Q1

a)

Since we know that Q is a orthogonal matrix, we have: $Q^TQ = I$

1. inner products. needs to show: $(Qu)^T \cdot (Qv) = u^Tv$

$$(Qu)^T \cdot (Qv) = u^TQ^TQv = u^TIv = u^Tv$$

2. norms. needs to show: $||Qu||_2 = ||u||_2$

$$\begin{aligned} ||Qu||_2 &= | < Qu, Qu >| \\ &= |(Qu)^T \cdot (Qu)| \\ &= |u^TQ^TQu| = |u^Tu| \\ &= | < u, u > | = ||u||_2 \end{aligned}$$

   then we have: $||Qu||_2 = ||u||_2$

3. distances. needs to show $||Qu - Qv||_2 = ||u - v||_2$

$$\begin{aligned} ||Qu - Qv||_2 &= | < Qu - Qv, Qu - Qv >| \\ &= | < Qu, Qu - Qv > - < Qv, Qu - Qv >| \\ &= |(< Qu, Qu > - < Qv, Qu >) - (< Qu, Qv > - < Qv, Qv >)| \\ &= |(u^Tu - v^Tu) - (u^Tv - v^Tv)| \\ &= |(< u, u > - < v, u >) - (< u, v > - < v, v >)| \\ &= | < u - v, u > - < u - v, v >| \\ &= | < u, u - v > - < v, u - v >| \\ &= | < u - v, u - v >| \\ &= ||u - v||_2 \end{aligned}$$

then we have: $||Qu - Qv||_2 = ||u - v||_2$

4. angles. needs to show: $\angle(Qu, Qv) = \angle(u, v)$

$$\theta = arccos(\frac{< Qu, Qv >}{||Qu||||Qv||})$$

$$= arccos(\frac{u^T Q^T Q v}{||Qu||_2||Qv||_2}) = arccos(\frac{u^T v}{||u||_2||v||_2})$$

which implies that: $\angle(Qu, Qv) = \angle(u, v)$

b)

1) show that the orthogonal projection of u onto v, is the vector in span(v) closest to u

Let $av \in span(v), a \in \mathbb{R}$, then $d(u, av) = ||u - av||$

The orthogonal projection of u onto v is denoted by $Proj_v(u)$, which is colinear with av. By definition, residual $r = u - Proj_v(u)$ and $u - Proj_v(u) \perp v$.

Since av and $Proj_v(u)$ are colinear, we have $Proj_v(u) - av$ being colinear with av as well.

Therefore, we have $(u - Proj_v(u)) \perp (Proj_v(u) - av)$, by Pythagoras theorem, we have:

$$||u - Proj_v(u) + Proj_v(u) - av||^2$$
$$= ||u - Proj_v(u)||^2 + ||Proj_v(u) - av||^2$$

$= ||u - av||^2$, which is the squared distance between u and any vector from span(v)

such value is minized when $av = Proj_v(u)$:

$$||u - Proj_v(u)||^2 + ||Proj_v(u) - av||^2$$
$$= ||u - Proj_v(u)||^2 + ||Proj_v(u) - Proj_v(u)||^2$$
$$= ||u - Proj_v(u)||^2 = ||u - av||^2$$

Therefore, the orthogonal projection of u onto v is the vector in span(v) closest to u.

2) show that the orthogonal projection of u onto W is the vector inside W closest to u

Let $x \in W$, Since W is a subspace and $Proj_W(u) \in W$, we can find
$y = Proj_W(u) + x \in W$

$$||u - y||^2 = ||u - (Proj_W(u) + x)||^2 = ||(u - Proj_W(u)) - x||^2$$

By definition of residual, $u - Proj_W(u) \perp x$, then by Pythagoras theorem, we have:

$$||(u - Proj_W(u)) - x||^2 = ||u - Proj_W(u)||^2 + ||x||^2$$

where $||(u - Proj_W(u)) - x||^2$ is the squared distance from u to subspace W. The distance is minimized if we have $||x||^2 = 0$, which implies that
$||u - y||^2 = ||u - Proj_W(u)||^2 \implies ||u - y|| = ||u - Proj_W(u)||$

Therefore, the orthogonal projection of u onto W is the vector inside W closest to u

The closet vector is not unique because

3) uniquness of closest vector

Assume there are $x, y \in W$ that are both orthogonal projection of u onto W, and $||u - x|| \leq ||u - y||$

Since x, y are both orthogonal projections of u onto W, we have:

$$||u - x|| = ||u - Proj_W(u)||$$
$$||u - y|| = ||u - Proj_W(u)||$$

Plus $||u - y||$ on both sides of $||u - x|| \leq ||u - y||$, we obtain:

$$||u - x|| + ||u - y|| \leq ||u - y|| + ||u - y||$$
$$2||u - Proj_W(u)|| \leq 2||u - Proj_W(u)|| \implies ||u - Proj_W(u)|| \leq ||u - Proj_W(u)||$$

Since $||u - Proj_W(u)||$ is always non-negative, the above inequality implies that $||u - Proj_W(u)|| = 0$, which means that the closest vector is indeed unique.

