Question 1

a)

Let vectors $u = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0, v = b_nx^n + b_{n-1}x^{n-1} + \ldots + b_0$ where $u, v \in P_n(\mathbb{R}), a_i, b_j \in \mathbb{R}$, we also have scalars $m, l \in \mathbb{R}$, we can show the followings:

1. $$\begin{aligned} u + v &= (a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0) + (b_nx^n + b_{n-1}x^{n-1} + \ldots + b_0) \\ &= (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \ldots (a_0 + b_0) \\ &= c_nx^n + c_{n-1}x^{n-1} + \ldots c_0 \in P_n(\mathbb{R}), c_i \in \mathbb{R} \end{aligned}$$

2. $$\begin{aligned} mu &= m(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0) \\ &= ma_nx^n + ma_{n-1}x^{n-1} + \ldots + ma_0 \\ &= c_nx^n + c_{n-1}x^{n-1} + \ldots c_0 \in P_n(\mathbb{R}), c_i \in \mathbb{R} \end{aligned}$$

3. let vector $w = d_nx^n + d_{n-1}x^{n-1} + \ldots d_0, w \in P_n(\mathbb{R}), d_i \in \mathbb{R}$, then

$$\begin{aligned} (u + v) + w &= (a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 + b_nx^n + b_{n-1}x^{n-1} + \ldots + b_0) + d_nx^n + d_{b-1} \\ &= a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 + (b_nx^n + b_{n-1}x^{n-1} + \ldots + b_0 + d_nx^n + d_{n-1}x^{n-1} + \\ &= u + (v + w) \end{aligned}$$

4. $$\begin{aligned} m(lu) &= m(la_nx^n + la_{n-1}x^{n-1} + \ldots + la_0) \\ &= ml(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0) \\ &= (ml)u \end{aligned}$$

5. $$\begin{aligned} u + v &= (a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0) + (b_nx^n + b_{n-1}x^{n-1} + \ldots + b_0) \\ &= (b_nx^n + b_{n-1}x^{n-1} + \ldots + b_0) + (a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0) \\ &= v + u \end{aligned}$$

6. $$\begin{aligned} (m + l)u &= (m + l)a_nx^n + (m + l)a_{n-1}x^{n-1} + \ldots (m + l)a_0 \\ &= ma_nx^n + la_nx^n + ma_{n-1}x^{n-1} + la_{n-1}x^{n-1} + \ldots ma_0 + la_0 \\ &= (ma_nx^n + ma_{n-1}x^{n-1} + \ldots ma_0) + (la_nx^n + +la_{n-1}x^{n-1} + \ldots la_0) \\ &= mu + lu \end{aligned}$$

7. $$\begin{aligned} m(u+v) &= m(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 + b_nx^n + b_{n-1}x^{n-1} + \ldots + b_0) \\ &= ma_nx^n + ma_{n-1}x^{n-1} + \ldots + ma_0 + mb_nx^n + mb_{n-1}x^{n-1} + \ldots + mb_0 \\ &= m(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0) + m(b_nx^n + b_{n-1}x^{n-1} + \ldots + b_0) \\ &= mu + mv \end{aligned}$$

8. let vector $w = d_nx^n + d_{n-1}x^{n-1} + \ldots d_0, w \in P_n(\mathbb{R}), d_i \in \mathbb{R}$, we know that $w = \vec{0} \in P_n(\mathbb{R})$ if $\forall d_i = 0$

$$\begin{aligned} w + u &= 0x^n + 0x^{n-1} + \ldots 0 + a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 \\ &= a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 \\ &= u \end{aligned}$$

Thus $\exists \vec{0} \in P_n(\mathbb{R})$

9. We know that $-1 \in \mathbb{R}$, thus we know if $u \in P_n(\mathbb{R}), \forall u$, then we must have $-1u \in P_n(\mathbb{R})$

$$\begin{aligned} u - 1u &= a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 - (a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0) \\ &= (a_n - a_n)x^n + (a_{n-1} - a_{n-1})x^{n-1} + \ldots (a_0 - a_0) \\ &= 0x^n + 0x^{n-1} + \ldots + 0 \\ &= \vec{0} \end{aligned}$$

10. We know $1 \in \mathbb{R}$

$$\begin{aligned} 1u &= 1(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0) \\ &= 1a_nx^n + 1a_{n-1}x^{n-1} + \ldots + 1a_0 \\ &= a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 \\ &= u \end{aligned}$$

