Tutorial Questions

Question 1.

Let (u_1, u_2, u_3) be a list of vectors of a vector space V.

a. *Show that if (u_1, u_2, u_3) span V then so does the list $(u_1 - u_2, u_2 - u_3, u_3)$.*

b. *Show that if (u_1, u_2, u_3) is linearly independent then so is the list $(u_1 - u_2, u_2 - u_3, u_3)$.*

Solution. a. We will show that $\operatorname{span}(u_1, u_2, u_3) = \operatorname{span}(u_1 - u_2, u_2 - u_3, u_3)$, even they are not necessarily equal to V. The $\supseteq$ inclusion is clear, so let us prove $\operatorname{span}(u_1, u_2, u_3) \subseteq \operatorname{span}(u_1 - u_2, u_2 - u_3, u_3)$. Let $v \in \operatorname{span}(u_1, u_2, u_3)$. There exists scalars λ_1, λ_2 and λ_3 such that

$$\begin{aligned} v &= \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \\ &= \lambda_1(u_1 - u_2) + (\lambda_2 + \lambda 1)(u_2 - u_3) + (\lambda_3 + \lambda_2)u_3. \end{aligned}$$

Since all of λ_1, $\lambda_2 + \lambda 1$ and $\lambda_3 + \lambda_2$ are in $\mathbf{F}$, we have just shown that v belongs to $\operatorname{span}(u_1 - u_2, u_2 - u_3, u_3)$, which finishes the proof.

b. Suppose that (u_1, u_2, u_3) is linearly independent and let λ_1, λ_2 and λ_3 be scalars such that

$$0 = \lambda_1(u_1 - u_2) + \lambda_2(u_2 - u_3) + \lambda_3 u_3.$$

We have to show that this implies that all the λ_i are zeros. We have

$$0 = \lambda_1 u_1 + (\lambda_2 - \lambda_1)u_2 + (\lambda_3 - \lambda_2)u_3.$$

By linear independence of (u_1, u_2, u_3), all the coefficients λ_1, $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_2$ are zeros. This directly implies that all the λ_i are 0, as required.

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Question 2.

Let V be a finite dimensional vector space and suppose that $U \subseteq V$ is a subspace. Prove that if $\dim(U) = \dim(V)$ then $U = V$.

Solution. By assumption, $\dim(U) = \dim(V)$. So there exists a finite basis $\mathscr{B} = (u_1, \dots, u_n)$ of U, where $n = \dim(V)$. Then $\mathscr{B}$ is a linearly independent family of vectors of U, hence also of vectors of V. Since it has $\dim(V)$ elements it is spanning in V. So $U = \operatorname{span}(\mathscr{B}) = V$. ■

Question 3.

Consider the subspace $U = \{p \in \mathcal{P}(\mathbf{R})_2 \mid p(1) = 0\}$ *of* $\mathcal{P}(\mathbf{R})_2$.

a. *Find a basis for* U. *(Hint: if a polynomial* p *satisfies* $p(a) = 0$ *then* $(x-a)$ *is a factor of* $p(x)$.*)*

b. *Extend the basis you found to a basis of* $\mathcal{P}(\mathbf{R})_2$.

c. *Find a subspace* W *of* $\mathcal{P}(\mathbf{R})_2$ *such that* $U \oplus W = \mathcal{P}(\mathbf{R})_2$.

Solution. We will show the result for a general field $\mathbf{F}$.

a. On one hand, we know that $\mathcal{P}(\mathbf{F})_2$ is of dimension 3 (with basis $(1, x, x^2)$). On the other hand, $U \neq \mathcal{P}(\mathbf{F})_2$ (for example, because $x \notin U$). By Question 2, this implies that U is of dimension at most 2. So if we find a linearly independent family of length 2 we are done. For example, one can take the family $(p_1 = x-1, p_2 = x^2-1)$. We claim that this family is linearly independent. Indeed, if $0 = \lambda p_1 + \mu p_2$, then

$$0 = \mu x^2 + \lambda x + (-\lambda - \mu)$$

and therefore $\mu = 0 = \lambda$, which proves the claim.

b. We already have a length 2 linearly independent family. So for any $q \in \mathcal{P}(\mathbf{F})_2$ *not* in $\operatorname{span}(p_1, p_2) = U$, the family (p_1, p_2, q) is linearly independent by the linear dependence lemma. Since it is a length 3 linear independent family in a dimension 3 vector space, it is linearly independent.

   For a concrete example, one can take $q = 1$ the constant polynomial 1. But $q = x$ or $q = x^2$ also work.

c. Let $q \notin U$ be any polynomial in $\mathcal{P}(\mathbf{F})_2$ but not in U. We claim that for $W := \operatorname{span}(q)$ we have $U \oplus W = \mathcal{P}(\mathbf{F})_2$. It is clear from above that $U + W = \mathcal{P}(\mathbf{F})_2$. It remains to prove that the sum is direct. Let p be in $U \cap W$. We need to show that $p = 0$. We have $p = \lambda q \in U$. If $p \neq 0$, then $q = \lambda^{-1} p \in U$; a contradiction.

