CS 541: Homework2

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1. Answer to question 1:

   Prove that $w_1 = w_2 = 0, b = 0$ is the solution of XOR function using equation 6.1 in textbook.

$$J(\theta) = \frac{1}{4}\sum(f^*(x) - f(x,\theta))^2 \tag{1}$$

$$\mathbf{W} = \begin{pmatrix} w1 \\ w2 \end{pmatrix}.$$

$$\mathbf{X} = \begin{pmatrix} x1 \\ x2 \end{pmatrix}. = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

$$f^*(x) = x1 or x2 = \begin{pmatrix} 0 & 1 & 1 & 1 \end{pmatrix}.$$

$$f(x,\theta) = \mathbf{x}^T\mathbf{w} + \mathbf{b} = \begin{pmatrix} b & w_2 + b & w_1 + b & w_1 + w_2 \end{pmatrix}.$$

$$\begin{aligned} J(\theta) &= \frac{1}{4}\sum(f^*(x) - f(x,\theta))^2 \\ &= \frac{1}{4}(b^2 + (w_2 + b - 1)^2 + (w_1 + b - 1)^2 + (w_2 + w_1 + b - 1)^2) \\ &= \frac{1}{4}(4b^2 - 6b + 3 + 2w_1^2 + 2w_2^2 + 4(w_1 + w_2)(b - 1) + 2w_1w_2) \end{aligned} \tag{2}$$

$$\begin{aligned} \nabla J(\theta) &= 0 \\ 8b - 4 + 4(w_1 + w_2) &= 0 \\ 4w_1 + 4(2b - 1) + 2w_2 &= 0 \\ 4w_2 + 4(2b - 1) + 2w_1 &= 0 \end{aligned}$$

   Solution:

$$\begin{aligned} w_1 &= 0 \\ w_2 &= 0 \\ b &= 0.5 \end{aligned}$$

2. Answer to question 3:

$$J(\theta) = \frac{1}{2m}\sum(\hat{y}^j - y^j)^2 + \frac{1}{2}w^T w$$

1. Add Extra component:

$$\hat{\mathbf{w}} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \\ 1 \end{pmatrix}.$$

2. Add $J(\theta)$ the L2 penalty term $Penalty$:

$$J(\theta) = \frac{1}{2m}\sum(\hat{y}^j - y^j)^2 + \frac{1}{2}\hat{w}^T\hat{w} + Penalty^T\hat{w}$$

$$\mathbf{Penalty} = \begin{pmatrix} p \\ \vdots \\ p \\ 0 \end{pmatrix}.$$

3. Add $J(\theta)$ training example weight w_l :

$$w_l = [w_l^1 \cdots w_l^j \cdots w_l^m]$$

$$J(\theta) = \frac{1}{2m}\sum w_l^j(\hat{y}^j - y^j)^2 + \frac{1}{2}\hat{w}^T\hat{w} + Penalty^T\hat{w} \tag{-4}$$

Derive:

$$\nabla J(\theta) = 0$$

$$\nabla_{\hat{w}_i}(\frac{1}{2m}\sum w_l^j(\hat{y}^j - y^j)^2 + \frac{1}{2}\hat{w}^T\hat{w} + Penalty^T\hat{w}) = 0$$

$$\nabla_{\hat{w}_i}(\frac{1}{2m}\sum w_l^j(\hat{y}^j - y^j)^2) + \hat{w}_i + Penalty_i = 0$$

$$\frac{1}{2m}\sum w_l^j\nabla_{\hat{w}_i}((\hat{y}^j - y^j)^2) + \hat{w}_i + Penalty_i = 0$$

$$\frac{1}{2m}\sum w_l^j\hat{w}_i(2\hat{y}^j - 2y^j)\nabla\hat{y}^j + \hat{w}_i + Penalty_i = 0$$

$$\frac{1}{m}\sum_{j=1}^{j=m} w_l^j\hat{w}_i(\hat{y}^j - y^j)x^j + \hat{w}_i + Penalty_i = 0$$

$$\nabla_{\hat{w}_i}J(\theta) = \frac{1}{m}\sum_{j=1}^{m} w_l^j\hat{w}_i(\hat{y}^j - y^j)x^j + \hat{w}_i + Penalty_i \tag{-10}$$

3. Answer to question 4:

Set $\omega = \begin{bmatrix}\omega_1 & \omega_2\end{bmatrix}$, $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$. The equation can be derived as following:

$$\omega^T S\omega = \begin{bmatrix}\omega_1 & \omega_2\end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix}$$

$$= \begin{bmatrix} \omega_1 S_{11} + \omega_2 S_{21} & \omega_1 S_{12} \omega_2 S_{21} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$= {\omega_1}^2 S_{11} + \omega_1 \omega_2 (S_{21} + S_{12}) + {\omega_2}^2 S_{22}$$

To ensure the symmetric, thus $(\omega_1 - \omega_2)^2 = 0$. Then $S_{11} = S_{22}$.

Assume $S_{11} = S_{22} = 1$, $S_{21} + S_{12} = -2$. Thus S could be: $S = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

4. Answer to question 5:

   Proof:

$$P(x_t|y_1, y_2, ..., y_t) \propto P(y_t|x_t) \prod_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}|y_1, ..., y_{t-1})$$

   According to the Bayes' rule, $P(a|b,c) = \frac{P(b|a,c)}{P(a|c)}$, $P(x_t|y_1, y_2, ..., y_t)$ can be derived as:

$$P(x_t|y_1, y_2, ..., y_t) = P(y_t|x_t, y1, ..., y_{t-1}) P(x_t|y1, ..., y_{t-1}) / P(y_t|y_1, ..., y_{t-1})$$

$$= P(y_t|x_t) . \frac{P(x_t|y_1, ..., y_{t-1})}{P(y_t|y_1, ..., y_{t-1})}$$

$$= P(y_t|x_t) P(y_{t-1}|x_{t-1}) \frac{P(x_{t-1}|y_1, ..., y_{t-1})}{P(y_t|y_1, ..., y_{t-1}) P(y_{t-1}|y_1, ..., y_{t-2})}$$

$$\propto P(y_t|x_t) P(y_{t-1}|x_{t-1}) P(x_{t-1}|y_1, ..., y_{t-1}) \tag{-17}$$

$$\propto P(y_t|x_t) \prod_{i=1}^{t-1} P(x_t|x_{t-1}) P(x_{t-1}|y_1, ..., y_{t-1}) \tag{-17}$$