Tutorial Questions

Question 1.

Determine whether the following functions are linear maps.

a. $T_1 : \mathbf{R}^2 \to \mathbf{R}$ *given by* $T_1(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]) = xy$.

b. $T_2 : \mathbf{C} \to \mathbf{C}$ *given by* $T_2(z) = \bar{z}$, *where* $\mathbf{C}$ *is regarded as a real vector space* *(*$\overline{a+bi} = a - bi$ *is the complex conjugation).*

c. $T_2 : \mathbf{C} \to \mathbf{C}$ *given by* $T_2(z) = \bar{z}$, *where* $\mathbf{C}$ *is regarded as a complex vector space.*

d. $T_3 : \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ *given by* $T_3(p) = p(2)$.

Solution. a. T_1 is not linear. Indeed, $T_1\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right) = T_1\left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\right) = 1 \cdot 1 = 1$ while $T_1\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) + T_1\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right) = 1 \cdot 0 + 0 \cdot 1 = 0 \neq 1$.

To go further

We have actually showed for any $\mathbf{F}$ the map $T_1 : \mathbf{F}^2 \to \mathbf{F}$, $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \mapsto xy$ is not linear.

b. T_2 is $\mathbf{R}$-linear. Indeed, for $z, w \in \mathbf{C}$ and $\lambda \in \mathbf{R}$ we have $T_2(\lambda z + w) = \overline{\lambda z + w} = \overline{\lambda}\overline{z} + \overline{w} = \lambda\overline{z} + \overline{w} = \lambda T_2(z) + T_2(w)$. The above equalities can be verified explicitly by writing $z = a + bi$ and $w = c + di$.

c. T_2 is not $\mathbf{C}$-linear. Indeed, $iT_2(1) = i\overline{1} = i$, but $T_2(i1) = \overline{i} = -i \neq i$.

d. T_3 is linear. Indeed, let p and q by two polynomials in $\mathcal{P}(\mathbf{R})$ and $\lambda \in \mathbf{R}$. Then $T_3(\lambda p + q) = (\lambda p + q)(2) = \lambda p(2) + q(2) = \lambda T(p) + T(q)$.

To go further

The same proof shows that for any field $\mathbf{F}$, and any element $a \in \mathbf{F}$, the map $\mathcal{P}(\mathbf{F}) \to \mathbf{F}, p \mapsto p(a)$ is linear. Even more generally, if S is any set then the map $\mathbf{F}^S \to \mathbf{F}, f \mapsto f(a)$ is linear. This map is called the **evaluation at** a.

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For a linear map $T \in \mathcal{L}(U, V)$, we write $\ker(T) = \mathrm{null}(T)$ for its kernel (also called nullspace): $\ker(T) = \{u \in U \,|\, T(u) = 0\}$. We write $\mathrm{Im}(T) = \mathrm{range}(T)$ for the image of T (also called range): $\mathrm{Im}(T) = \{v \in V |\, \exists u \in U : T(u) = v\}$.

Question 2.

Let U, V, and W be vector spaces, and suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are linear maps.

a. Prove that $\ker(T)$ is a subspace of $\ker(ST)$.

b. Prove that $\operatorname{Im}(ST)$ is a subspace of $\operatorname{Im}(S)$.

Solution. We have

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, "S"] & W. \end{tikzcd}$$

a. Both T and ST have domain U, so both $\ker(T)$ and $\ker(ST)$ are subspaces of U. It is thus enough to prove that $\ker(T)$ is a *subset* of $\ker(ST)$. Let u be any element of $\ker(T)$, so $T(u) = 0$. Then $(ST)(u) = S(T(u)) = S(0) = 0$ and hence u is in $\ker(ST)$ as desired.

b. Both S and ST have codomain W, so both $\operatorname{Im}(S)$ and $\operatorname{Im}(ST)$ are subspaces of W. It is thus enough to prove that $\operatorname{Im}(ST)$ is a *subset* of $\operatorname{Im}(S)$. Let w be any element in $\operatorname{Im}(ST)$. So there exists $u \in U$ with $(ST)u = w$. In particular, there exists $v = T(u)$ in V (the domain of S) such that $S(v) = S(T(u)) = (ST)(u) = w$ and hence w is in $\operatorname{Im}(S)$ as desired.

Remark 0.1.