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Question 4.

Let U *and* W *be 3-dimensional subspaces of* $\mathbf{R}^5$. *What are the possible values of* $\dim(U \cap W)$*? Can* $U + W$ *be a direct sum?*

Solution. Once again, we will prove the statement for a general $\mathbf{F}$.
We have

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U + W) \leq \dim(\mathbf{R}^5) = 5.$$

In other words, $\dim(U \cap W) = \dim(U) + \dim(V) - \dim(U + W) \geq 3 + 3 - 5 = 1$. But we also have $\dim(U + W) \leq \dim(U) = 3$. So, $\dim(U + W)$ belongs to $\{1, 2, 3\}$. We now show that all this value can happens.

If $U = W_1$, for example $U = W_1 = \{[x_1, x_2, x_3, 0, 0]^\intercal \mid x_i \in \mathbf{F}\}$, then $\dim(U \cap W_1) = 3$.

For U as above and $W_2 = \{[0, x_2, x_3, x_4, 0]^\intercal \mid x_i \in \mathbf{F}\}$ the subspace $U \cap W_2 = \{[0, x_2, x_3, 0, 0]^\intercal \mid x_i \in \mathbf{F}\}$ is of dimension 2.

Finally, for U as above and $W_3 = \{[0, 0, x_3, x_4, x_5]^\intercal \mid x_i \in \mathbf{F}\}$ the subspace $U \cap W_3 = \{[0, 0, x_3, 0, 0]^\intercal \mid x_i \in \mathbf{F}\}$ is of dimension 1.

Since $\dim(U \cap W) \geq 1$, the sum $U + W$ is never direct. Indeed, the sum $U + W$ is direct if and only if $U \cap W = \{0\}$, if and only if $\dim(U \cap W) = 0$. ■

Question 5.

Let V be a vector space. Suppose that for each integer $m > 0$, there exists a list of m linearly independent vectors in V. Prove that V is infinite dimensional.

Solution. By Grassman's exchange lemma, if V has a linearly independent family of length m, then any spanning family has length at least m. In particular, any base of V has at least m elements, for all $m \in \mathbf{N}$. We conclude that a basis cannot be finite. ■

Further Questions

Question 6.

Prove that $\mathbf{R}^{\mathrm{N}}$ is infinite dimensional. (More generally, if S is an infinite set then $(\mathbf{F}^S)_0$ and $\mathbf{F}^S$ are infinite dimensional.)

Solution. By Question 5, it is enough to exhibit arbitrary long linearly independent lists. For any $i \in \mathbf{N}_{\geq 1}$, define $e_i = (0, \dots, 0, 1, 0, \dots)$ to be list with a 1 in coordinate i and 0 everywhere else. We claim that for any m, the family $(e_1, \dots, e_m)$ is linearly independent. Indeed, let λ_i be scalars such that

$$0 = \sum_{i=1}^{m} \lambda_i e_i.$$

Then

$$(0, 0, \dots) = (\lambda_1, \dots, \lambda_m, 0, \dots)$$

forcing all the λ_i to be 0.

A similar proof shows that $(\mathbf{F}^S)_0$ (and thus also $\mathbf{F}^S$) are infinite dimensional as soon as S is infinite. Indeed, for $s \in S$ one can define a map $\chi_S \colon S \to \mathbf{F}$ by

$$\chi_s(t) = \begin{cases} 1 & \text{if} t = s \\ 0 & \text{otherwise.} \end{cases}$$

We now claim that for any finite subset $A \subseteq S$, the family $(\chi_a)_{a\in A}$ is linearly independent. Indeed, let $(\lambda_a)_{a\in A}$ be a family of scalars such that

$$0 = \sum_{a\in A} \lambda_a \chi_a.$$

Then, for $a \in A$ one can evaluate both sides of the above equality on a to obtain $0 = \lambda_a \cdot 1$, and thus $\lambda_a = 0$.

To go further

It is in fact possible to show that for any S, finite or infinite, we have $\dim((\mathbf{F}^S)_0) = \#S$. Recall that for finites S, one have $(\mathbf{F}^S)_0 = \mathbf{F}^S$.

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Question 7.

Explain why all the subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, all straight lines through the origin, all planes through the origin, and $\mathbf{R}^3$.

