Assignment 2:
Orthogonalization and the QR Decomposition
Math 327/397 Winter 2023

Due: Tuesday Feb 21, 2023

Instructions

Submit a copy on myCourses by midnight on the due date or on paper in class. Include a printout of your Matlab m-files as well as output. Questions $1-5$ are for everyone. Question 6 is for Math 397.

Question 1: Orthogonal Matrices

(a) (4pts) Let $u, v \in \mathbb{R}^n$, $< u, v >= u^T v$ denote the standard inner product, and $\| \cdot \|$ denote the induced (2-)norm. If $Q \in \mathbb{R}^{n \times n}$ is orthogonal, show that Q preserves inner products, norms, distances, and angles.

(b) (6pts) Show that the orthogonal projection of u onto v, is the vector in $span(v)$ closest to u. Extend the result to subspaces, that is, if $W \subseteq \mathbb{R}^n$ is a subspace, show that the orthogonal projection of u onto W is the vector inside W closest to u. Is the closest vector unique? Justify.

Question 2: Classical Gram-Schmidt

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

In constructing the QR decomposition, ensure that R has non-negative diagonal elements. For by-hand computations, round to at 3 least significant digits after every step. (Please do not use root expressions as this will complicate your by-hand calculations.)

(a)(3pts) Compute by hand the reduced QR decomposition of A using the Classical Gram-Schmidt method. Show your work.

(b)(3pts) Assuming the input is full rank, write the pseudo-code for the Classical Gram-Schmidt (you may refer to the class notes.)

(c)(3pts) Implement the Classical Gram-Schmidt into a Matlab m-file.

(d)(1pts) Run your algorithm on A and paste the output.

Question 3: Modified Gram-Schmidt

(a) (3 pts) If q_1, q_2 are orthonormal and a_3 is linearly independent from q_1, q_2, we've seen that $v_3 = a_3 - (q_1^T a_3)q_1 - (q_2^T a_3)q_2$ produces v_3 orthogonal to both q_1 and q_2. Does that still hold if q_1, q_2 are unit length but not orthogonal to each other? Either prove or explain (with a clear argument or a specific counterexample) why v_3, as constructed, is not orthogonal to q_1 and q_2.

(b) – (d) Repeat parts (b) – (d) of Q2 using the MGS instead of CGS.

Question 4: Givens Rotations

Repeat question 2 with the Givens Rotations method. In this case, construct the full QR decomposition instead of the reduced one.

Question 5: Householder Reflections

Repeat question 2 with the Householder Reflections method. Again, construct the full QR decomposition.

Question 6 (Math 397): Equivalency of CGS and MGS

Show formally that the Classical and Modified Gram-Schmidt methods are mathematically equivalent, i.e. produce the same orthogonalization sequence q_k (assuming no rounding errors occur). Assume that A is full column-rank.