The inclusion $\ker(T) \subseteq \ker(ST)$ might be interpreted as ST *is less injective than* T. We recover the particular case: if ST is injective then so is T.
The inclusion $\operatorname{Im}(ST) \subseteq \operatorname{Im}(S)$ might be interpreted as ST *is less surjective than* S. We recover the particular case: if ST is surjective then so is S.

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Question 3.

Let $T \in \mathcal{L}(V, W)$ be an injective linear map. Suppose that $(v_1, \ldots, v_k)$ is a linearly independent list of vectors in V. Prove that $(Tv_1, \ldots, Tv_k)$ is a linearly independent list in W.

Solution. Let $\lambda_1,\ldots, \lambda_k$ be scalars such that

$$0 = \sum_{j=1}^{k} \lambda_j T v_j.$$

We need to prove that all the λ_j are 0. By linearity of T we have $0 = T\Big(\sum_{j=1}^{k} \lambda_j v_j\Big)$. By injectivity of T one obtains

$$0 = \sum_{j=1}^{k} \lambda_j v_j.$$

By linear independence of the $(v_1, \dots, v_k)$ we conclude that all the λ_j are 0. ■

Question 4.

Let V, W be vector spaces such that W is finite-dimensional. Let $T \in \mathcal{L}(V, W)$ be a linear map.

a. *Prove that T is surjective if and only if there exists a linear map $S \in \mathcal{L}(W, V)$ such that $TS = \mathrm{Id}_W$. (Hint: Define S on any basis of W then extend linearly.)*

b. *Prove that T is injective if and only if there exists a linear map $S \in \mathcal{L}(W, V)$ such that $ST = \mathrm{Id}_V$. (Hint: Similar to the above, but choose a basis of W carefully.)*

Solution. First of all, we know that for functions $f\colon X \to Y$ and $g\colon Y \to Z$ we have:

- if $g \circ f$ is injective then f is injective;
- if $g \circ f$ is surjective then g is surjective.

Since Id_X is bijective, this directly shows:

a. if $ST = \mathrm{Id}_V$ then T is injective;

b. if $TS = \mathrm{Id}_W$ then T is surjective.

Observe that we don't need to suppose that W is finite dimensional, or that S is a linear map (S a function is enough).

For the other direction, we will use bases to translate our problem for linear maps on vector spaces to a problem about functions on sets. We can do this, because a linear map $T\colon V \to W$ is uniquely determined by what it does on a given basis $\mathscr{B}$ of V.

a. Let $n = \dim W$. Suppose that T is surjective and let $\mathscr{C} = (w_1, \dots, w_n)$ be any basis of W. For any $w_j \in \mathscr{C}$, choose a preimage $v_j \in V$ so $T(v_j) = w_j$. There exists at least one such v_j by surjectivity. We define $S\colon \mathscr{C} \to V$ by $S(w_j) = v_j$ and extend it to a linear map $S \in \mathcal{L}(W, V)$. Then for any w_j we have $TSw_j = w_j$. Since TS is the identity on the basis $\mathscr{C}$, it is the identity everywhere: $TS = \mathrm{Id}_W$.

b. Suppose that T is injective and let $\mathscr{D} = (w_1, \dots, w_k)$ be a basis of $\mathrm{Im}(T)$. Since T is injective, for every $j \in \{1, \dots, k\}$ there exists a unique $v_j \in V$ with $Tv_j = w_j$. We claim that $\mathscr{A} = (v_1, \dots, v_k)$ is a basis of V.

   First, we prove that $\mathscr{A}$ is a spanning family for V. Let v be any vector in V. Then $Tv \in \mathrm{Im}(T) = \mathrm{span}(\mathscr{D})$ so there exists λ_j, $j \in \{1, \dots, k\}$ such that

$$T(v) = \sum_{j=1}^{k} \lambda_j w_j = \sum_{j=1}^{k} \lambda_j T v_j = T\Big(\sum_{j=1}^{k} \lambda_j v_j\Big).$$

   By injectivity, $v = \sum_{j=1}^{k} \lambda_j v_j$ is in $\mathrm{span}(\mathscr{A})$.

We now prove that $\mathscr{A}$ is linearly independent. Suppose that

$$0 = \sum_{j=1}^{k} \lambda_j v_j$$

for some λ_j. Applying T on both sides, one obtain $0 = \sum_{j=1}^{k} \lambda_j w_j$ and conclude that all the λ_j are 0, proving linear independence of $\mathscr{A}$.

