Tutorial Questions

Question 1.

Let $X = \{a\}$ and $Y = \{b, c\}$.

a. *List all possible functions from X to Y (you may draw a diagram).*

b. *List all functions from Y to X.*

c. *Compose a function from the first list with a function from the second list (in either order). What are all the different functions that arise in this way?*

Solution. a. To specify a function $f\colon X \to Y$, we need to describe the image of each element of X. Since X has only one element a, any function $f\colon X \to Y$ is uniquely determined by $f(a) \in Y = \{b, c\}$. That is, there are exactly two functions f_1 and f_2 in Y^X, which are defined by $f_1(a) = b$and $f_2(a) = c$.

b. To define a function $g\colon Y \to X$ is equivalent to choose the value of $g(y) \in X$ for all $y \in Y$. Since $X = \{a\}$ has only one element, we are forced to choose $g(y) = a$ for all $y \in Y$. So there exists a unique function $g\colon Y \to X$, defined by $g(b) = g(c) = a$.

c. Let us give the details for $f_1 \circ g$:

$$\begin{aligned} f_1 \circ g\colon Y &\xrightarrow{g} X \xrightarrow{f_1} Y \\ b &\longmapsto a \longmapsto b \\ c &\longmapsto a \longmapsto b. \end{aligned}$$

So $f_1 \circ g$ is the constant function b. That is: $(f_1 \circ g)(y) = b$ for all $y \in Y$. Similarly, $f_2 \circ g$ is the constant function c. Composing in the other direction we obtain that both $g \circ f_1$ and $g \circ f_2$ are the constant function a.

Observe that we have $g \circ f_1 = g \circ f_2 = \mathrm{Id}_X$ with $f_1 \neq f_2$, but $f_1 \circ g \neq f_2 \circ g$ and none of them is equal to Id_Y.

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Question 2.

Let X, Y, Z be sets and suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are functions.

a. *Show that if f and g are injective then $g \circ f$ is injective.*

b. *Show that if f and g are surjective then $g \circ f$ is surjective.*

c. *If $g \circ f$ is injective then what can you say about f or g? What if $g \circ f$ is surjective?*

Solution. a. Suppose that both f and g are injective and let $a, b \in X$ be two elements of X such that $(g \circ f)(a) = (g \circ f)(b)$. We need to prove that in this case one necessary have $a = b$. From $(g \circ f)(a) = (g \circ f)(b)$ one obtain $g(f(a)) = g(f(b))$ and so $f(a) = f(b)$ by injectivity of g and finally $a = b$ by injectivity of f.

b. Suppose that both f and g are surjective and let $z \in Z$ be any element of Z. By surjectivity of g, there exists $y \in Y$ such that $g(y) = z$. By surjectivity of f, there exists $x \in X$ such that $f(x) = y$. But then $(g \circ f)(x) = g(y) = z$, proving surjectivity of $g \circ f$.

c. We will show that $g \circ f$ injective implies f injective, but g might not be injective; and that $g \circ f$ surjective implies g surjective, but f might not be surjective.

We start by providing a counterexample to the two incorrect implications. Let f_1 and g be the functions from the solution to Exercise 1.a. Then $g \circ f_1 = \mathrm{Id}_X$ is both injective and surjective. However, g is not injective as $g(b) = g(c)$ even if $b \neq c$, and f_1 is not surjective as $c \notin \mathrm{Im}(f_1)$.

Suppose now that $g \circ f$ is injective. Let $a, b \in X$ be any two elements with $f(a) = f(b)$. By applying g on both sides we have $(g \circ f)(a) = (g \circ f)(b)$ and hence $a = b$ by injectivity of $g \circ f$. We have just proved injectivity of f.

Finally, suppose that $g \circ f$ is surjective. Let $z \in Z$ be any element. Then, by surjectivity of $g \circ f$, there exists $x \in X$ such that $z = (g \circ f)(x) = g(f(x))$. Since $y := f(x)$ is in Y, we have $z \in \mathrm{Im}(g)$, proving surjectivity of g.

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Question 3.

Recall that $\mathbf{F}$ *stands for* $\mathbf{R}$ *or* $\mathbf{C}$. *Let* m *be an integer. Prove that* $\mathbf{F}^m$ *satisfies the* distributive properties, *that is, for all* $a, b \in \mathbf{F}$ *and* $u, v \in \mathbf{F}^m$ *we have:*

1. $a(u + v) = au + av$;
2. $(a + b)v = av + bv$.

