Question 1

a)

To show H_v is an orthogonal matrix, which means we need to show: $H_v^T H_v = I$

$$H_v^T = (I - 2\frac{vv^T}{v^T v})^T = I - 2\frac{(vv^T)^T}{(v^T v)^T} = I - 2\frac{vv^T}{v^T v}$$

$$H_v = I - 2\frac{vv^T}{v^T v} = H_v^T$$

$$\begin{aligned} H_v^T H_v &= (I - 2\frac{vv^T}{v^T v})(I - 2\frac{vv^T}{v^T v}) \\ &= I - 4\frac{vv^T}{v^T v} + 4\frac{vv^T vv^T}{v^T vv^T v} \\ &= I - 4\frac{vv^T}{v^T v} + 4\frac{v^T vvv^T}{v^T vv^T v} \\ &= I - 4\frac{vv^T}{v^T v} + 4\frac{vv^T}{v^T v} \\ &= I \end{aligned}$$

b)

To show H_v is symmetric, we need to show: $H_v = H_v^T$

$$H_v^T = (I - 2\frac{vv^T}{v^T v})^T = I - \frac{2}{v^T v}(vv^T)^T = I - 2\frac{vv^T}{v^T v}$$

$$H_v = I - 2\frac{vv^T}{v^T v} = H_v^T$$

c)

From b, we know that $H_v = H_v^T$, thus $H_v^2 = H_v^T H_v$

From a, we know that $H_v^T H_v = I$

Thus by combining the result from a and b, we know that $H_v^2 = I$

d)

Expand a orthogonal basis $\{v, v_2, \dots v_m\}$ of $\mathbb{R}^m$, we have the following:

$$v^T v = 1, H_v v = v - 2\frac{vv^T v}{v^T v} = v - 2v = -v$$

Then the eigen value is -1 and the eigen vector associated with it is v

$$v^T v_j = 0, H_v v_j = v_j - 2\frac{vv^T v_j}{v^T v} = v_j - 0 = v_j, v_j \in \{v_2, \dots v_m\}$$

The eigen value is 1, and there are $m-1$ corresponding eigen vectors since that's the cardinality of $\{v_2, \dots v_m\}$ when we execlude v out of it.

e)

We know that $det(H_v)$ equals to the product of the eigenvalues of H_v, thus combine with d, we have:

$$det(H_v) = -1 \times 1 = -1$$

Question 2

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a)

Given $v = \begin{bmatrix} 0 \\ \tilde{v} \end{bmatrix}$, then we have $v^T v = \tilde{v}^T \tilde{v}, vv^T = \begin{bmatrix} 0_{k \times k} & 0_{k \times n} \\ 0_{n \times k} & \tilde{v}\tilde{v}^T \end{bmatrix}$ by the nature of matrix multiplication.

$$H_v = I_{n+k} - 2\frac{vv^T}{v^T v} = I_{n+k} - 2\frac{vv^T}{\tilde{v}^T \tilde{v}}$$

since we have $vv^T = \begin{bmatrix} 0_{k\times k} & 0_{k\times n} \\ 0_{n\times k} & \tilde{v}\tilde{v}^T \end{bmatrix}$ being a square matrix with dimension $n+k$ by $n+k$, we can rewrite the above equation to:

$$H_v = I_{n+k} - 2\frac{vv^T}{\tilde{v}^T\tilde{v}} = \begin{bmatrix} I_{k\times k} & 0_{k\times n} \\ 0_{n\times k} & I_{n\times n} - 2\frac{\tilde{v}\tilde{v}^T}{\tilde{v}^T\tilde{v}} \end{bmatrix} = \begin{bmatrix} I_{k\times k} & 0_{k\times n} \\ 0_{n\times k} & H_{\tilde{v}} \end{bmatrix}$$

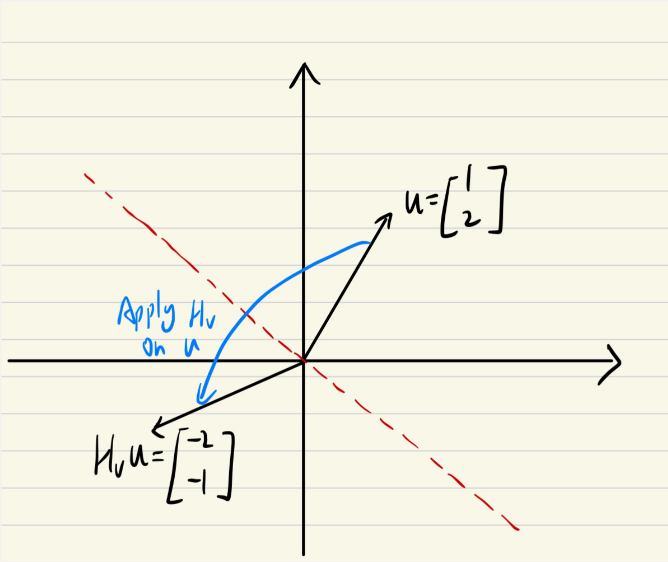
b)?