Now, since $\mathscr{D}$ is a basis of $\mathrm{Im}(T) \subseteq W$, we can extend it to a basis $\mathscr{C} = (w_1, \dots, w_k, w_{k+1}, \dots, w_m)$ of W. We define S on $\mathscr{C}$ by:

$$S(w_j) = \begin{cases} v_j & \text{if } j \in \{1, \dots, k\} \\ 0 & \text{otherwise} \end{cases}$$

and then extend it to a linear map $S \in \mathcal{L}(W, V)$. One have $ST(v_j) = v_j$ for all $j \in \{1, \dots, k\}$ and so $ST = \mathrm{Id}_V$.

Infinite dimensional vector spaces

The statements remain true for a general, not necessarily finite dimensional, vector space W if we assume (AC). We use (AC) in two different places. Firstly, for the existence of a basis of W, and secondly when T is injective and we *choose* a preimage of w_j.

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Question 5.

Rephrase the following problems in terms of linear maps between vector spaces. (Do not solve the problems.)

a. *Find all smooth functions $f\colon \mathbf{R} \to \mathbf{R}$ that satisfy $f'' - 3f' + 2f = 0$.*

b. *Find all real sequences $(x_0, x_1, \dots)$ that satisfy $x_{n+2} = x_{n+1} + x_n + 1$ for all $n \in \mathbf{N}$.*

Solution. a. The space of smooth functions is $\mathcal{C}^\infty(\mathbf{R}) = \{f\colon \mathbf{R} \to \mathbf{R} \mid f \text{ smooth}\} = \{f\colon \mathbf{R} \to \mathbf{R} \mid \forall n : f^{(n)} \text{ exists}\}$. Let $D\colon \mathcal{C}^\infty(\mathbf{R}) \to \mathcal{C}^\infty(\mathbf{R}), f \mapsto f'$ be the differentiation operator. This is a linear map. Define $T := D^2 - 3D + 2\,\mathrm{Id}$, where $D^2 = D \circ D$. This is also a linear map from $\mathcal{C}^\infty(\mathbf{R})$ to itself and $T(f) = f'' - 3f' + 2f$. We have $f'' - 3f' + 2f = 0$ if and only if $f \in \ker(T)$.

b. The space of real sequences is $\mathbf{R}^{\mathbf{N}} = \{(x_0, x_1, x_2, \dots) \mid \forall j : x_j \in \mathbf{R}\}$. Let $S\colon \mathbf{R}^{\mathbf{N}} \to \mathbf{R}^{\mathbf{N}}$ be the backward shift: $S(x_0, x_1, \dots) = (x_1, x_2, \dots)$. Finally, let $c = (1, 1, \dots)$ be the constant sequence 1. We want to find all sequences $x \in \mathbf{R}^{\mathbf{N}}$ such that $S^2(x) = S(x) + x + c$. So if we define $T = S^2 - S - \mathrm{Id}$, a sequence x satisfies the questions if and only if $T(x) = c$.

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Further Questions

Question 6.

Let U, V, and W be vector spaces, and suppose $S_1, S_2 \in \mathcal{L}(U,V)$ and $T \in \mathcal{L}(V,W)$ are linear maps. Prove that the following distributive property holds: $T(S_1+S_2) = TS_1 + TS_2$.

Solution. This is similar to the proof of $(T_1+T_2)S = T_1S + T_2S$ viewed in class. Let u be any vector in U. Then

$$\begin{aligned}(T(S_1+S_2))u &= T((S_1+S_2)u)\\ &= T(S_1u + S_2u)\\ &= T(S_1u) + T(S_2u)\\ &= (TS_1)u + (TS_2)u = (TS_1+TS_2)u\end{aligned}$$

and thus $T(S_1+S_2) = TS_1 + TS_2$. ■

Question 7.

Let $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ be linear maps. Show that $ST = 0 \in \mathcal{L}(U,W)$ (the zero map) if and only if $\operatorname{Im}(T)$ is a subspace of $\ker S$.

Solution. "$\Rightarrow$" Suppose $ST = 0$. Let v be any element in $\operatorname{Im}(T)$. By assumption, there exists $u \in U$ such that $Tu = v$. But then, $Sv = S(Tu) = (ST)u = 0u = 0$ and thus v is in $\ker(S)$ as desired.

"$\Leftarrow$" Suppose that $\operatorname{Im}(T) \subseteq \ker(S)$. Then for any $u \in U$ we have $(ST)(u) = S(Tu) = 0$, showing that ST is the 0 map. ■