With all the above rules being satisfied, we can conclude that $(P_n(\mathbb{R}), \mathbb{R}, +, \cdot)$ equipped with the usual polynomial addition and scalar multiplication is a vector space.

b)

1. For $N(T)$, let $u, v \in N(T)$, then $T(u) = T(v) = 0$. We know that $N(T) = \{v \in V : T(v) = 0 \in W\}$, thus $N(T) \subseteq V$
    a. Since $T : V \to W$ is a linear transformation, we have: $T(u+v) = T(u) + T(v) = 0 + 0 = 0$, thus $u + v \in N(T)$
    b. Let $a \in \mathbb{R}$, then we have $T(au) = aT(u) = a0 = 0$, thus $au \in N(T)$
    c. Since V, W are vector spaces, both of them must contain the zero vector. $\vec{0}_v \in V, \vec{0}_w \in W$. We thus have $T(\vec{0}_v) = \vec{0}_w$, which implies that $\vec{0} \in N(T)$

Therefore, $N(T)$ is the subspace of V.

2. For $Im(T)$, let $u, v \in Im(T)$, then there must be $u', v' \in V$ such that $u = T(u'), v = T(v')$
    a. $T(u' + v') = T(u') + T(v') = u + v$, thus $u + v \in Im(T)$
    b. Let $a \in \mathbb{R}$, then $T(au') = aT(u') = au$, thus $au \in Im(T)$
    c. Since V, W are vector spaces, we have $\vec{0}_v \in V, \vec{0}_w \in W, \vec{0}_w = T(\vec{0}_v)$, thus $\vec{0} \in Im(T)$

   Therefore, $Im(T)$ is the subspace of W.

Question 2

a)

The set $\{v_1, v_2, v_3\}$ are linearly independent if the following equation has one trivial solution where $x_1 = x_2 = x_3 = 0$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we can write the equation in the following format:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] \overset{RREF}{=} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

There is no free variable, which implies that the set $\{v_1, v_2, v_3\}$ are linearly independent.

b)

Since the set $\{v_1, v_2, v_3\}$ has a degree of 3 and it is linearly independent, thus $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$

c)

Since the set $\{v_1, v_2, v_3\}$ is linearly independent and it spans $\mathbb{R}^3$, thus it is a basis of $\mathbb{R}^3$.

d)

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] \overset{RREF}{=} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus we have:

$$v = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = -11v_1 + 5v_2 + v_3$$

Question 3

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a)

1. $||x||_1 = \sum_{i=1}^{n} |x_i|$
   a. Need to prove: $x = \vec{0} \iff ||x||_1 \geq 0, ||x||_1 = 0$
      i. $x = \vec{0} \implies ||x||_1 \geq 0, ||x||_1 = 0$

         If we have $x = \vec{0}$, then $x_i = 0 \forall x_i \in \mathbb{R}$, which then implies that $\sum_{i=1}^{n} |x_i| \geq 0, \sum_{i=1}^{n} |x_i| = 0 \implies ||x||_1 \geq 0, ||x||_1 = 0$
      ii. $||x||_1 \geq 0, ||x||_1 = 0 \implies x = \vec{0}$, since we have $|x_i| \geq 0 \forall x_i \in \mathbb{R}$ and $\sum_{i=1}^{n} |x_i| = 0$, if we have $x_j > 0, \exists j \in [1, n]$, then we will have $||x||_1 = \sum_{i=1}^{n} |x_i| > 0$, which contradicts with our assumption. Thus there must be no $x_j > 0$, combined with $|x_i| \geq 0 \forall x_i \in \mathbb{R}$, we have $x_i = 0 \forall i \in [1, n]$, thus $x = \vec{0}$

      After proving 2 directions, we have $x = \vec{0} \iff ||x||_1 \geq 0, ||x||_1 = 0$
   b. Need to prove: $||ax||_1 = |a|||x||_1, a \in \mathbb{R}$

$$||ax||_1 = \sum_{i=1}^{n} |ax_i| = \sum_{i=1}^{n} |a||x_i| = |a| \sum_{i=1}^{n} |x_i| = |a|||x||_1$$

c. Need to prove: $||x+y||_1 \leq ||x||_1 + ||y||_1$ where $y = (y_1, y_2, \dots y_n)^T \in \mathbb{R}^n$

$$||x+y||_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + |y_i| \text{ by triangle inequality}$$

we know that

$$\sum_{i=1}^{n} |x_i| + |y_i| = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = ||x||_1 + ||y||_1$$

Thus, we have shown $||x+y||_1 \leq ||x||_1 + ||y||_1$

With all 3 rules being satisfied, we know that $||\cdot||_1$ defines norms.