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2, H_v = I - \frac{2vv^T}{v^Tv} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

The space that the Householder matrix H_v reflect across is $span(v^\perp)$. Say we have $w = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \perp v$

then $v^Tw = w_0 + w_1 = 0, \implies w_0 = -w_1 \implies w = \begin{bmatrix} -w_1 \\ w_1 \end{bmatrix}$. Thus H_v reflect across $span(\begin{bmatrix} -1 \\ 1 \end{bmatrix})$

Visually, it is drew as the red dotted line below.



$$H_v u = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Q3

a)

$$\begin{bmatrix} 5.1 \\ 5.79 \\ 6.53 \\ 7.45 \\ 8.44 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1.25 \\ 1 & 1.5 \\ 1 & 1.75 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

$$\Updownarrow$$

$$y = A\alpha$$

The normal equations will be:

$$A^T(y - A\alpha) = 0$$
$$A^Ty - A^TA\alpha = 0$$
$$\alpha = (A^TA)^{-1}A^Ty$$

$$\alpha = \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1.25 & 1.5 & 1.75 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1.25 \\ 1 & 1.5 \\ 1 & 1.75 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1.25 & 1.5 & 1.75 & 2 \end{bmatrix} \begin{bmatrix} 5.1 \\ 5.79 \\ 6.53 \\ 7.45 \\ 8.44 \end{bmatrix}$$
$$= \begin{bmatrix} 1.658 \\ 3.336 \end{bmatrix}$$

thus $y \approx 1.658 + 3.336x$

b)

$$A^T(y - A\alpha) = 0$$
$$A^Ty - A^TA\alpha = 0$$

$\alpha = (A^TA)^{-1}A^Ty$

$$= \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1.25 & 1.5 & 1.75 & 2 \\ 1 & 1.5625 & 2.25 & 3.0625 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.25 & 1.5625 \\ 1 & 1.5 & 2.25 \\ 1 & 1.75 & 3.0625 \\ 1 & 2 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.25 & 1.5 & 1.75 \\ 1 & 1.5625 & 2.25 & 3.0625 \end{bmatrix}$$

$$= \begin{bmatrix} 3.5523 \\ 0.6617 \\ 0.8914 \end{bmatrix}$$

thus $y \approx 3.5523 + 0.6617x + 0.8914x^2$

Question 4

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a)

$$Q^T(y - A\alpha) = 0, A = QR$$
$$Q^Ty - Q^TQR\alpha = 0$$
$$Q^Ty = R\alpha$$

From matlab function call: `[Q,R] = qr(A, 0)` with $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.25 & 1.5625 \\ 1 & 1.5 & 2.25 \\ 1 & 1.75 & 3.0625 \\ 1 & 2 & 4 \end{bmatrix}$

we obtain:

$$Q = \begin{bmatrix} 0.4472 & -0.6325 & 0.5345 \\ 0.4472 & -0.3162 & -0.2673 \\ 0.4472 & 0 & -0.5345 \\ 0.4472 & 0.3162 & -0.2673 \\ 0.4472 & 0.6325 & 0.5345 \end{bmatrix}, Q^T = \begin{bmatrix} 0.4472 & 0.4472 & 0.4472 & 0.4472 & 0.447 \\ -0.6325 & -0.3162 & 0 & 0.3162 & 0.632 \\ 0.5345 & -0.2673 & -0.5345 & -0.2673 & 0.534 \end{bmatrix}$$

$$R = \begin{bmatrix} 2.2361 & 3.3541 & 5.3107 \\ 0 & 0.7906 & 2.3717 \\ 0 & 0 & 0.2339 \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$Q^T y = \begin{bmatrix} 0.4472 & 0.4472 & 0.4472 & 0.4472 & 0.4472 \\ -0.6325 & -0.3162 & 0 & 0.3162 & 0.6325 \\ 0.5345 & -0.2673 & -0.5345 & -0.2673 & 0.5345 \end{bmatrix} \begin{bmatrix} 5.1 \\ 5.79 \\ 6.53 \\ 7.45 \\ 8.44 \end{bmatrix}$$

$$= \begin{bmatrix} 14.8967 \\ 2.6373 \\ 0.2085 \end{bmatrix}$$

Thus we have:

$$\begin{cases} 0.2339\alpha_2 = 0.2085 \\ 2.3717\alpha_2 + 0.7906\alpha_1 = 2.6373 \\ 5.3107\alpha_2 + 3.3541\alpha_1 + 2.2361\alpha_0 = 14.8967 \end{cases} \implies \begin{cases} \alpha_2 = 0.8914 \\ \alpha_1 = 0.6617 \\ \alpha_0 = 3.5523 \end{cases}$$

