Numerrical Analysis

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1 Assignments

1.1 Problem 1

Assume that $p(x) = ax^3 + bx^2 + cx + d$.We have $p(0) = s(0) = 0; p(1) = (2-1)^3 = 1; p^{'}(1) = s^{'}(1) = -3; p^{''}(1) = s^{''}(1) = 6$,from which we can conclude that $d = 0; a+b+c+d = 1; 3a+2b+c = -3; 6a+2b = 6$; from which we can obtain that $a = 7; b = -18; c = 12; d = 0$.If $s(x)$ is a natural cubic spline, which implies that $s^{''}(0) = s^{''}(2) = 0$, but $s^{''}(0) = p^{''}(0) = -18$, from which we obtain the contradiction.

1.2 Problem 2

Assume that $s_1(x) = a_1x^2+b_1x+c_1, ..., s_{n-1}(x) = a_{n-1}x^2+b_{n-1}x+c_{n-1}$. If we want to guarantee these coefficients unique, we need at least $3(n-1)$ equations. But actually we can only obtain $3n-4$ equations without extra information, $2+2(n-2)$ of these from the values of these knots and $n-2$ of these from the derivative of these konts.

Assume that $p_i(x) = a_ix^2 + b_ix + c_i$. From the conditions we have $m_i = 2a_ix_i + b_i; f_i = a_ix_i^2 + b_ix_i + c_i; f_{i+1} = a_{i+1}x_{i+1}^2 + b_{i+1}x_{i+1} + c_{i+1}$, from which we can obtain $a_i = \frac{f_{i+1}-f_i}{(x_{i+1}-x_i)^2} - \frac{m_i}{x_{i+1}-x_i}; b_i = m_i\frac{x_{i+1}+x_i}{x_{i+1}-x_i} - 2x_i\frac{f_{i+1}-f_i}{(x_{i+1}-xi)^2}; c_i = \frac{x_i^2(f_{i+1}-f_i)+f_i(x_{i+1}-x_i)^2}{(x_{i+1}-x_i)^2} - \frac{m_ix_ix_{i+1}}{x_{i+1}-x_i}$.

From the previous discussion, we konw that $p_i^{'}(x) = 2a_ix + b_i = 2(\frac{f_{i+1}-f_i}{(x_{i+1}-x_i)^2} - \frac{m_i}{x_{i+1}-x_i})x + m_i\frac{x_{i+1}+x_i}{x_{i+1}-x_i} - 2x_i\frac{f_{i+1}-f_i}{(x_{i+1}-x_i)^2}$. And let $x = x_{i+1}$, we have $m_{i+1} = 2\frac{f_{i+1}-f_i}{x_{i+1}-x_i} - m_i$. From the recursive relation we can prove the question.

1.3 Problem 3

Assume that $s_2(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. We already konw that $s_1(x) = 1 + c(x+1)^3$, so $s(0) = s_1(0) = 1 + c = s_2(0) = a_0; s_1^{'}(0) = s^{'}(0) = 3c = s_2^{'}(0) = a_1; s_1^{''}(x) = s^{''}(0) = 6c = s_2^{''}(0) = 2a_2$. And $s(x)$ is a natural cubic spline tells us that $s^{''}(1) = s_2^{''}(1) = 6a_3 + 2a_2 = 0$. From these equations, we know that $s_2(x) = 1 + c + 3cx + 3cx^2 - cx^3 = -1$. If we want $s(1) = s_2(1) = 1+c+3c+3c-c = -1$, from which we can obtain that $c = -\frac{1}{3}$.

1.4 Problem 4

Because $f(x) = cos(\frac{\pi}{2}x)$, we have $f(-1) = 0; f(0) = 1; f(1) = 0$. Assume that $s_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1; s_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2$. From the $s(-1) = s_1(-1) = 0; s_1(0) = s_2(0) = s(0) = 1; s_2(1) = s(1) = 0; s^{''}(-1) = s_1^{''}(-1) = 0; s^{''}(1) = s_2^{''}(1) = 0; s_1^{'}(0) = s_2^{'}(0); s_1^{''}(0) = s_2^{''}(0)$, we have $-a_1 + b_1 - c_1 + d_1 = 0; d_1 = 1; d_2 = 1; a_2 + b_2 + c_2 + d_2 = 0; c_1 = c_2; -6a_1 + 2b_1 = 0; 6a_2 + 2b_2 = 0; b_1 = b_2$. By solving these euqations, we have $a_1 = -\frac{1}{2}; b_1 = -\frac{3}{2}; c_1 = 0; d_1 = 1; a_2 = \frac{1}{2}; b_2 = -\frac{3}{2}; c_2 = 0; d_2 = 1$. So $s(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$,if $x \in [-1, 0]$; $s(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$, if $x \in (0, 1]$.

