Tutorial Questions

Question 1.

Let V be an inner product space. Suppose that $u, v \in V$ are vectors satisfying $\langle u, u\rangle = 4$, $\langle v, v\rangle = 6$ and $\langle u, v\rangle = -2$. What is $\|3u - 2v\|$?

Solution. We have

$$\begin{aligned} \|3u - 2v\|^2 &= \langle 3u - 2v, 3u - 2v\rangle = 9\langle u, u\rangle - 6\langle u, v\rangle - 6\langle v, u\rangle + 4\langle v, v\rangle \\ &= 9 \cdot 4 - 6 \cdot (-2) - 6 \cdot \overline{-2} + 4 \cdot 6 = 84. \end{aligned}$$

So $\|3u - 2v\| = \sqrt{84} = 2\sqrt{21}$. ∎

Question 2.

Consider the vector space $\mathcal{P}(\mathbf{R})$ equipped with the inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$. Write the polynomial x as a sum $p + q$, where p is a scalar multiple of 1 and q is orthogonal to 1.

Solution. Recall that if V is an inner product space and u and v are two vectors with $v \neq 0$ we have $u = \frac{\langle u,v\rangle}{\langle v,v\rangle} v + w$ with w orthogonal to v. In our case, we have $p = \frac{\langle x,1\rangle}{\langle 1,1\rangle} 1$ and $q = x - p$. To find p we compute

$$\langle x, 1\rangle = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2} \qquad \text{and} \qquad \langle 1, 1\rangle = \int_0^1 1\,dx = [x]_0^1 = 1.$$

So $p = \frac{1}{2}$ and $q = x - \frac{1}{2}$. ∎

Question 3.

Prove the converse of the Pythagorean Theorem:

If u and v are vectors in a real inner product space V satisfying $\|u\|^2 + \|v\|^2 = \|u + v\|^2$ then u and v must be orthogonal.

Explain why the assumption that V is a real inner product space is necessary.

Solution. We have

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v\rangle = \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\ &= \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \langle v, u\rangle. \end{aligned}$$

Therefore, the equality $\|u\|^2 + \|v\|^2 = \|u + v\|^2$ implies (in fact is equivalent to) $0 = \langle u, v\rangle + \langle v, u\rangle = \langle u, v\rangle + \overline{\langle u, v\rangle}$. If V is a real inner product space we conclude $0 = \langle u, v\rangle$. In other words, u and v are orthogonal.

The above proof does not work for complex inner product spaces. But what about the conclusion? It also fails for any complex inner product space $V \neq \{0\}$. Indeed, let $v \neq 0$ be a non-zero vector and let $u = \mathrm{i}v$. Then u and v are not orthogonal since $\langle u, v\rangle = \mathrm{i} \cdot \langle v, v\rangle = \neq 0$. However, we have $\langle u, v\rangle + \langle v, u\rangle = \mathrm{i}\langle v, v\rangle + \bar{\mathrm{i}}\langle v, v\rangle = 0$ and thus $\|u\|^2 + \|v\|^2 = \|u + v\|^2$. ∎

Question 4.

*Recall that the **mean** of n real numbers $x_1, \dots, x_n$ is equal to $\frac{1}{n}\sum_{i=1}^{n} x_n$. Prove that the square of the mean of $x_1, \dots, x_n$ is less than or equal to the mean of $x_1^2, \dots, x_n^2$.*

Solution. We want to prove that

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_n\right)^2 \leq \frac{1}{n}\sum_{i=1}^{n} x_n^2,$$

or equivalently that

$$\left(\sum_{i=1}^{n} x_n\right)^2 \leq n\sum_{i=1}^{n} x_n^2.$$

In order to do that, we will use the Cauchy-Schwarz inequality: $|\langle u, v\rangle|^2 \leq \|u\|^2\|v\|^2$ for $u = [x_1, \dots, x_n]^{\mathsf{T}}$ and $v = [1, \dots, 1]^{\mathsf{T}}$ in $\mathbf{R}^n$ and the standard dot product and the corresponding norm $\|v\| = \sqrt{v \bullet v}$. One easily check

$$(u \bullet v)^2 = (x_1 + \dots + x_n)^2,$$
$$\|u\|^2 = x_1^2 + \dots + x_n^2,$$
$$\|v\|^2 = n$$

and thus the desired formula follows. ∎

Question 5.