$y \approx 3.5523 + 0.6617x + 0.8914x^2$

b)

```
function alpha = solveLS(y, A)
[Q,R] = qr(A, 0);
n = size(R,1);
qTy = Q' * y;
alpha = zeros(n,1);
alpha(n) = qTy(n)/R(n,n);
for i = n-1:-1:1
    alpha(i) = (qTy(i) - R(i,i+1:n)*alpha(i+1:n))/R(i,i);
end

```

```
A = [1 1 1; 1 1.25 1.25^2; 1 1.5 1.5^2; 1 1.75 1.75^2; 1 2 4]
```

```
A = 5×3
    1.0000    1.0000    1.0000
    1.0000    1.2500    1.5625
    1.0000    1.5000    2.2500
    1.0000    1.7500    3.0625
    1.0000    2.0000    4.0000
```

```
y = [5.1;5.79;6.53;7.45;8.44]
```

```
y = 5×1
    5.1000
    5.7900
    6.5300
    7.4500
    8.4400
```

```
alpha = solveLS(y, A)
```

```
alpha = 3×1
    3.5523
    0.6617
    0.8914
```

Question 5

I first assume that the polynomial relationship between the execution time y of running $inv(A)$ where A is a $n \times n$ matrix is as follow:

$$y = \alpha_0 + \alpha_1 n + \alpha_2 n^2$$

Considering the computation resources limitations, I ran 10 experiments of executing $inv(A)$ and kept track of the time of each run. A is randomly initialized for each run of experiment, and the size of it is increasing as experiments go. Here is my whole experiement code:

```
for i=1:10
  A = randn(i * 3000, i * 3000);
  tic
  inv(A);
  toc
end
```

I obtain:

$$\begin{bmatrix} 0.614477 \\ 4.100077 \\ 12.074519 \\ 30.684360 \\ 59.414220 \\ 103.630225 \\ 161.962510 \\ 242.159408 \\ 353.229978 \\ 533.150282 \end{bmatrix} = \begin{bmatrix} 1 & 3000 & 9000000 \\ 1 & 6000 & 36000000 \\ 1 & 9000 & 81000000 \\ 1 & 12000 & 144000000 \\ 1 & 15000 & 225000000 \\ 1 & 18000 & 324000000 \\ 1 & 21000 & 441000000 \\ 1 & 24000 & 576000000 \\ 1 & 27000 & 729000000 \\ 1 & 30000 & 900000000 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Where:

$$y = \begin{bmatrix} 0.614477 \\ 4.100077 \\ 12.074519 \\ 30.684360 \\ 59.414220 \\ 103.630225 \\ 161.962510 \\ 242.159408 \\ 353.229978 \\ 533.150282 \end{bmatrix}, A = \begin{bmatrix} 1 & 3000 & 9000000 \\ 1 & 6000 & 36000000 \\ 1 & 9000 & 81000000 \\ 1 & 12000 & 144000000 \\ 1 & 15000 & 225000000 \\ 1 & 18000 & 324000000 \\ 1 & 21000 & 441000000 \\ 1 & 24000 & 576000000 \\ 1 & 27000 & 729000000 \\ 1 & 30000 & 900000000 \end{bmatrix}$$

Using the method I defiend above: `solveLS(y, A)`, I solved for α, which is:

$$\alpha = \begin{bmatrix} 58.6698 \\ -0.0160 \\ 0.0000 \end{bmatrix}$$

But due to rounding errors, it’s not a satisfying solution.

I realized that my assumption should be adjusted since the size of the matrix A is $n \times n$, thus it makes more sense to have only the square relationship:

$$y \approx \alpha_0 n^2$$

$$y = \begin{bmatrix} 0.614477 \\ 4.100077 \\ 12.074519 \\ 30.684360 \\ 59.414220 \\ 103.630225 \\ 161.962510 \\ 242.159408 \\ 353.229978 \\ 533.150282 \end{bmatrix}, A = \begin{bmatrix} 9000000 \\ 36000000 \\ 81000000 \\ 144000000 \\ 225000000 \\ 324000000 \\ 441000000 \\ 576000000 \\ 729000000 \\ 900000000 \end{bmatrix}$$

solve for α again, I got:

$$\alpha = [4.8770 \times 10^{-7}]$$

Which is way better than the previous estimation, as $y \approx 4.8770 \times 10^{-7} n^2$