$

$$u_3 = cos(2arccosx) = 2x^2 - 1, ||u_2||^2 = \frac{\pi}{2}, u_3^* = \sqrt{\frac{2}{\pi}}(2x^2 - 1)$$

Problem 3

由第二题知 $u_1^* = \sqrt{\frac{1}{\pi}}, u_2^* = \sqrt{\frac{2}{\pi}}x, u_3^* = \sqrt{\frac{2}{\pi}}(2x^2 - 1)$。

$$b_0 =< f, u_1^* >= \int_{-1}^{1} \sqrt{\frac{1}{\pi}}dx = \sqrt{\frac{4}{\pi}}, b_1 =< f, u_2^* >= \int_{-1}^{1} \sqrt{\frac{2}{\pi}}xdx = 0,$$

$$b_3 =< f, u_3^* >= \int_{-1}^{1} \sqrt{\frac{2}{\pi}}(2x^2 - 1)dx = -\frac{2}{3}\sqrt{\frac{2}{\pi}}$$

$$\hat{\phi}_1 = \frac{2}{\pi}, \hat{\phi}_2 = \frac{2}{\pi} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}(2x^2 - 1) = -\frac{8}{3\pi}x^2 + \frac{10}{3\pi}$$

$$G<1,x,x^2>=\begin{bmatrix}<1,1> & <1,x> & <1,x^2>\\ <x,1> & <x,x> & <x,x^2>\\ <x^2,1> & <x^2,x> & <x^2,x^2>\end{bmatrix}=\begin{bmatrix}\pi & 0 & \frac{\pi}{2}\\ 0 & \frac{\pi}{2} & 0\\ \frac{\pi}{2} & 0 & \frac{3\pi}{8}\end{bmatrix},c=\begin{bmatrix}<f,1>\\ <f,x>\\ <f,x^2>\end{bmatrix}=\begin{bmatrix}2\\ 0\\ \frac{2}{3}\end{bmatrix}$$

由此解得 $a_0=\frac{10}{3\pi},a_1=0,a_2=-\frac{8}{3\pi},\hat{\phi}=-\frac{8}{3\pi}x^2+\frac{10}{3\pi}$。

Problem 4

$$u_1=1,v_1=1,||v_1||^2=2\sqrt{3},u_1^*=\frac{\sqrt{3}}{6}$$

$$u_2=x,v_2=x-<x,\frac{\sqrt{3}}{6}>\frac{\sqrt{3}}{6}=x-\frac{13}{x},<v_2,v_2>=143,||v_2||=\sqrt{143},u_2^*=\frac{x-\frac{13}{2}}{\sqrt{143}}$$

$$u_3=x^2,v_3=x^2-<x^2,\frac{\sqrt{3}}{6}>\frac{\sqrt{3}}{6}-<x^2,\frac{x-\frac{13}{2}}{\sqrt{143}}>\frac{x-\frac{13}{2}}{\sqrt{143}}=x^2-13x+\frac{416}{3}$$

$$<v_3,v_3>=1300,u_3^*=\frac{x^2-13x+78}{10\sqrt{13}}$$

$$\hat{\phi}=<f,u_1^*>u_1^*+<f,u_2^*>u_2^*+<f,u_3^*>u_3^*=386-113.427x+9.042x^2$$

显而易见这与讲义例子所提供的数据一致。这样计算的理由是通过第一问我们已经找到了由 $(1,x,x^2)$ 构成的一组标准化的正交基，再由 Theorem 5.25 知，其最小误差是由函数与标准正交基做内积决定的，由此可以得出上述函数。

在标准正交化多项式方法中，u_1^*,u_2^*,u_3^* 重复利用了，b_0,b_1,b_2 不能被重复利用。在 normal equation 方法中系数矩阵 G 被重复利用了，但是 $<y,1>,<y,2>,<y,3>$ 不能被重复利用。由此可以看出前者相较于后者的优势在于前者只需要计算不同向量在这组标准正交基上面的投影即可，而后者不仅要重新计算方程组右端的值，还需要重新解这个线性方程组，无疑增大了运算量。