Tutorial Questions

Question 1.

Let $P \in \mathcal{L}(V)$ be a projection, i.e. $P^2 = P$. Let $b \in V$. Show that there exists a solution $x \in V$ to $Px = b$ if and only if $Pb = b$.

Solution. If $Pb = b$, then $x = b$ is a solution to $Px = b$.

Suppose that x is a solution to $b = Px$. Then applying P to both sides we have $Pb = P^2x = Px = b$. ■

Question 2.

Suppose that $P \in \mathcal{L}(V)$ is a projection. Prove that $\mathrm{Id} - P \in \mathcal{L}(V)$ is also a projection (it is sometimes called the complementary projection *to P). What is the relationship between the image and kernel of P and $\mathrm{Id} - P$?*

Solution. The identity function $\mathrm{Id} = \mathrm{Id}_V$ is in $\mathcal{L}(V)$. So P is in $\mathcal{L}(V)$ if and only if $\mathrm{Id} - P$ is in $\mathcal{L}(V)$, which proves that $\mathrm{Id} - P$ is linear. We have

$$(\mathrm{Id} - P)^2 = \mathrm{Id}^2 - \mathrm{Id}\,P - P\,\mathrm{Id} + P^2 = \mathrm{Id} - 2P + P = \mathrm{Id} - P,$$

showing that $\mathrm{Id} - P$ is a projection.

A vector v is in $\ker(P)$ if and only if $P(v) = 0$, if and only if $(\mathrm{Id} - P)v = v$, if and only if (by Question 1) v is in the image of $\mathrm{Id} - P$. We conclude that $\ker(P) = \mathrm{Im}(\mathrm{Id} - P)$.

Since $\mathrm{Id} - P$ is a projection, we have $\ker(\mathrm{Id} - P) = \mathrm{Im}(\mathrm{Id} - (\mathrm{Id} - P)) = \mathrm{Im}(P)$. ■

Question 3.

Show that the matrix

$$A = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

represents a projection map on $\mathbf{F}^3$. Use this to determine whether the equations $Ax = b$ and $Ax = c$ are consistent, where $b = [1, 1, 0]^\top$ and $c = [1, -1, -1]^\top$. Write b and c in the form $u + u'$ where $u \in \operatorname{col} A$ and $u' \in \operatorname{null} A$.

Solution. Recall that a $m \times n$ matrix A, represents the linear map $\mathrm{L}_A \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ defined by $\mathrm{L}_A(v) = Av$. The linear map L_A is a projection if and only if $A^2 = A$.

We need to show that $A^2 = A$. A simple computation gives us

$$A^2 = \frac{1}{9}\begin{bmatrix} 6 & 3 & 3 \\ 3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix} = A.$$

We verify

$$Ab = \frac{1}{3}\begin{bmatrix}3\\3\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} = b \qquad \text{and} \qquad Ac = \frac{1}{3}\begin{bmatrix}0\\0\\0\end{bmatrix} \neq c.$$

By Question 1, $Ax = b$ is consistent, but $Ax = c$ is not.

We have $\mathrm{col}(A) = \mathrm{Im}(\mathrm{L}_A)$ and $\mathrm{null}\, A = \ker(\mathrm{L}_A)$. It immediately follows that $b = b + 0$ and $c = 0 + c$ as b belongs to $\mathrm{Im}(\mathrm{L}_A)$ and c belongs to $\ker(\mathrm{L}_A)$. ∎

Question 4.

Let $\mathbf{F}$ be either $\mathbf{R}$ or $\mathbf{C}$ and let V be an $\mathbf{F}$-vector space. Suppose that $P \in \mathcal{L}(V)$ is an operator. Let $T = 2P - \mathrm{Id} \in \mathcal{L}(V)$.

a. *Prove that P is a projection if and only if $T^2 = \mathrm{Id}$.*

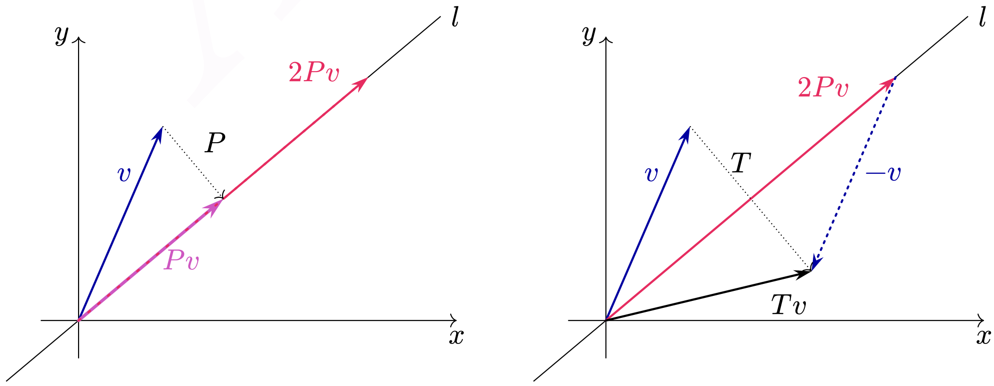
b. *Suppose that P is an orthogonal projection to a line L through the origin in $\mathbf{R}^2$. Give a geometric description of the operator T.*

