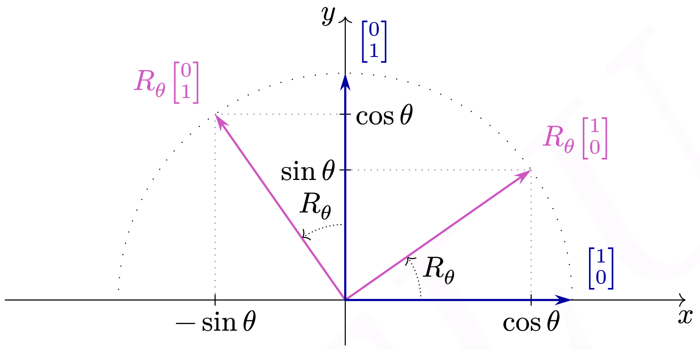
Tutorial Questions

Question 1.

Let $R_\theta\colon \mathbf{R}^2 \to \mathbf{R}^2$ *be a rotation about the origin through an angle of* θ*. Find the matrix* $[R_\theta]^{\mathscr{E}}_{\mathscr{E}}$*, where* $\mathscr{E}$ *is the standard basis for* $\mathbf{R}^2$*. Use this to prove that* $R_\theta \circ R_\phi = R_{\theta+\phi}$. (Hint: use the trigonometric formulas $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.)

Solution. We have $\mathscr{E} = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$. Simple trigonometry allows us to compute the coordinates of $R_\theta\begin{bmatrix}1\\0\end{bmatrix}$ and $R_\theta\begin{bmatrix}0\\1\end{bmatrix}$.



So we have

$$R_\theta\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \quad \text{and} \quad R_\theta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix},$$

which implies

$$[R_\theta]^{\mathscr{E}}_{\mathscr{E}} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

We then compute

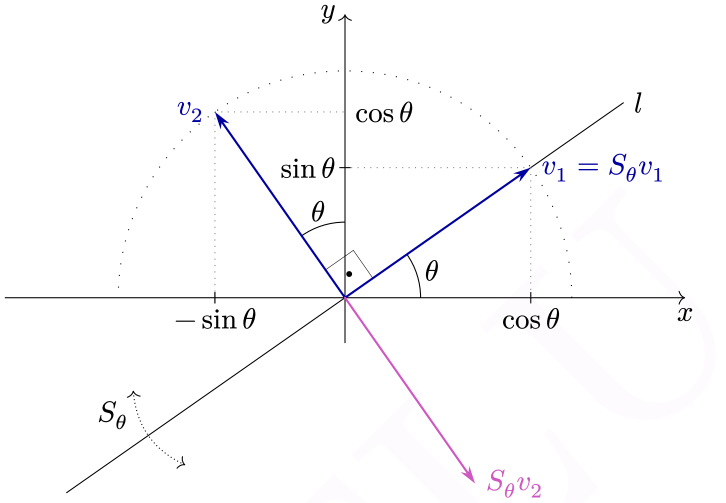
$$\begin{aligned}
[R_\theta \circ R_\phi]^{\mathscr{E}}_{\mathscr{E}} &= [R_\theta]^{\mathscr{E}}_{\mathscr{E}}[R_\phi]^{\mathscr{E}}_{\mathscr{E}}\\
&= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi\\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi\end{bmatrix}\\
&= \begin{bmatrix}\cos(\theta+\phi) & -\sin(\theta+\phi)\\ \sin(\theta+\phi) & \cos(\theta+\phi)\end{bmatrix} = [R_{\theta+\phi}]^{\mathscr{E}}_{\mathscr{E}}.
\end{aligned}$$

We conclude $R_\theta \circ R_\phi = R_{\theta+\phi}$ as desired. ■

Question 2.

Let $S_\theta \colon \mathbf{R}^2 \to \mathbf{R}^2$ *be a reflection across the line through the origin which makes an angle of* θ *with the positive* x*-axis. Let* $\mathscr{B}$ *be the basis of* $\mathbf{R}^2$ *given by* $v_1 = \left[\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right]$ *and* $v_2 = \left[\begin{smallmatrix}-\sin\theta\\ \cos\theta\end{smallmatrix}\right]$. *Find* $[S_\theta]_{\mathscr{B}}^{\mathscr{B}}$. *Then find* $[S_\theta]_{\mathscr{E}}^{\mathscr{E}}$ *by applying the change of basis formula, where* $\mathscr{E}$ *is the standard basis of* $\mathbf{R}^2$.

