Tutorial Questions

Question 1.

Let $D\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ be the differentiation operator. Is D invertible?

Solution. The operator D is not injective and hence not invertible. Indeed, both the constant polynomial 0 and the constant polynomial 1 are sent onto 0. ■

Question 2.

Let U be the subspace of $\mathcal{C}^\infty(\mathbf{R})$ spanned by the basis $\mathscr{B}$ given by (e^x, xe^x, x^2e^x). Let $D \in \mathcal{L}(U)$ be the differentiation operator. Find the matrix $[D]_{\mathscr{B}}^{\mathscr{B}}$.

Solution. We compute each column of $[D]_{\mathscr{B}}^{\mathscr{B}}$.

Firstly, $D(e^x) = e^x = 1 \cdot e^x + 0 \cdot xe^x + 0 \cdot x^2e^x$ so $[D(e^x)]_{\mathscr{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$. Then, $D(xe^x) = e^x + xe^x$ and therefore $[D(xe^x)]_{\mathscr{B}} = \begin{bmatrix}1\\1\\0\end{bmatrix}$. Finally, $D(x^2e^x) = 2xe^x + x^2e^x$ and therefore $[D(xe^x)]_{\mathscr{B}} = \begin{bmatrix}0\\2\\1\end{bmatrix}$.

Altogether, we have

$$[D]_{\mathscr{B}}^{\mathscr{B}} = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}.$$

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Question 3.

Let $T \in \mathcal{L}(V, W)$ be a linear map and suppose $S_1, S_2 \in \mathcal{L}(W, V)$ satisfy $S_1T = \mathrm{Id}_V$ and $TS_2 = \mathrm{Id}_W$. Prove that T is invertible. What is the inverse of T?

Solution. We have

$$S_1 = S_1 \mathrm{Id}_W = S_1TS_2 = Id_VS_2 = S_2.$$

Therefore $S_1 = S_2$ is the inverse of T, which is therefore invertible. ■

Question 4.

Let $T \in \mathcal{L}(V, W)$ be a surjective linear map. Suppose that U is a subspace of V such that $\ker(T) \oplus U = V$. Prove that the restriction $T|_U\colon U \to W$ is an isomorphism (recall that the restriction is defined by $T|_U(u) = Tu$ for all $u \in U$).

Solution. The map $T|_U$ is the restriction of a linear map, and thus linear. It remains to show that it is injective and surjective.

We first prove injectivity. We both have $\ker(T|_U) \subseteq U$ and $\ker(T|_U) \subseteq \ker(T)$. We conclude $\ker(T|_U) \subseteq U \cap \ker(T)$. But since the sum $U \oplus \ker(T)$ is direct, we have $U \cap \ker(T) = \{0\}$, proving that $T|_U$ is injective.

We now prove surjectivity. Let $w \in W$ be any vector. By surjectivity of T, there exists $v \in V$ such that $T(v) = w$. Since $V = \ker(T) \oplus U$, there exist (unique) $u_1 \in \ker(T)$ and $u_2 \in U$ such that $v = u_1 + u_2$. So $w = T(v) = T(u_1 + u_2) = T(u_1) + T(u_2) = 0 + T(u_2) = T|_U(u_2)$, proving surjectivity of $T|_U$. ■

Question 5.

Let $x_1, \dots, x_{n+1} \in \mathbf{R}$ be distinct. Define a linear map $T\colon \mathcal{P}(\mathbf{R})_n \to \mathbf{R}^{n+1}$ by $T(p) := [p(x_1), \dots, p(x_{n+1})]^\mathsf{T}$.

a. *Prove that T is injective. (Hint: By the division algorithm, any non-zero degree n polynomial has at most n distinct roots.)*

b. *Show that for any $y_1, \dots, y_{n+1} \in \mathbf{R}$, there exists a unique polynomial p of degree at most n such that $p(x_i) = y_i$ for all $1 \leq i \leq n+1$. (Hint: Show that T is an isomorphism.)*

Solution.

a. Let p be any polynomial in $\ker(T)$. So $p(x_1) = \dots = p(x_{n+1}) = 0$. That is, p is a polynomial of degree at most n with (at least) $n+1$ distinct roots. We conclude that p is the zero polynomial and so that $\ker(T) = \{0\}$.

b. Both $\mathcal{P}(\mathbf{R})_n$ and $\mathbf{R}^{n+1}$ have dimension $n+1$. Therefore T being injective is also surjective, and hence bijective. In particular, for any $y_1, \dots, y_{n+1} \in \mathbf{R}$ there exists a unique polynomial $p \in \mathcal{P}(\mathbf{R})_n$ such that $T(p) = [y_1, \dots, y_{n+1}]^\mathsf{T}$.