2. $||x||_2 = (x^T x)^{1/2}$

a. Need to prove: $x = \vec{0} \iff ||x||_2 \geq 0, ||x||_2 = 0$

i. $x = \vec{0} \implies ||x||_2 \geq 0, ||x||_2 = 0$

If we have $x = \vec{0}$, then we know $x_i = 0 \forall i \in [1, n]$, which results in $x^T x = x_1^2 + x_2^2 + \dots x_n^2 = 0$. Thus $||x||_2 = (x^T x)^{1/2} = 0$

ii. $||x||_2 \geq 0, ||x||_2 = 0 \implies x = \vec{0}$

If $||x||_2 = (x^T x)^{1/2} = 0$, then we must have: $x_1^2 + x_2^2 + \dots x_n^2 = 0$, since $x_i^2 \geq 0$ and if we assume $\exists j \; s.t \; x_j^2 > 0$, then $x_1^2 + x_2^2 + \dots x_n^2 > 0$, which is contradictory to our prior condition. Thus $x_i^2 = 0 \implies x_i = 0 \forall i \in [1, n]$, thus we have $x = \vec{0}$

b. Need to prove: $||ax||_2 = |a| ||x||_2, a \in \mathbb{R}$

$$||ax||_2 = (ax^T ax)^{1/2} = (a^2)^{1/2} (x^T x)^{1/2} = |a| (x^T x)^{1/2} = |a| ||x||_2$$

c. Need to prove: $||x+y||_2 \leq ||x||_2 + ||y||_2$ where $y = (y_1, y_2, \dots y_n)^T \in \mathbb{R}^n$

By Cauchy-Schwarz's inequality, we have:

$$\begin{gathered} ||x+y||_2^2 = < x+y, x+y > \\ = ||x||_2^2 + < x, y > + < y, x > + ||y||_2^2 \\ \leq ||x||_2^2 + 2| < x, y > | + ||y||_2^2 \\ \leq ||x||_2^2 + 2||x||_2 ||y||_2 + ||y||_2^2 \text{ by Cauchy-Schwarz's inequality} \\ \leq (||x||_2 + ||y||_2)^2 \end{gathered}$$

By taking square root on both sides, we obtain:

$$||x+y||_2 \leq ||x||_2 + ||y||_2$$

With all 3 rules being satisfied, we know that $||\cdot||_2$ defines norms.

3. $||x||_\infty = \max_i |x_i|$

    a. Need to prove: $x = \vec{0} \iff ||x||_\infty \geq 0, ||x||_\infty = 0$

        i. $x = \vec{0} \implies ||x||_\infty \geq 0, ||x||_\infty = 0$

        $x = \vec{0} \implies x_i = 0 \forall i \in [1, n]$, therefore
        $\max_i |x_i| = 0, \max_i |x_i| \geq 0$ by the definition of $\max_i$.

        ii. $||x||_\infty \geq 0, ||x||_\infty = 0 \implies x = \vec{0}$

        Given $\max_i |x_i| = 0, \max_i |x_i| \geq 0$, then if $\exists x_j > 0, j \in [1, n]$,

        then $\max_i |x_i| > 0$, which is contradictary to our assumption.

        Therefore $x_i = 0 \forall i \in [1, n]$, which implies that $x = \vec{0}$

    b. Need to prove: $||ax||_\infty = |a|||x||_\infty, a \in \mathbb{R}$

$$||ax||_\infty = \max_i |ax_i| = \max_i |a||x_i| = |a| \max_i |x_i| = |a|||x||_\infty$$