Solution.



Since S_θ is a reflection along $l = \operatorname{span}(v_1)$ and $v_2 \perp v_1$ we have $S_\theta(v_1) = v_1$ and $S_\theta(v_2) = v_2$. Therefore, $[S_\theta(v_1)]_{\mathscr{B}} = \left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]$ and $[S_\theta(v_2)]_{\mathscr{B}} = \left[\begin{smallmatrix}0\\ -1\end{smallmatrix}\right]$. Altogether we obtain

$$[S_\theta]_{\mathscr{B}}^{\mathscr{B}} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

The change of basis formula gives us:

$$[S_\theta]_{\mathscr{E}}^{\mathscr{E}} = [\mathrm{Id}]_{\mathscr{B}}^{\mathscr{E}} [R_\theta]_{\mathscr{B}}^{\mathscr{B}} [\mathrm{Id}]_{\mathscr{E}}^{\mathscr{B}}.$$

So we need to compute $[\mathrm{Id}]_{\mathscr{B}}^{\mathscr{E}}$ and $[\mathrm{Id}]_{\mathscr{E}}^{\mathscr{B}} = [\mathrm{Id}]_{\mathscr{B}}^{\mathscr{E}\,-1}$. We have $\mathrm{Id}\, v_1 = \cos\theta e_1 + \sin\theta e_2$ and $\mathrm{Id}\, v_2 = -\sin\theta e_1 + \cos\theta e_2$. This gives us

$$[\mathrm{Id}]_{\mathscr{B}}^{\mathscr{E}} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \qquad \text{and} \qquad [\mathrm{Id}]_{\mathscr{E}}^{\mathscr{B}} = ([\mathrm{Id}]_{\mathscr{B}}^{\mathscr{E}})^{-1} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}.$$

Finally, one have

$$\begin{aligned}
[S_\theta]_{\mathscr{E}}^{\mathscr{E}} &= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}\\
&= \begin{bmatrix}\cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta\\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta\end{bmatrix} = \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}.
\end{aligned}$$

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Question 3.

Consider a basis $\mathscr{B}$ of $\mathbf{R}^2$ given by $v_1 = \left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]$ and $v_2 = \left[\begin{smallmatrix}-4\\3\end{smallmatrix}\right]$. Let T be a transformation that stretches in the v_1-direction by a factor of 2, and in the v_2-direction by a factor of 3. Compute $[T]^{\mathscr{E}}_{\mathscr{E}}$.

Solution. We have $Tv_1 = 2v_1$ and $Tv_2 = 3v_2$, which gives $[T]^{\mathscr{B}}_{\mathscr{B}} = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$. The change of basis matrices are given by

$$[\mathrm{Id}]^{\mathscr{E}}_{\mathscr{B}} = \begin{bmatrix}3 & -4\\4 & 3\end{bmatrix} \qquad \text{and} \qquad [\mathrm{Id}]^{\mathscr{B}}_{\mathscr{E}} = ([\mathrm{Id}]^{\mathscr{E}}_{\mathscr{B}})^{-1} = \frac{1}{25}\begin{bmatrix}3 & 4\\-4 & 3\end{bmatrix}.$$

Finally,

$$[T]^{\mathscr{E}}_{\mathscr{E}} = \begin{bmatrix}3 & -4\\4 & 3\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\frac{1}{25}\begin{bmatrix}3 & 4\\-4 & 3\end{bmatrix} = \frac{1}{25}\begin{bmatrix}3 & -4\\4 & 3\end{bmatrix}\begin{bmatrix}6 & 8\\-12 & 9\end{bmatrix} = \frac{1}{25}\begin{bmatrix}66 & -12\\-12 & 59\end{bmatrix}.$$

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Question 4.

Let U be the subspace of $\mathcal{C}^\infty(\mathbf{R})$ spanned by the basis $\mathscr{B}$ given by (e^x, xe^x, x^2e^x). Let $D \in \mathcal{L}(U)$ be the differentiation operator. Recall from Tutorial 5 that

$$[D]^{\mathscr{B}}_{\mathscr{B}} = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}.$$

Is D invertible? If so, compute $[D^{-1}]^{\mathscr{B}}_{\mathscr{B}}$ and use this to find the unique function $f \in U$ satisfying $f' = x^2e^x$.

