15. Prove $\nabla_x(x^T a) = \nabla_x(a^T x) = a$,

$$x^T a = \sum_i^n x_i a_i$$

$$\nabla_{x_i}(x^T a) = a_i$$

$$\therefore \nabla_x(x^T a) = a$$

Same as above,

$$a^T x = \sum_i^n a_i x_i$$

$$\nabla_{x_i}(a x^T) = a_i$$

$$\therefore \nabla_x(a x^T) = a$$

16. Prove $\nabla_x(x^T A x) = (A + A^T)x$,

$$x^T A x = \sum_j^n x_j \sum_i^n x_i A_{ij} = \sum_j^n \sum_i^n x_i x_j A_{ij}$$

$$\nabla_{x_i}(x^T A x) = \nabla_{x_i}\left(\sum_i^n \sum_j^n x_i x_j A_{ij}\right) = \sum_j^n x_j A_{ij} + \sum_j^n x_j A_{ji}$$

$$= \sum_j^n x_j\left(A_{ij} + A_{ji}\right)$$

$$(A + A^T)x = \begin{matrix} \cdots & \cdots & (A_{1n} + A_{n1})x_1 \\ \vdots & (A_{ij} + A_{ji})x_i & \vdots \\ \cdots & \cdots & \cdots \end{matrix} = \nabla_x(x^T A x)$$

17. Prove $\nabla_x(x^T A x) = 2Ax$,

$$x^T A x = \sum_j^n x_j \sum_i^n x_i A_{ij} = \sum_j^n \sum_i^n x_i x_j A_{ij}$$

$$\nabla_{x_i}(x^T A x) = \nabla_{x_i}\left(\sum_i^n \sum_j^n x_i x_j A_{ij}\right) = \sum_j^n x_j A_{ij} + \sum_j^n x_j A_{ji}$$

$$= \sum_j^n 2A_{ij} x_j \left(A_{ij} = A_{ji}, when\ A\ is\ symmetric\ matrix\right)$$

$$(2A)x = \begin{matrix} \cdots & \cdots & (2A_{1n})x_1 \\ \vdots & (2A_{ij})x_i & \vdots \\ \cdots & \cdots & \ldots \end{matrix} = \nabla_x(x^T A x)$$

18. $\nabla_x\big((Ax+b)^T(Ax+b)\big) = 2A^T(Ax+b)$,

$$(Ax+b)^T(Ax+b) = \sum_j^n \left(\sum_i^n x_i A_{ij} + b\right)^2$$

$$\nabla_{x_i}\big((Ax+b)^T(Ax+b)\big) = \nabla_{x_i}\left(\sum_j^n \left(\sum_i^n x_i A_{ij} + b\right)^2\right)$$

$$= \sum_j^n 2{A_{ij}}^2 x_i + 2bA_{ij}$$

$$2A^T(Ax+b) = \begin{matrix} \cdots & \cdots & \ldots \\ \vdots & 2{A_{ij}}^2 x_i + 2bA_{ij} & \vdots \\ \cdots & \cdots & \ldots \end{matrix} = \nabla_x\big((Ax+b)^T(Ax+b)\big)$$