Assignment 3: Householder Matrices and Linear Least Squares
Math 327/397 Winter 2023

Due: Monday 6 March, 2023

Instructions

Submit a copy on myCourses by midnight on the due date or on paper in class. Include a printout of your Matlab m-files as well as output. Questions $1-5$ are for everyone. Question 6 is for Math 397.

Question 1: Householder Matrices I

Let $v \in \mathbb{R}^m$ be a non-zero vector. For the Householder matrix $H_v = I - 2\frac{vv^T}{v^T v}$, prove the following properties:

(a)(2pts) H_v is an orthogonal matrix

(b)(2pts) H_v is symmetric

(c)(2pts) $H_v^2 = I_m$

(d)(2pts) H_v has eigenvalues -1 with one corresponding eigenvector, and +1 with (m-1) corresponding eigenvectors

(e)(2pts) $\det H_v = -1$

Question 2: Householder Matrices II

(a)(5pts) Let $\tilde{v} \in \mathbb{R}^n$ with corresponding Householder matrix $H_{\tilde{v}}$, and let $v = \begin{bmatrix} 0 \\ \tilde{v} \end{bmatrix}$ where the zero block has k entries. Here, both n and k are positive integers. Show that the Householder matrix corresponding to v takes the form

$$H_v = \begin{bmatrix} I_{k\times k} & 0_{k\times n} \\ 0_{n\times k} & H_{\tilde{v}} \end{bmatrix}.$$

(b)(5pts) Consider the vector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Describe the space that the Householder matrix H_v reflects across? Write the answer both in terms of a span of a vector as well as by drawing it on the 2-dimensional cartesian plane. Show the action H_v has on the vector $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ both as a calculation and also by drawing it on the same plane.

Question 3: Least Squares - Normal Equations

Experimental data produces the following table

x_i	y_i
1.00	5.10
1.25	5.79
1.50	6.53
1.75	7.45
2.00	8.44

(a)(5pts) It is believed that $y \approx \alpha_0 + \alpha_1 x$. Formulate a linear least squares problem for this model and solve for α_0, α_1 using the normal equations. Write down your intermediate steps. You may use Matlab for any routine calculations.

(b)(5pts) Repeat part (a) for the model $y \approx \alpha_0 + \alpha_1 x + \alpha_2 x^2$.

Question 4: Least Squares - QR Decomposition

(a)(5pts) Repeat Question 3(b) this time by using the QR Decomposition method. You may use a routine you've already implemented or Matlab's function call [Q,R] = qr(A). Show your intermediate steps. You may refer to parts of Question 3 so as to avoid duplication of work.

(b)(5pts) Write a Matlab function *alpha = solveLS(y,A)* which takes input vector y and data matrix A and returns the Least Squares solution vector *alpha* using the QR Decomposition method. You may reuse one which you have implemented in Assignment 2 or Matlab's $[Q, R] = qr(A, 0)$ which computes the reduced version of the decomposition. Run your function on the input matrix above and display the output.

Question 5: Open Modeling Question

You would like to find out the running time polynomial of Matlab's matrix inversion function *inv(A)*. In other words, for a matrix of size $n \times n$, what polynomial in n does the algorithm take to run? But you have no visibility as to how the algorithm is implemented. How would you go about finding out?

Write down clearly your thinking process, your assumptions if any, as well as your full solution together with any Matlab supporting functionality (excluding routine matrix multiplications). You may find the following Matlab commands handy: *randn(n,n)*, *tic*, *toc*.

Question 6 (Math 397): Eigenvalue Decomposition of a Householder Matrix

Let $v \in \mathbb{R}^m$ be nonzero. What is the eigenvalue value decompositions of the Householder matrix $H_v = I - 2\frac{vv^T}{v^Tv}$? Justify.