Tutorial Questions

Question 1.

Let V be a vector space. Prove that $(-1)v = -v$ for all $v \in V$. (Before starting, think about what the minus sign on each side means.)

Solution. The left hand side of the equation is $-1_{\mathbf{F}} \cdot_{\mathbf{F}} v$. That is: the scalar multiplication of $-1 \in \mathbf{F}$ and $v \in V$. The right hand side of the equation is $-_V v$, that is the additive inverse of v.

We have

$$v + (-1) \cdot v = 1 \cdot v + (-1) \cdot v = (1 + (-1))v = 0 \cdot v = 0.$$

We conclude that $(-1)v$ is an additive inverse of v and hence $(-1)v = -v$ by unicity of the additive inverse. ■

Question 2.

Determine if the following subsets of $\mathbf{R}^3$ are subspaces. Give reasons.

a. $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3 \,\middle|\, 2x - 3y + z = 0 \right\}.$

b. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3 \,\middle|\, xyz = 0 \right\}.$

Solution. a. U is a subspace of $\mathbf{R}^3$. Firstly, $2 \cdot 0 - 3 \cdot 0 + 0 = 0$ so $[0, 0, 0]^\intercal$ is in U. Secondly, let $u_1 = [x_1, y_1, z_1]^\intercal$ and $u_2 = [x_2, y_2, z_2]^\intercal$ be two elements of U and let $\lambda \in \mathbf{R}$ be a scalar. By assumption, we have

$$2x_1 - 3y_1 + z_1 = 0 \qquad \text{and} \qquad 2x_2 - 3y_2 + z_2 = 0.$$

By multiplying the first equation by λ and adding the result to the second equation, we obtain

$$\begin{aligned} 0 &= \lambda(2x_1 - 3y_1 + z_1) + (2x_2 - 3y_2 + z_2) \\ &= 2(\lambda x_1 + x_2) - 3(\lambda y_1 + y_2) + (\lambda z_1 + z_2). \end{aligned}$$

Therefore, $\lambda u_1 + u_2$ is in U, finishing the proof that U is a subspace.

One can show that U is in a plane containing the origin in $\mathbf{R}^3$.

b. W is not a subspace of $\mathbf{R}^3$. Indeed, both $[1, 1, 0]^\intercal$ and $[0, 0, 1]^\intercal$ belong to W, but there sum $[1, 1, 0]^\intercal + [0, 0, 1]^\intercal = [1, 1, 1]^\intercal$ is not in W.

Geometrically, W is the union of the three coordinates planes (the xy-plane, the xz-plane and the yz plane), see Figure 1.

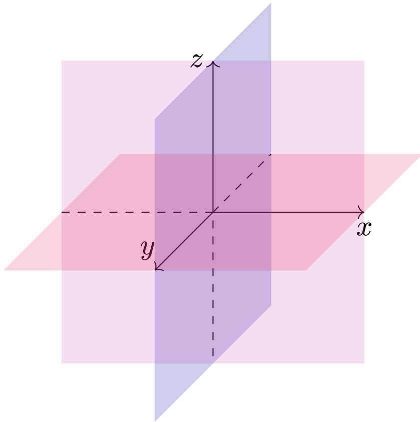

Figure 1: The union of the three coordinates planes.

Observe that the fact that U is a subspace of $\mathbf{F}^3$ but W is not remains true for an arbitrary field $\mathbf{F}$. ■

Question 3.

A sequence $(x_1, x_2, x_3, \dots) \in \mathbf{R}^\infty$ is increasing *if $x_m \leq x_{m+1}$ for all $m \geq 1$. Does the set X of all increasing sequences form a subspace of $\mathbf{R}^\infty$? Give reasons.*

Solution. No, because X is not closed under scalar multiplication. For example, $(0, 1, 1, \dots)$ is in X, but $-1 \cdot (0, 1, 1, \dots) = (0, -1, -1, \dots)$ is not. ■

Question 4.

Let V be a vector space and $U \subseteq V$ be a subspace. Suppose that $u \in U$ and $v \in V$. Prove that $v \in U$ if and only if $u + v \in U$.

Solution. If v is in U, then $u + v$ is also in U because U is a subspace.

If $u + v$ is in U, then $v = -1 \cdot u + (u + v)$ is also in U because U is a subspace. ■

Question 5.

Let U be the xy-plane in $\mathbf{R}^3$. Find distinct subspaces W_1, W_2 of $\mathbf{R}^3$ such that $U \oplus W_1 = U \oplus W_2 = \mathbf{R}^3$.

Solution. Let $W_1 := \{[0, 0, z]^\mathsf{T} \mid z \in \mathbf{R}\}$ and let $W_2 := \{[0, z, z]^\mathsf{T} \mid z \in \mathbf{R}\}$. Then $U \oplus W_1 = U \oplus W_2 = \mathbf{R}^3$.