4. Need to prove: $||x+y||_\infty \leq ||x||_\infty + ||y||_\infty$ where $y = (y_1, y_2, \dots y_n)^T \in \mathbb{R}^n$

$$||x+y||_\infty = \max_i |x_i + y_i|$$

$$\leq \max_i (|x_i| + |y_i|) \text{ by triangle inequality}$$

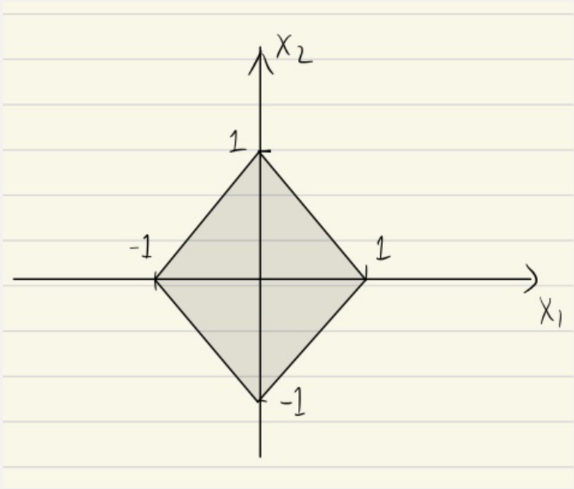
$$\leq \max_i |x_i| + \max_i |y_i|$$

$$\leq ||x||_\infty + ||y||_\infty$$

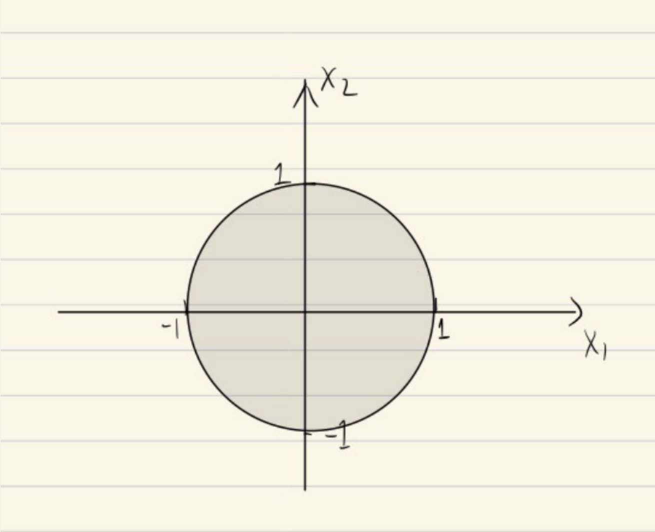
With all 3 rules being satisfied, we know that $||\cdot||_\infty$ defines norms.

b)

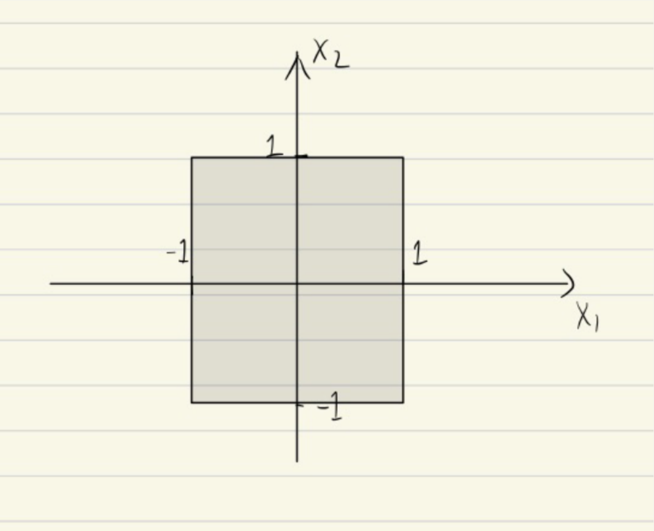
1. $||x||_1 = \sum_{i=1}^n |x_i|$



2. $||x||_2 = (x^T x)^{1/2}$



3. $||x||_\infty = \max\limits_i |x_i|$



Question 4

a)

1. Let $x = (x_1, x_2 ... x_n)^T, y = (y_1, y_2 ... y_n)^T, x, y \in \mathbb{R}^n$

$$< x, y >= x^T y = x_1 y_1 + x_2 y_2 + ... x_n y_n$$
$$< y, x >= y^T x = y_1 x_1 + y_2 x_2 + ... y_n x_n$$

we know that $x_i y_i = y_i x_i \forall x_i, y_i \in \mathbb{R}$, therefore we have $< x, y >=< y, x >$

2. Let $a \in \mathbb{R}$, then

$$< ax, y >= (ax^T)y$$
$$= ax_1y_1 + ax_2y_2 + \dots . ax_ny_n$$
$$= a(x_1y_1 + x_2y_2 + \dots . x_ny_n)$$
$$= a(x^Ty) = a < x, y >$$