*[From Calculus] Let $\gamma\colon \mathbf{R} \to \mathbf{R}^n$ be a **smooth path**, i.e., $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ where each $\gamma_i\colon \mathbf{R} \to \mathbf{R}$ is a smooth function. Its derivative is given by $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t))$. Prove that if γ and δ are smooth paths in $\mathbf{R}^n$, then for any t we have $(\gamma(t) \bullet \delta(t))' = (\gamma'(t) \bullet \delta(t)) + (\gamma(t) \bullet \delta'(t))$ for the standard dot product on $\mathbf{R}^n$.*

Solution. For a fixed t, the dot product is $\gamma(t) \bullet \delta(t) = \sum_{i=1}^{n} \gamma_i(t)\delta_i(t)$. So the function $t \mapsto (\gamma(t) \bullet \delta(t))$ is a smooth function from $\mathbf{R}$ to $\mathbf{R}$ and

$$\begin{aligned}(\gamma(t) \bullet \delta(t))' &= \left(\sum_{i=1}^{n} \gamma_i(t)\delta_i(t)\right)' = \sum_{i=1}^{n} (\gamma_i(t)\delta_i(t))' \\ &= \sum_{i=1}^{n} (\gamma_i'(t)\delta_i(t) + \gamma_i(t)\delta_i'(t)) = \sum_{i=1}^{n} \gamma_i'(t)\delta_i(t) + \sum_{i=1}^{n} \gamma_i(t)\delta_i'(t) \\ &= (\gamma'(t) \bullet \delta(t)) + (\gamma(t) \bullet \delta'(t)).\end{aligned}$$ ∎

Further Questions

Question 6.

[Continuing from Question 5] Suppose that $\gamma\colon \mathbf{R} \to \mathbf{R}^n$ is a smooth path of constant speed, i.e. $\|\gamma'(t)\|$ is constant. Prove that $\gamma'(t)$ and $\gamma''(t)$ are orthogonal for all t. (This shows that the velocity and acceleration along a constant speed path are always orthogonal.)

Solution. Since $\|\gamma'(t)\|^2 = \gamma'(t) \bullet \gamma'(t)$ is constant, we have

$$0 = (\gamma'(t) \bullet \gamma'(t))' = (\gamma''(t) \bullet \gamma'(t)) + (\gamma'(t) \bullet \gamma''(t)) = 2\gamma''(t) \bullet \gamma'(t)$$

since $\gamma''(t) \bullet \gamma'(t)$ is a real number. Therefore $0 = \gamma''(t) \bullet \gamma'(t)$ for every $t \in \mathbf{R}$. ■

Question 7.

*[From Statistics] Suppose that we have some data given as n pairs of real numbers $(x_1, y_1), \dots, (x_n, y_n)$. The **sample correlation coefficient** is given by the following formula:*

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}},$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_n$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^n y_n$. Prove that $-1 \le r \le 1$. What can you say about the value of r if:

a. A fixed constant is added to all x values (or all y values)?

b. We scale each x value (or each y value) by a fixed positive constant?

Solution. Let $x = [x_1, \dots, x_n]^\mathsf{T}$, $y = [y_1, \dots, y_n]^\mathsf{T}$ and $u = [1, \dots, 1]^\mathsf{T}$ be vectors in $\mathbf{R}^n$. Then

$$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i = \frac{x \bullet u}{u \bullet u} \qquad \text{and} \qquad \bar{y} = \frac{1}{n}\sum_{i=1}^n y_i = \frac{y \bullet u}{u \bullet u}.$$

Now, let

$$\tilde{x} = x - \frac{x \bullet u}{u \bullet u}u = [x_1 - \bar{x}, \dots, x_n - \bar{x}]^\mathsf{T} \quad \text{and} \quad \tilde{y} = y - \frac{y \bullet u}{u \bullet u}u = [y_1 - \bar{y}, \dots, y_n - \bar{y}]^\mathsf{T}.$$

We therefore have

$$r = \frac{\tilde{x} \bullet \tilde{y}}{\|\tilde{x}\|\|\tilde{y}\|}.$$

By the Cauchy-Schwarz inequality we have $|\tilde{x} \bullet \tilde{y}| \le \|\tilde{x}\|\|\tilde{y}\|$, that is: $-1 \le r \le 1$.

a. Adding a fixed constant λ to all x_i is the same as replacing x by $x + \lambda u$. But then $\overline{x + \lambda u} = \bar{x} + \lambda$ and $\tilde{x}$ is replaced by

$$\widetilde{x + \lambda u} = [(x_1 + \lambda) - (\bar{x} + \lambda), \dots, (x_n + \lambda) - (\bar{x} + \lambda)]^\mathsf{T} = \tilde{x}.$$

So r remains the same.

b. Scaling each value by a positive λ is the same as replacing x by $\lambda x = [\lambda x_1, \dots, \lambda x_n]^\mathsf{T}$. We have $\overline{\lambda x} = \lambda \bar{x}$ and thus $\widetilde{\lambda x} = \lambda \tilde{x}$. Therefore r becomes

$$r = \frac{\lambda \tilde{x} \bullet \tilde{y}}{\|\lambda \tilde{x}\| \|\tilde{y}\|} = \frac{\lambda \tilde{x} \bullet \tilde{y}}{|\lambda| \|\tilde{x}\| \|\tilde{y}\|} = \frac{\tilde{x} \bullet \tilde{y}}{\|\tilde{x}\| \|\tilde{y}\|}.$$

So r remains the same (observe that it is crucial that λ is positive).

■