Solution. a. We have $T^2 = (2P - \mathrm{Id})^2 = 4P^2 - 4P + \mathrm{Id} = \mathrm{Id} + 4(P^2 - P)$. Therefore, $T^2 = \mathrm{Id}$ if and only if $P^2 = P$.

To go further

In any field $\mathbf{F}$, if P is a projection, then $(2P - \mathrm{Id})^2 = \mathrm{Id}$. The converse is not true in general. Indeed, it might happens that $4 = 0$ in $\mathbf{F}$. This is the case in $\mathbf{F}_2 = \{0, 1\}$ for example. If $4 = 0$ in $\mathbf{F}$ then one can show that $2 = 0$ and thus $-1 = 1$. This implies $T = \mathrm{Id}$ is independent of P.
For a concrete example, let $V = (\mathbf{F}_2)^2$. This is a $\mathbf{F}_2$-vector space with 4 elements: $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. Then the map $S\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}y\\x\end{smallmatrix}\right]$ is a linear map but not a projection as $S^2 = \mathrm{Id} \neq S$. However, $(2S - \mathrm{Id})\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}2y-x\\2x-y\end{smallmatrix}\right] = \left[\begin{smallmatrix}-x\\-y\end{smallmatrix}\right] = \left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$.

b. T is the orthogonal reflection across L as demonstrated in the following pictures.



■

Question 5.

Let $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, V)$. Given $b \in W$, consider the linear problem $Tx = b$. Prove the following:

a. *If $TS = \mathrm{Id}_W$ then $x = Sb$ is a solution to $Tx = b$ (in particular, the problem is consistent for all $b \in W$).*

b. *If $ST = \mathrm{Id}_V$ then $Tx = b$ is consistent if and only if $TSb = b$; in which case there is a unique solution $x = Sb$.*

Solution. a. We have $T(Sb) = (TS)b = \mathrm{Id}\, b = b$.

b. If $b = TSb = T(Sb)$, the system is consistent.

If the system is consistent, then $Tx = b$ for some x. Applying S on both sides we obtain $Sb = STx = x$. This shows both that Sb is a solution and also that it is the unique solution.

■

Further Questions

Question 6.

Let $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ be the differentiation operator. Find a right-inverse for D. Does D have a left-inverse?

Solution. Intuitively, we want the right inverse to be the integration map $\int p(x)\,\mathrm{d}x$. Integration is defined up to an additive constant only. Since we want our map to be linear we probably want to impose $T0 = 0$ and more generally that the constant coefficient of Tp is 0 for all polynomial.

Formally, let $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ be the unique linear map such that $T(x^n) = \frac{1}{n+1}x^{n+1}$. We have $T(a_0 + \cdots + a_n x^n) = a_0 x + \cdots + \frac{1}{n+1}x^{n+1}$. One easily verify that we have $DT(p) = p$ for any polynomial.

The map D has no left-inverse, as the equality $SD = \mathrm{Id}$ would implies that D is injective, which is absurd. ■

Question 7.

Suppose $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, V)$ satisfy $TST = T$ and $STS = S$. Show that TS and ST are both projections. What are their images/kernel? (Hint: The answers can be chosen from $\ker(S)$, $\operatorname{Im}(S)$, $\ker(T)$, $\operatorname{Im}(T)$.)

Solution. We always have $\ker(T) \subseteq \ker(ST)$ and $\mathrm{Im}(ST) \subseteq \mathrm{Im}(S)$ (Tutorial 4, Question 2).

Suppose that $TST = T$. Then $(ST)^2 = STST = S(TST) = ST$ and so ST is a projection. A vector v is in $\ker(ST)$ if and only if $STv = 0$. This implies $0 = TSTv = Tv$. We conclude that $\ker(ST) \subseteq \ker(T)$ as soon as $TST = T$.

Suppose now that $STS = S$. Then $(ST)^2 = STST = (STS)T = ST$ is a projection. Finally, let $v \in \mathrm{Im}(S)$. Then there exists w such that $v = Sw$. But then $STv = STSw = Sw = v$ and therefore $\mathrm{Im}(S) = \mathrm{Im}(ST)$.

The proofs for TS are similar. ∎