3. Need to show: $< x, x > \geq 0, < x, x >= 0 \iff x = \vec{0}$

   a. $< x, x > \geq 0, < x, x >= 0 \implies x = \vec{0}$

$$< x, x >= x^Tx = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$$
$$x^Tx = x_1^2 + x_2^2 + \dots + x_n^2 = 0$$

   then we must have $x_i = 0 \forall i \in [1, n]$. Therefore, we have $x = (0_1, 0_2, 0_3 \dots 0_n)^T = \vec{0}$

   b. $x = \vec{0} \implies < x, x > \geq 0, < x, x >= 0$

   $x = \vec{0} \implies x = (0_1, 0_2, 0_3 \dots 0_n)^T$, then we have $< x, x >= x^Tx = 0 \cdot 0 + 0 \cdot 0 + \dots + 0 \cdot 0 = 0, < x, x > \geq 0$

   Thus, we proved that $< x, x > \geq 0, < x, x >= 0 \iff x = \vec{0}$

With all 3 rules being true, we proved that $< x, y >= x^Ty$ defines an inner product.

b)

If $v = \vec{0}$, then we have $|< u, v >| = ||u||||v|| = 0$

If $v \neq \vec{0}$, we have the residual $r = u - Proj_v(u) = u - \frac{<u,v>}{<v,v>}v, u = r + \frac{<u,v>}{<v,v>}v$

$$||u||^2 = ||r + \frac{< u, v >}{< v, v >}v||^2$$

$$= ||r||^2 + ||\frac{< u, v >}{< v, v >}v||^2 \text{ by Pythagoras therorem since v is orthogonal to r}$$

$$= \frac{|< u, v >|^2}{|< v, v >|^2}||v||^2 + ||r||^2$$

$$= \frac{|< u, v >|^2}{(||v||^2)^2}||v||^2 + ||r||^2$$

$$= \frac{|< u, v >|^2}{||v||^2} + ||r||^2$$

$$\geq \frac{|< u, v >|^2}{||v||^2}$$

By rewriting the above equation, we obtain:

$$|< u,v > |^2 \leq ||u||^2||v||^2 \implies < u,v >\leq ||u||||v||$$

For such inequality to be equality, we must have following 2 directions being true:

1. $u = av \implies | < u,v > | = ||u||\ ||v||$

   $$u = av \implies r = av - \frac{a < v,v >}{< v,v >}v = av - av = 0$$

   If $r = 0$, then we will have:

   $$||u||^2 = \frac{|< u,v >|^2}{||v||^2} + ||r||^2 = \frac{|< u,v >|^2}{||v||^2}$$
   $$\implies | < u,v > | = ||u||\ ||v||$$

2. $|< u,v > | = ||u||\ ||v|| \implies u = av$

   Similarly, if we know $|< u,v > | = ||u||\ ||v||$, then we must have $r = 0$ from above eqution.

   With residual $r = 0$, we know that u, v must be collinear, which is equivalent with $u = av, a \in \mathbb{R}$

Therefore, we have proved that $|< u,v > | = ||u||\ ||v|| \iff u = av$

c)

Given $||v|| =< v,v >^{1/2}$

1. Need to prove: $||v|| \geq 0, ||v|| = 0 \iff v = \vec{0}$

   a. $||v|| \geq 0, ||v|| = 0 \implies v = \vec{0}$

   $$||v|| =< v,v >^{1/2}= (v^Tv)^{1/2} \geq 0, (v^Tv)^{1/2} = 0$$

   since $v_i^2 \geq 0$ and if we assume $\exists j\ s.t\ v_j^2 > 0$, then $v_1^2 + v_2^2 + ... = (v^Tv)^{1/2} > 0$, which is contradictory to our prior condition. Thus $v_i^2 = 0 \implies v_i = 0$, thus we have $v = \vec{0}$

   b. $v = \vec{0} \implies ||v|| \geq 0, ||v|| = 0$

   we have $v_i = 0 \forall v_i \in \mathbb{R}$, then we know that $||v|| = (v^Tv)^{1/2} = (v_1^2 + v_2^2 + ...)^{1/2} = 0$, and $||v|| \geq 0$

   Therefore, we proved that $||v|| \geq 0, ||v|| = 0 \iff v = \vec{0}$

2. Need to prove: Let $a \in \mathbb{R}$, then we have $||av|| = |a|||v||$

   $$||av|| =< av,av >^{1/2}= (a^2 < v,v >)^{1/2} = |a| < v,v >^{1/2}= |a|||v||$$

3. Need to prove: $||u+v|| \leq ||u|| + ||v||$ where $v \in V$

$$||u+v||^2 =< u+v, u+v >$$
$$= ||u||^2 + < u, v > + < v, u > + ||v||^2$$
$$\leq ||u||^2 + 2| < u, v > | + ||v||^2$$
$$\leq ||u||^2 + 2||u||||v|| + ||v||^2 \text{ by Cauchy-Schwarz's inequality}$$
$$\leq (||u|| + ||v||)^2$$

With all 3 rules being satisfied, we know that $||v|| =< v, v >^{1/2}$ is a norm.