Solution. Let $u = [u_1, \dots, u_m]^\intercal$ and $v = [v_1, \dots, v_m]^\intercal$. Then

$$\begin{aligned} a(u+v) &= a\left(\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}\right) = a\begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_m + v_m \end{bmatrix} \\ &= \begin{bmatrix} a(u_1 + v_1) \\ \vdots \\ a(u_m + v_m) \end{bmatrix} = \begin{bmatrix} au_1 + av_1 \\ \vdots \\ au_m + av_m \end{bmatrix} \\ &= \begin{bmatrix} au_1 \\ \vdots \\ au_m \end{bmatrix} + \begin{bmatrix} av_1 \\ \vdots \\ av_m \end{bmatrix} = au + av, \end{aligned}$$

where we used the distributivity property $a(u_1 + v_1) = au_1 + av_1$ in $\mathbf{F}$. ■

Recall that the **identity map** on a set X is the function $\mathrm{Id}_X\colon X \to X$ given by $\mathrm{Id}_X(x) = x$ for all $x \in X$.

Question 4.

Let $f\colon X \to Y$ and $g\colon Y \to X$ be functions. Suppose that $g \circ f = \mathrm{Id}_X$. Does it follow that $f \circ g = \mathrm{Id}_Y$?

Solution. No. See the solution of Question 1 for a counterexample. ■

Further Questions

Question 5.

Let X and Y be finite sets and $f\colon X \to Y$ be a function.

a. *Assume that f is injective. Find a function $g\colon Y \to X$ such that $g \circ f = \mathrm{Id}_X$.*

b. *Assume that f is surjective. Find a function $g\colon Y \to X$ such that $f \circ g = \mathrm{Id}_Y$.*

Solution. a. Suppose X non-empty and let x_0 be any element of X. Define $g\colon Y \to X$ by:

$$g(y) := \begin{cases} \text{the unique } x \in X \text{ such that } f(x) = y & \text{if } y \in \mathrm{Im}(f) \\ x_0 & \text{otherwise.} \end{cases}$$

It is then straightforward that $g \circ f = \mathrm{Id}_X$.

To go further

If $X = \emptyset$ and $Y \neq \emptyset$, then there is no function from X to Y. If $X = \emptyset = Y$, then there is a unique function from X to Y: the empty function, which is its own inverse.

Observe that we did not use that X and Y are finite. That is, we actually proved a bit more than what was stated. Moreover, one can check that the g we constructed is surjective. So we have proved: if $f\colon X \to Y$ is an injective function between arbitrary sets, there exists a surjective function $g\colon Y \to X$ with $g \circ f = \mathrm{Id}_X$.

b. If f is surjective, then $Y = \mathrm{Im}(f)$. One can define $g\colon Y \to X$ by for each $y \in Y = \mathrm{Im}(f)$ choosing a $x \in X$ such that $f(x) = y$. One can check that such a g is injective and $f \circ g = \mathrm{Id}_Y$. So we have proved: if $f\colon X \to Y$ is an surjective function with Y finite, there exists an injective function $g\colon Y \to X$ with $f \circ g = \mathrm{Id}_Y$.

To go further

If Y is infinite, we need the axiom of choice (AC) to be able to make infinitely many simultaneous choices. So, if we assume (AC) we have: if $f\colon X \to Y$ is an surjective function between arbitrary sets, there exists an injective function $g\colon Y \to X$ with $f \circ g = \mathrm{Id}_Y$. Actually, this statement is equivalent to (AC). This is one of the reason to assume (AC): it implies that injective and surjective functions behave "dually".

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The following questions are intended as revision for Year 1 Linear Algebra.

Question 6.

Solve the system of linear equations

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4\\ x_1 + 4x_2 - 8x_3 &= 7\\ -3x_1 - 7x_2 + 9x_3 &= -6. \end{aligned}$$

Solution. By doing the row transformations $r_2 \mapsto r_2 - r_1$ and $r_3 \mapsto r_3 + 3r_1$ we obtain the equivalent system

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4\\ x_2 - 3x_3 &= 3\\ 2x_2 - 6x_3 &= 6. \end{aligned}$$

Doing one last row transformation $r_3 \mapsto r_3 - 2r_2$ one have the, also equivalent, system

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4\\ x_2 - 3x_3 &= 3\\ 0 &= 0. \end{aligned}$$

We conclude that x_3 is free, $x_2 = 3 + 3x_3$ and $x_1 = 4 - 3x_2 + 5x_3 = -5 - 4x_3$. ■

Question 7.