For each W_i, we need to prove three things: 1. that it is a subspace of $\mathbf{R}^3$, 2. that $U + W_i = \mathbf{R}^3$, 3. and finally that $U + W_i$ is a direct sum. We will only check the last two properties, and only do it for W_2. The proof for W_1 is similar (and even simpler) and the proof that the W_i are subspaces of $\mathbf{R}^3$ is elementary.

Let $[x, y, z]^\top$ be any vector of $\mathbf{R}^3$. Then $[x, y, z]^\top = [x, y-z, 0]^\top + [0, z, z]^\top$ is in $U + W_2$ and so $U + W_2 = \mathbf{R}^3$. Finally, suppose that $0 = u + w$ with $u \in U$ and $w \in W_2$. So $[0, 0, 0]^\top = [x, y, 0]^\top + [0, z, z]^\top$ for some $x, y, z \in \mathbf{R}$. But this implies

$$\begin{aligned} 0 &= x \\ 0 &= y + z \\ 0 &= z \end{aligned}$$

and we hence conclude that both u and w are the 0 vector. This shows that the sum $U + W_2$ is direct.

Observe that a similar proof shows that $U \oplus W = \mathbf{R}^3$ if W is any line containing the origin and not include in U.

To go further

The above result remains true if we replace $\mathbf{R}$ by $\mathbf{F}$, in which case a *line* simply means a subspace of dimension 1. That is, a *line containing the origin* is any subspace of the form $\{[\lambda_1 z, \lambda_2 z, \lambda_3 z]^\top \mid z \in \mathbf{F}\}$ for some $\lambda_1, \lambda_2, \lambda_3$ with at least one $\lambda_i \neq 0$.

■

Further Questions

Question 6.

Is addition of subspaces commutative? That is, for subspaces U, W of V, is it true that $U + W = W + U$? Explain.

Solution. Addition of subspaces is commutative, because addition of vectors is. Indeed, a vector $v \in V$ belongs to $U + W$ if and only if there exists $u \in U$ and $w \in W$ such that $v = u + w$. By commutativity of *vectors addition*: $u + w = w + u$. So $v \in V$ belongs to $U + W$ if and only if there exists $u \in U$ and $w \in W$ such that $v = w + u$, that is if and only if v belongs to $W + U$.

Observe that we did not used the fact that U and W are subspaces. Therefore, addition of subsets (of a vector space) is commutative. ■

Question 7.

Does every subspace of V have an additive inverse*? That is, for each subspace U of V, does there exist a subspace W such that $U + W = \{0\}$?*

Solution. No. This is the case if and only if $U = \{0\}$, in which case $W = \{0\}$. Indeed, we always have $U \subseteq U + W$. So if $U + W = \{0\}$ for some W, then $U = \{0\}$. ■

Question 8.

Recall that a function $f\colon \mathbf{R} \to \mathbf{R}$ is even *if $f(-x) = f(x)$ for all $x \in \mathbf{R}$; and* odd *if $f(-x) = -f(x)$ for all $x \in \mathbf{R}$. Let U_e and U_o respectively denote the set of even and odd functions in $\mathbf{R}^{\mathbf{R}}$. Prove that $\mathbf{R}^{\mathbf{R}} = U_e \oplus U_o$.*

Solution. Once again, we need to prove 3 things: 1. that both U_e and U_o are subspaces of $\mathbf{R}^{\mathbf{R}}$, 2. that $U_e + U_o = \mathbf{R}^{\mathbf{R}}$, 3. and finally that $U_e + U_o$ is a direct sum.

Firstly, the constant function 0 is both even and odd. Moreover, if f and g are two even functions (respectively odd functions) and $\lambda \in \mathbf{R}$ a scalar, one easily verify that $\lambda f + g$ is still even (respectively odd). Indeed, this directly follows from $(\lambda f + g)(x) = \lambda f(x) + g(x)$. We conclude that both U_e and U_o are subspaces of $\mathbf{R}^{\mathbf{R}}$.

Then, let f be any function of $\mathbf{R}^{\mathbf{R}}$. Define two functions f_e and f_o by:

$$f_e(x) := \frac{f(x) + f(-x)}{2}, \qquad f_o(x) := \frac{f(x) - f(-x)}{2}.$$

On directly have $(f_e + f_o)(x) = f(x)$ for all $x \in \mathbf{R}$ and so $f_e + f_o = f$. We also have

$$f_e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x)$$

and

$$f_o(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x).$$

So f_e is even and f_o is odd, showing that f is in $U_e + U_o$. This finishes the proof that $U_e + U_o = \mathbf{R}^{\mathbf{R}}$.

Finally, let f be in $U_e \cap U_o$. Then for every $x \in \mathbf{R}$, let $y = -x$. We have:

$$f(-y) = f(-y) = -f(-y).$$

But this implies that $f(x) = f(-y) = -f(-y) = -f(x)$ and so $f(x) = 0$. Therefore, f is the zero function, $U_e \cap U_o = \{0\}$ and the sum is direct. ■