Solution. Since $\det([D]^{\mathscr{B}}_{\mathscr{B}}) = 1$, the matrix $[D]^{\mathscr{B}}_{\mathscr{B}}$ is invertible and so D is invertible. We compute

$$\left[\begin{array}{c|c}\underbrace{\begin{matrix}1 & 1 & 0\\0 & 1 & 2\\0 & 0 & 1\end{matrix}}_{[D]^{\mathscr{B}}_{\mathscr{B}}} & \underbrace{\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}}_{\mathrm{Id}}\end{array}\right] \xrightarrow[r_2 \mapsto r_2 - 2r_3]{r_1 \mapsto r_1 - r_2 + 2r_3} \left[\begin{array}{c|c}\underbrace{\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}}_{\mathrm{Id}} & \underbrace{\begin{matrix}1 & -1 & 2\\0 & 1 & -2\\0 & 0 & 1\end{matrix}}_{([D]^{\mathscr{B}}_{\mathscr{B}})^{-1}}\end{array}\right].$$

So

$$([D]^{\mathscr{B}}_{\mathscr{B}})^{-1} = \begin{bmatrix}1 & -1 & 2\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}$$

and f is given by

$$[f]_{\mathscr{B}} = ([D]_{\mathscr{B}}^{\mathscr{B}})^{-1}[x^2e^x]_{\mathscr{B}} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

In other words, $f = 2e^x - 2xe^x + x^2e^x$. ■

Further Questions

Question 5.

[Continuing from Question 2] Compute $[S_\theta \circ S_\phi]_{\mathscr{E}}^{\mathscr{E}}$. *Explain what the transformation* $S_\theta \circ S_\phi$ *does geometrically.*

Solution.

$$\begin{aligned} [S_\theta \circ S_\phi]_{\mathscr{E}}^{\mathscr{E}} &= [S_\theta]_{\mathscr{E}}^{\mathscr{E}}[S_\phi]_{\mathscr{E}}^{\mathscr{E}} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi & \cos 2\theta \sin 2\phi - \sin 2\theta \cos 2\phi \\ \sin 2\theta \cos 2\phi - \cos 2\theta \sin 2\phi & \sin 2\theta \sin 2\phi + \cos 2\theta \cos 2\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\theta - 2\phi) & -\sin(2\theta - 2\phi) \\ \sin(2\theta - 2\phi) & \cos(2\theta - 2\phi) \end{bmatrix} = [R_{2\theta-2\phi}]_{\mathscr{E}}^{\mathscr{E}}. \end{aligned}$$

We conclude that the linear map $S_\theta \circ S_\phi$ is the rotation of angle $2\theta - 2\phi$. ■

Question 6.

Let B *and* D *be invertible* $n \times n$ *matrices over* $\mathbf{F}$. *Let* $\mathscr{B} = (b_1, \dots, b_n)$ *and* $\mathscr{D} = (d_1, \dots, d_n)$ *be bases of* $\mathbf{F}^n$, *where* b_i *and* d_i *are respectively the* i^{th} *column of* B *and* D. *Explain how you can compute* $[\mathrm{Id}_{\mathbf{F}^n}]_{\mathscr{B}}^{\mathscr{D}}$ *from* B *and* D.

Solution. Let $\mathscr{E}$ be the standard basis of $\mathbf{F}^n$. Since the b_is and d_is are column vectors in $\mathbf{F}^n$ we have $b_i = [b_i]_{\mathscr{E}}$ and $d_i = [d_i]_{\mathscr{E}}$. So

$$B = [b_1\ b_2\ \dots b_n] = [[b_1]_{\mathscr{E}}\ [b_2]_{\mathscr{E}}\ \dots\ [b_n]_{\mathscr{E}}] = [\mathrm{Id}]_{\mathscr{B}}^{\mathscr{E}} \qquad \text{and} \qquad D = [\mathrm{Id}]_{\mathscr{D}}^{\mathscr{E}}.$$

Since D is invertible, we obtain

$$[\mathrm{Id}_{\mathbf{F}^n}]_{\mathscr{B}}^{\mathscr{D}} = [\mathrm{Id}]_{\mathscr{E}}^{\mathscr{D}}[\mathrm{Id}]_{\mathscr{B}}^{\mathscr{E}} = D^{-1}B.$$

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