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Further Questions

Question 6.

[Continuing from Question 5, if you want to solve it explicitly] Let $(e_1, \dots, e_{n+1})$ be the standard basis for $\mathbf{R}^{n+1}$ and $p_i = T^{-1}(e_i) \in \mathcal{P}(\mathbf{R})_n$. In other words, p_i is the unique polynomial of degree at most n satisfying $p_i(x_i) = 1$ and $p_i(x_j) = 0$ for $i \neq j$.

a. *Find an explicit formula for $p_i(x)$. (Hint: a non-zero polynomial p satisfies $p(a) = 0$ if and only if $(x - a)$ is a factor of p.)*

b. *Write down $T^{-1}[y_1, \dots, y_{n+1}]^\mathsf{T}$ in terms of the p_i's and y_i's. (This is known as Lagrange's Interpolation Formula.)*

Solution. a. We first construct polynomials that satisfy $q_i(x_j) = 0$ if $i \neq j$, but might not have the correct value for $q_i(x_i) = 0$. This is quite easy.

Let $q_1(x) := (x - x_2) \cdots (x - x_{n+1})$ and define similarly $q_2 := (x - x_1)(x - x_3) \cdots (x - x_{n+1})$ and $q_3, \dots, q_n$ up to $q_{n+1}(x) := (x - x_1)(x - x_2) \cdots (x - x_{n+1} + 1)$. In other words,

$$q_i = \prod_{\substack{1 \le k \le n+1 \\ k \neq i}} (x - x_k).$$

Then for any $1 \le j \le n+1$ one have

$$q_i(x_j) = \begin{cases} 0 & \text{if } i \neq j, \\ \prod_{\substack{1 \le k \le n+1 \\ k \neq i}} (x_i - x_k) \neq 0 & \text{if } i = j. \end{cases}$$

It is now easy to construct the p_i, by normalising their value at x_i:

$$p_i = \frac{\prod_{\substack{1 \le k \le n+1 \\ k \neq i}} (x - x_k)}{\prod_{\substack{1 \le k \le n+1 \\ k \neq i}} (x_i - x_k)} = \prod_{\substack{1 \le k \le n+1 \\ k \neq i}} \frac{(x - x_k)}{(x_i - x_k)}.$$

b. We have $[y_1, \dots, y_{n+1}]^\intercal = y_1 e_1 + \cdots y_{n+1} e_{n+1}$. By linearity

$$T^{-1}[y_1, \dots, y_{n+1}]^\intercal = \sum_{i=1}^{n+1} y_i T^{-1}(e_i) = \sum_{i=1}^{n+1} y_i p_i = \sum_{i=1}^{n+1} y_i \prod_{\substack{1 \le k \le n+1 \\ k \neq i}} \frac{(x - x_k)}{(x_i - x_k)}.$$

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Question 7.

Prove that V is isomorphic to $\mathcal{L}(\mathbf{F}, V)$. (Hint: consider the function $F\colon \mathcal{L}(\mathbf{F}, V) \to V$ defined by $F(T) = T(1)$. This function is called evaluation at 1.)

Solution. We need to prove that F is a bijective linear map.

We first prove injectivity. Let T be an element of the kernel of F. That is: $T\colon \mathbf{F} \to V$ is a linear map such that $T(1) = 0$. But then, for every $\lambda \in \mathbf{F}$ we have $T(\lambda) = T(\lambda \cdot 1) = \lambda T(1) = \lambda \cdot 0 = 0$. This proves that $\ker(F) = \{0\}$ and so F is injective.

For surjectivity, let v be any vector in V. Since 1 is a basis of $\mathbf{F}$, there exists a unique linear map $T\colon \mathbf{F} \to V$ satisfying $T(1) = v$. We have $F(T) = v$ as desired.

Alternatively, one can prove bijectivity of F by finding an inverse. Let $G\colon V \to \mathcal{L}(\mathbf{F}, V)$ be the map defined by $G(v)\colon \mathbf{F} \to V, \lambda \mapsto \lambda.v$. That is, each v is send by G onto the map $G(v) = T$ from the surjectivity proof. One easily verify that $(G \circ F)(T) = T$ and $(F \circ G)(v) = v$, proving that G is the inverse of F.

Finally, we show that F is linear. Let $S, T \mathcal{L}(\mathbf{F}, V)$ be two linear maps and let $\lambda \in \mathbf{F}$ be a scalar. Then $F(\lambda S + T) = (\lambda S + T)(1) = \lambda \cdot S(1) + T(1) = \lambda F(S) + F(T)$ as desired. ■