Let $A = \begin{bmatrix} 2 & -2 & 1\\ 3 & 5 & 0\\ -1 & 3 & 4 \end{bmatrix}$. *Compute* $\det(A)$ *and* A^{-1}.

Solution. There are many ways to compute the determinant. We will compute it by developing the determinant according to the last column, giving:

$$\begin{aligned}\det(A) &= 1 \cdot \det \begin{bmatrix} 3 & 5 \\ -1 & 3 \end{bmatrix} + 0 + 4 \cdot \det \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix} \\ &= (9+5) + 4(10+6) = 78.\end{aligned}$$

They are also many methods to compute the inverse of A. We will show to of them. Firstly, we will use the formula

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

where $\operatorname{adj}(A) = ((-1)^{i+j} \det(\operatorname{cofact}_{ij}))_{ij}^{\mathsf{T}}$. Recall that $\operatorname{cofact}_{ij}$ is the matrix obtained from A by erasing the i-row and j-column. Let us unravel this for our specific example. So we have

$$\operatorname{adj}(A)^{\mathsf{T}} = ((-1)^{i+j} \det(\operatorname{cofact}_{ij}))_{ij} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

where we need to compute the coefficients a to i. For example,

$$a = 1 \cdot \det \begin{bmatrix} 5 & 0 \\ 3 & 4 \end{bmatrix} = 20 \qquad b = -1 \cdot \det \begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix} = -12.$$

By computing all the coefficients, one obtain

$$\operatorname{adj}(A)^{\mathsf{T}} = \begin{bmatrix} 20 & -12 & 14 \\ 11 & 9 & -4 \\ -5 & 3 & 16 \end{bmatrix},$$

And finally

$$A^{-1} = \frac{1}{78} \begin{bmatrix} 20 & 11 & -5 \\ -12 & 9 & 3 \\ 14 & -4 & 16 \end{bmatrix}.$$

While the above method always work, it is lengthy. Another way to compute the inverse is to use operations on rows. The idea is to start with $[A \mid \mathrm{Id}]$ and to do raw transformations to obtain $[\mathrm{Id} \mid B]$. The matrix B obtained with this process is A^{-1}. If we apply that to our concrete example, we obtain

$$A^{-1} = \begin{bmatrix} 10/39 & 11/78 & -5/78 \\ -2/13 & 3/26 & 1/26 \\ 7/39 & -2/39 & 8/39 \end{bmatrix} = \frac{1}{78} \begin{bmatrix} 20 & 11 & -5 \\ -12 & 9 & 3 \\ 14 & -4 & 16 \end{bmatrix},$$

which is identical to the answer obtained via the cofactors method. The details are shown on the next page.

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 1 & 1 & 0 & 0 \\ 3 & 5 & 0 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xmapsto{\substack{r_1 \mapsto 1/2\, r_1 \\ r_2 \mapsto r_2 - 3/2\, r_1 \\ r_3 \mapsto r_3 + 1/2\, r_1}} \left[\begin{array}{ccc|ccc} 1 & -1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 8 & -3/2 & -3/2 & 1 & 0 \\ 0 & 2 & 9/2 & 1/2 & 0 & 1 \end{array}\right]$$

$$\xmapsto{\substack{r_2 \mapsto 1/8\, r_2 \\ r_3 \mapsto r_3 - 1/4\, r_2}} \left[\begin{array}{ccc|ccc} 1 & -1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -3/16 & -3/16 & 1/8 & 0 \\ 0 & 0 & 39/8 & 7/8 & -1/4 & 1 \end{array}\right] \xmapsto{r_3 \mapsto 8/39\, r_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -3/16 & -3/16 & 1/8 & 0 \\ 0 & 0 & 1 & 7/39 & -2/39 & 8/39 \end{array}\right]$$

$$\xmapsto{\substack{r_1 \mapsto r_1 - 1/2\, r_3 \\ r_2 \mapsto r_2 + 3/16\, r_3}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 16/39 & 1/39 & -4/39 \\ 0 & 1 & 0 & -2/13 & 3/26 & 1/26 \\ 0 & 0 & 1 & 7/39 & -2/39 & 8/39 \end{array}\right] \xmapsto{r_1 \mapsto r_1 + r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10/39 & 11/78 & -5/78 \\ 0 & 1 & 0 & -2/13 & 3/26 & 1/26 \\ 0 & 0 & 1 & 7/39 & -2/39 & 8/39 \end{array}\right].$$

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