Solution. First of all, let us remind that a line in $\mathbf{R}^3$ is a subset of the form

$$L_{u,v} = \{\lambda u + v \mid \lambda \in \mathbf{R}\}$$

for some vectors v and $u \neq 0$. A line $L_{u,v}$ goes through the origin if and only if $v = 0$. Let us fix $u \in \mathbf{R}^3$. One easily check that $0 \in L_{u,0}$ and that $L_{u,0}$ is closed under addition and scalar multiplication. In consequence $L_{u,0}$ is a subspace of $\mathbf{R}^3$, of dimension 1 ((u) is a basis).

Similarly, a plane in $\mathbf{R}^3$ is a subset of the form

$$P_{u_1,u_2,v} = \{\lambda_1 u_1 + \lambda_2 u_2 + v \mid \lambda_1, \lambda_2 \in \mathbf{R}\}$$

for some vectors v and non-collinear (that is linearly independent) u_1 and u_2. Once again, a plane $P_{u_1,u_2,v}$ goes through the origin if and only if $v = 0$ and $P_{u_1,u_2,0}$ is a subspace of $\mathbf{R}^3$, of dimension 2 ((u_1, u_2) is a basis).

Finally, it is clear that both $\{0\}$ and $\mathbf{R}^3$ are subspaces of $\mathbf{R}^3$ (of respective dimension 0 and 3). We will now prove that any subspace U of $\mathbf{R}^3$ is of the form $\{0\}$, $L_{u,0}$ (for some non-zero u), $P_{u_1,u_2,0}$ (for some linearly independent u_1 and u_2) or $\mathbf{R}^3$. To do this, we will do a case-by-case analysis depending on $\dim(U) \in \{0, \dots, 3\}$.

If $\dim(U) = 0$, then $U = \{0\}$.

If $\dim(U) = 3 = \dim(\mathbf{R}^3)$, then $U = \mathbf{R}^3$ by Question 2.

Suppose now that $\dim(U = 1)$ and take any $u \neq 0$ in U. Then $\dim(\mathrm{span}(u)) = 1 = \dim(U)$ and thus $U = \mathrm{span}(u) = \{\lambda u \mid \lambda \in \mathbf{R}\} = L_{u,0}$ is a line through the origin. We also have the converse: any line through the origin is a subspace of dimension 1.

Suppose finally that $\dim(U) = 2$. Take any $u_1 \neq 0$ in U. Since $\dim(\mathrm{span}(u)) = 1 < 2 = \dim(U)$, there exists $u_2 \in U \setminus \mathrm{span}(u_1)$. By the linear dependence lemma, the family (u_1, u_2) is linearly independent and hence a basis of U. So $U = \mathrm{span}(u_1, u_2) = \{\lambda_1 u_1 + \lambda_2 u_2 \mid \lambda_i \in \mathbf{R}\} = P_{u_1,u_2,0}$ is a plane containing the origin. We also have the converse: any plane containing the origin is a subspace of dimension 2.

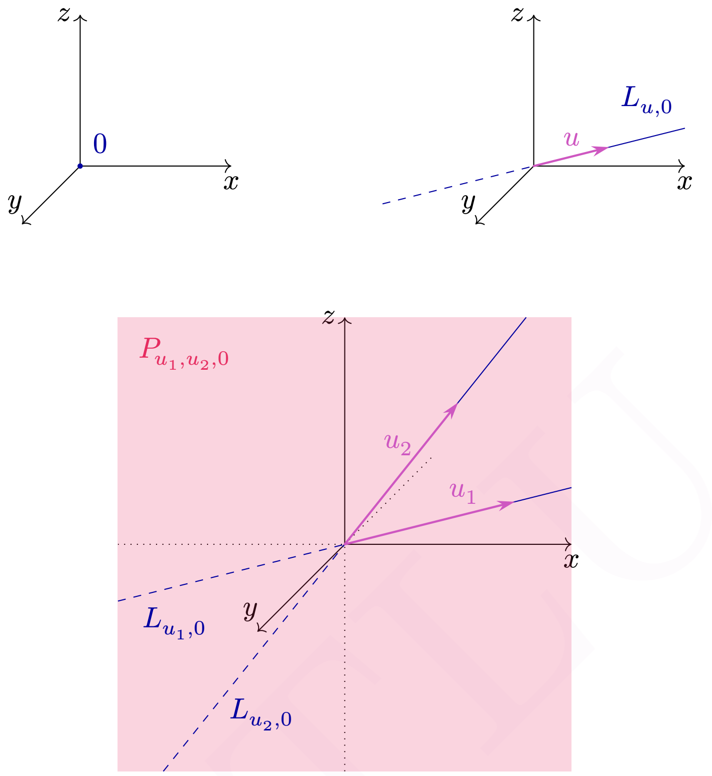

Figure 1: Subspaces of dimension 0 (the origin), 1 (line through the origin) and 2 (plane through the origin) of $\mathbf{R}^3$. The last kind of subspace is $\mathbf{R}^3$ itself.

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