We know that the parallelogram law is implicitly dervied from inner product with L2 norm.

$$||x||^2 + ||y||^2 = ||x+y||^2 + ||x-y||^2$$

Counter Example: take $||x||_\infty = \max_i |x_i|$, we will show that the parallelogram law does not hold with $||x||_\infty$.

$$||x+y||_\infty^2 + ||x-y||_\infty^2 = (\max_i |x_i + y_i|)^2 + (\max_i |x_i - y_i|)^2$$
$$= \max_i (x_i + y_i)^2 + \max_i (x_i - y_i)^2$$
$$= \max_i (x_i^2 + 2x_i y_i + y_i^2) + \max_i (x_i^2 - 2x_i y_i + y_i^2)$$
$$= \max_i x_i^2 + \max_i x_i^2 + \max_i y_i^2 + \max_i y_i^2$$
$$= 2\max_i x_i^2 + 2\max_i y_i^2$$
$$= 2\max_i |x_i|^2 + 2\max_i |y_i|^2$$
$$= 2(||x||_\infty^2 + ||y||_\infty^2)$$
$$\neq ||x||_\infty^2 + ||y||_\infty^2$$

therefore, we found a counter example $|| \cdot ||_\infty$ which can not be induced by an inner product in such a way.

Question 5

a)

Let $u = (u_1, u_2, u_3)^T_{\beta_3}, v = (v_1, v_2, v_3)^T_{\beta_3}, u, v \in \mathbb{R}^3$, take $a \in \mathbb{R}$

We know that $T((x_1, x_2, x_3)^T_{\beta_3}) = (2x_1 + 3x_2 - x_3, x_1 + 2x_3)^T_{\beta_2}$

1.
$$\begin{gathered} T(u+v) = T((u_1+v_1, u_2+v_2, u_3+v_3)^T_{\beta_3}) \\ = (2(u_1+v_1) + 3(u_2+v_2) - (u_3+v_3), u_1+v_1+3(u_3+v_3))^T_{\beta_2} \\ = ((2u_1+3u_2-u_3) + (2v_1+3v_2-v_3), (u_1+3u_3)+(v_1+3v_3))^T_{\beta_2} \\ = (2u_1+3u_2-u_3, u_1+3u_3)^T_{\beta_2} + (2v_1+3v_2-v_3, v_1+3v_3)^T_{\beta_2} \\ = T(u) + T(v) \end{gathered}$$

2.
$$\begin{gathered} T(au) = (2au_1 + 3au_2 - au_3, au_1 + 3au_3)^T_{\beta_2} \\ = (a(2u_1+3u_2-u_3), a(u_1+3u_3))^T_{\beta_2} \\ = a(2u_1+3u_2-u_3, u_1+3u_3)^T_{\beta_2} \\ = aT(u) \end{gathered}$$

With the 2 rules above being satisfied, we can say that T is a linear transformation.

b)

$$T : \mathbb{R}^3 \to \mathbb{R}^2 := \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 2e_1 + 3e_2 - e_3 \\ e_1 + 2e_3 \end{bmatrix}$$

c)

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -5/3 & 0 \end{array}\right]$$

we thus have:

$$\begin{gathered} e_1 = -2e_3 \\ e_2 = \frac{5}{3}e_3 \\ e_3 \text{ is free} \end{gathered}$$

$N(T) = \{a(-2, \frac{5}{3}, 1)^T : a \in \mathbb{R}\}$, since there is 1 free variable in $N(T)$, $Nullity(T) = 1$

Since we observe 2 pivots in the reduced matrix form, we can express the last column as a linear combination of the other two columns: $(-1, 2)^T = 2(2, 1)^T - \frac{5}{3}(3, 0)^T$, thus $Im(T) = span(\{(2, 1)^T, (3, 0)^T\}), rank(T) = 2$.