We already have $\int_{-1}^{1}[s^{''}(x)]^2dx = \int_{-1}^{0}[s^{''}(x)]^2dx + \int_{0}^{1}[s^{''}(x)]^2dx = 9$. For the first question, we assume that $g(x) = ax^2 + bx + c$. let $x = -1, 0, 1$, and we have $a - b + c = 0; c = 1; a + b + c = 0$, from which we can deduce that $a = -1; b = 0; c = 1$. So $g(x) = -x^2 + 1$, and $\int_{-1}^{1}[g^{''}(x)]^2dx = 8 > 6$. For the second question, we have $\int_{-1}^{1}[f^{''}(x)]^2dx = \frac{\pi^4}{16} > 6$.

1.5 Problem 5

$$B_i^0(x) = \begin{cases} 1, x \in (t_{i-1}, t_i] \\ 0, others \end{cases} \quad B_{i+1}^0(x) = \begin{cases} 1, x \in (t_i, t_{i+1}] \\ 0, others \end{cases} \quad B_{i+2}^0(x) = \begin{cases} 1, x \in (t_{i+1}, t_{i+2}] \\ 0, others \end{cases} \tag{1}$$

From the recursive definition of B-splines, we have

$$B_i^1(x) = \begin{cases} \dfrac{x - t_{i-1}}{t_i - t_{i-1}}, x \in (t_{i-1}, t_i] \\ \dfrac{t_{i+1} - x}{t_{i+1} - t_i}, x \in (t_i, t_{i+1}] \\ 0, others \end{cases} \quad B_{i+1}^1(x) = \begin{cases} \dfrac{x - t_i}{ti - 1 - t_i}, x \in (t_i, t_{i+1}] \\ \dfrac{t_{i+2} - x}{t_{i+2} - t_{i+1}}, x \in (t_{i+1}, t_{i+2}] \\ 0, others \end{cases} \tag{2}$$

So

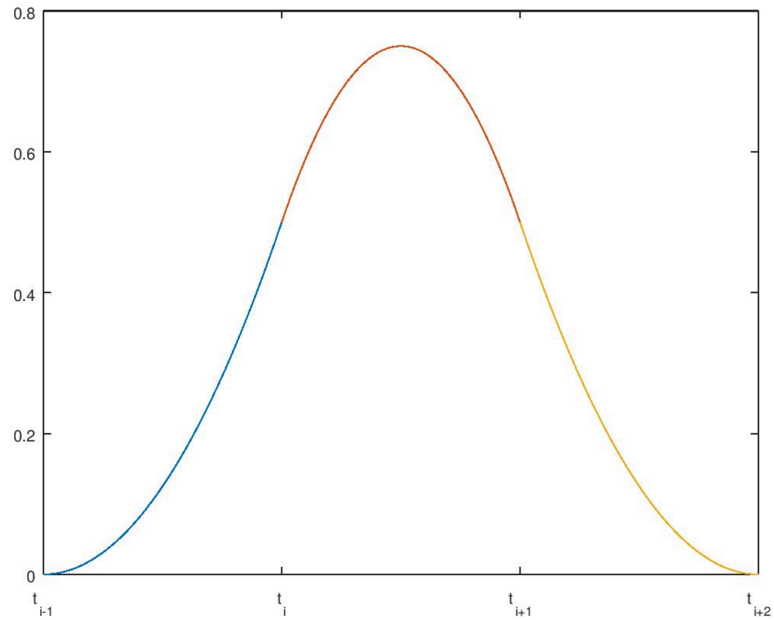
$$B_i^2(x) = \begin{cases} \dfrac{(x-t_{i-1})^2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}, x \in (t_{i-1}, t_i] \\ \dfrac{t_{i+1}-x}{t_{i+1}-t_i}\dfrac{x-t_{i-1}}{t_{i+1}-t_{i-1}} + \dfrac{t_{i+2}-x}{t_{i+2}-t_i}\dfrac{x-t_i}{t_{i+1}-t_i}, x \in (t_i, t_{i+1}] \\ \dfrac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, x \in (t_{i+1}, t_{i+2}] \\ 0, others \end{cases} \tag{3}$$

When $x = t_i$, $f'_-(x) = \frac{2}{t_{i+1}-t_{i-1}}$ and $f'_+(x) = \frac{-2t_i+t_{i+1}+t_{i-1}}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})} + \frac{1}{t_{i+1}-t_i} = \frac{2}{t_{i+1}-ti-1} = f'_-(x)$. When $x = t_{i+1}$, $f'_+(x) = -\frac{2}{t_{i+2}-t_{i+1}}$ and $f'_-(x) = -\frac{-2t_{i+1}+t_{i+2}+t_i}{(t_{i+2}-t_i)(t_{i+1}-t_i)} - \frac{1}{t_{i+1}-t_i} = -\frac{2}{t_{i+2}-ti} = f'_+(x)$.

