Assignment 1: Linear Algebra Review
Math 327/397 Winter 2023

Due: Tuesday Jan 31, 2023

Instructions

Submit a complete paper copy of your solutions in class or to the math department by 5:00pm on the due date. Questions 1 – 5 are for everyone. Question 6 is for Math 397 only.

Question 1: Vector Spaces

(a) Show that $(P_n(\mathbb{R}), \mathbb{R}, +, \cdot)$, the set of polynomials of degree n over $\mathbb{R}$ equipped with the usual polynomial addition and scalar multiplication, is a vector space.

(b) Let V, W be vector spaces and $T : V \to W$ be a linear transformation. Show that the Null Space $N(T)$ and Image $Im(T)$ are subspaces of V and W respectively.

Question 2: Linear Combinations

Consider $\mathbb{R}^3$ with the usual vector addition and scalar multiplication and let:

$$v = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

(a) Are $\{v_1, v_2, v_3\}$ linearly independent? Justify.

(b) Do $\{v_1, v_2, v_3\}$ span $\mathbb{R}^3$? Justify.

(c) Are $\{v_1, v_2, v_3\}$ a basis for $\mathbb{R}^3$?

(d) Express v as a linear combination of v_1, v_2, v_3.

Question 3: Norms

Consider $\mathbb{R}^n$ with the usual vector addition and scalar multiplication. For $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ let:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = (x^T x)^{1/2}, \quad \|x\|_\infty = \max_i |x_i|.$$

(a) Show that $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ define norms. You may invoke outside results (e.g. Cauchy-Schwarz) if needed.

(c) Draw the unit ball for each of these norms in $\mathbb{R}^2$, i.e. the set of all x such that $\|x\| = 1$.

Question 4: Inner Products

(a) For $x, y \in \mathbb{R}^n$, show that $< x, y >= x^T y$ defines an inner product.

(b) Let V be a vector space, with a real inner product $< \cdot, \cdot >$. Prove the Cauchy-Schwarz Inequality: $| < u, v > | \leq \|u\|\|v\|$ with equality if and only if $u = av$ for some scalar a.

(c) Let $(V, \mathbb{R}, +, \cdot, < \cdot, \cdot >)$ be a real inner product space. Show that the function $\|\cdot\| : V \to \mathbb{R}$ defined by $\|v\| =< v, v >^{1/2}$ is a norm. This is called the norm induced by the inner product. Also, give an example of a norm which cannot be induced by an inner product is such a way. Justify.

Question 5: Linear Transformations

(a) Let $\beta_3 = (e_1, e_2, e_3), \beta_2 = (e_1, e_2)$ represent the respective standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$. (Note that the symbols e_j are overloaded and their meaning depends on the context space.) Consider the operator $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T((x_1, x_2, x_3)^T_{\beta_3}) = (2x_1 + 3x_2 - x_3, x_1 + 2x_3)^T_{\beta_2}$. Show that T is a linear transformation.

(b) Write a matrix representation for the transformation in (a) relative to the standard bases β_3 and β_2.

(c) What are the Null Space, Image, Nullity and Rank of the linear transformation in (a)?

Question 6: Math 397 Only

Find an example of a distance function which cannot be induced by a norm. Justify in detail your claims.