$$s'(x) = \begin{cases} \dfrac{2(x-t_{i-1})}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}, x \in (t_{i-1}, t_i] \\ \dfrac{-2x+t_{i+1}+t_{i-1}}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})} + \dfrac{-2x+t_{i+2}+t_i}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, x \in (t_i, t_{i+1}] \\ -\dfrac{2(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, x \in (t_{i+1}, t_{i+2}] \end{cases} \tag{4}$$

It is obvious that $f'(x) = 0$ has no root when $x \in (t_{i-1}, t_i]$. We have $s'(t_i) > 0$ and $s'(t_{i+1}) < 0$, so there exist $x^* = \frac{t_{i+1}t_{i+2}-t_{i-1}t_i}{t_{i+2}+t_{i+1}-t_i-t_{i-1}} \quad s.t. s'(x^*) = 0$.

We already have $s'(x) > 0, x \in (t_{i-1}, x^*)$, $s'(x) < 0, x \in (x^*, t_{i+2})$ and $s(t_{i-1}) = s(t_{i+2}) = 0$. Let $x = x^*$, and we have $s(x^*) = \frac{(t_{i+2}t_{i+1}-t_it_{i-1})^2}{t_{i+2}+t_{i+1}-t_i-t_{i-1}} - t_{i+2}t_{i+1}t_{i-1} + t_{i+1}t_it_{i-1} - t_{i+2}t_{i+1}t_i + t_{i+2}t_it_{i-1} < 1$. So $s(x) \in [0, 1)$.



1.6 Problem 6

$$B_i^0 = (t_i - t_{i-1})[t_{i-1}, t_i](t-x)_+^0 \begin{cases} 1, x \in (t_{i-1}, t_i] \\ 0, others \end{cases} \tag{5}$$

$$B_i^1 = (t_{i+1} - t_{i-1})[t_{i-1}, t_i, t_{i+1}](t-x)_+ = [t_i, t_{i+1}](t-x)_+ - [t_{i-1}, t_i](t-x)_+ \begin{cases} \dfrac{x - t_{i-1}}{t_i - t_{i-1}}, x \in (t_{i-1}, t_i] \\ \dfrac{t_{i+1} - x}{t_{i+1} - t_i}, x \in (t_i, t_{i+1}] \\ 0, others \end{cases} \tag{6}$$

We have $B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} B_i^1(x) + \frac{t_{i+2}-x}{t_{i+2}-t_i} B_{i+1}^1 = (x-t_{i-1})[t_{i-1}, t_i, t_{i+1}](t-x)_+ + (t_{i+2})[t_i, t_{i+1}.t_{i+2}](t-x)_+ = [t_i, t_{i+1}, t_{i+2}](t-x)_+^2 - [t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = (t_{i+1} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2$.

1.7 Problem 7

We have $0 = B_i^n(x_{i+n}) - B_i^n(x_{i-1}) = \int_{t_{i-1}}^{t_{i+n}} \frac{d}{dx} B_i^n(x)dx = \frac{n}{t_{i+n-1}} \int_{t_{i-1}}^{t_{i+n-1}} B_i^{n-1}(x)dx - \frac{n}{t_{i+n}-t_i} \int_{t_i}^{t_{i+n}} B_(^{n-1} i+1)(x)dx$. So $\frac{1}{t_{i+n-1}} \int_{t_{i-1}}^{t_{i+n-1}} B_i^{n-1}(x)dx = \frac{1}{t_{i+n}-t_i} \int_{t_i}^{t_{i+n}} B_(^{n-1} i+1)(x)dx$.

1.8 Problem 8

x_i	x_i^4	
x_{i+1}	x_{i+1}^4	$(x_{i+1}^2 + x_i^2)(x_{i+1} + x_i)$
x_{i+2}	x_{i+2}^4	$x_{i+2}^2 + x_{i+1}^2 + x_i^2 + x_{i+2}x_i + x_{i+1}x_i + x_{i+2}x_{i+1}$

From the table, we have $\tau_2(x_i, x_{i+1}, x_{i+2}) = [x_i, x_{i+1}, x_{i+2}]x^4$.

We have $\tau_m(x_i) = [x_i]x^m$, and $\tau_{m-n-1}(x_i, ..., x_{i+n+1}) = \frac{\tau_{m-n}(x_{i+1},...,x_{i+n+1}) - \tau_{m-n}(x_i,...,x_{i+n})}{x_{i+n+1}-x_i} = \frac{[x_{i+1},...,x_{i+n+1}]x^m - [x_i,...,x_{i+n}]x^m}{x_{i+n+1}-x_i} = [x_i, ..., x_{i+n+1}]x^m$, which completes the proof.