Q2

---

a)

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$v_1 = a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, q_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \\ 0 \end{bmatrix}, ||v_1|| = 1.414$$

$$v_2 = a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 0 \end{bmatrix}, q_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} 0.6667 \\ 0.3333 \\ -0.6667 \\ 0 \end{bmatrix}, q_1^T a_2 = 1.414, ||v_2|| = 3$$

$$v_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2 = \begin{bmatrix} -0.1197 \\ 0.4786 \\ 0.1197 \\ 0.8615 \end{bmatrix}, q_3 = \frac{v_3}{||v_3||} = \begin{bmatrix} -0.1197 \\ 0.4786 \\ 0.1197 \\ 0.8615 \end{bmatrix}$$

$$q_1^T a_3 = 2.8284, q_2^T a_3 = -0.6667, ||v_3|| = 4.6428$$

$$Q = \begin{bmatrix} 0.7071 & 0.6667 & -0.1197 \\ 0 & 0.3333 & 0.4786 \\ 0.7071 & -0.6667 & 0.1197 \\ 0 & 0 & 0.8615 \end{bmatrix}$$

$$R = \begin{bmatrix} 1.414 & 1.414 & 2.8284 \\ 0 & 3 & -0.6667 \\ 0 & 0 & 4.6428 \end{bmatrix}$$

b)

- **Input:** $A = [a_1, \dots a_n] \in \mathbb{R}^{m \times n}, m \geq n,$ full column rank
- **Algorithm:**

for j = 1,....n

$v_j = a_j$

for i = 1, ... j-1

$R_{ij} = q_i^T a_j$

$v_j = v_j - R_{ij} q_j$

end

$r_{jj} = ||v_j||_2$

$q_j = v_j / r_{jj}$

end

- **<u>Output:</u>** R (result matrix) and $Q = [q_1, \ldots q_n]$

c)

```
function [Q, R] = myGS(A)
[m,n] = size(A);
R = zeros(n,n);
Q = zeros(m,n);
for j = 1:n
  v = A(1:m, j);
  for i = 1:j-1
    R(i,j) = Q(1:m,i)' * A(1:m,j);
    v = v - R(i,j) * Q(1:m,i);
  end
  R(j,j) = sqrt(v' * v);
  Q(1:m, j) = v/R(j,j);
end
```

d)

```
% Q2
A = [1 3 1; 0 1 2; 1 -1 3; 0 0 4]
```

```
A = 4×3
     1     3     1
     0     1     2
     1    -1     3
     0     0     4
```

```
[Q,R] = myGS(A)
```

```
Q = 4×3
    0.7071    0.6667   -0.1197
         0    0.3333    0.4786
    0.7071   -0.6667    0.1197
         0         0    0.8615
R = 3×3
    1.4142    1.4142    2.8284
         0    3.0000   -0.6667
         0         0    4.6428
```

Q3

a)

If v_3 is orthogonal to both q_1 and q_2, then we should have:

$$q_1^T v_3 = 0, q_2^T v_3 = 0$$

Take example of $q_1^T v_3$, given that $||q_1|| = ||q_2|| = 1$, we can express it as:

$$\begin{aligned} q_1^T v_3 &= q_1^T (a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2) \\ &= q_1^T a_3 - q_1^T q_1 (q_1^T a_3) - q_1^T q_2 (q_2^T a_3) \\ &= q_1^T a_3 - q_1^T a_3 - q_1^T q_2 (q_2^T a_3) \\ &= -q_1^T q_2 (q_2^T a_3) \end{aligned}$$

which is a non-zero term. That is contradictory to the fact that $q_1^T v_3 = 0$.

Therefore, if q_1, q_2 are not orthogonal, the $v_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$ constructed can not be orthogonal to both q_1 and q_2.

If q_1, q_2 are indeed orthogonal, then we have $q_1^T q_2 = 0$, which resolves the contradiction.

b)

- **Input:** $A = [a_1, \dots a_n] \in \mathbb{R}^{m \times n}, m \geq n$, full column rank
- **Algorithm:**

  for j = 1,....n

  $v_j = a_j$

  for i = 1, ... j-1

  $R_{ij} = q_i^T v_j$

  $v_j = v_j - R_{ij} q_j$

  end

$r_{jj} = ||v_j||_2$

$q_j = v_j / r_{jj}$

end

- **Output:** R (result matrix) and $Q = [q_1, \dots q_n]$

c)

```
function [Q, R] = myMGS(A)
[m,n] = size(A);
R = zeros(n,n);
Q = zeros(m,n);
for j = 1:n
  v = A(1:m, j);
  for i = 1:j-1
    R(i,j) = Q(1:m,i)' * v;
    v = v - R(i,j) * Q(1:m,i);
  end
  R(j,j) = norm(v,2);
  Q(1:m, j) = v/R(j,j);
end
```

d)

```
% Q3
A = [1 3 1; 0 1 2; 1 -1 3; 0 0 4]
```

```
A = 4×3
     1     3     1
     0     1     2
     1    -1     3
     0     0     4
```

```
[Q,R] = myMGS(A)
```

```
Q = 4×3
    0.7071    0.6667   -0.1197
         0    0.3333    0.4786
    0.7071   -0.6667    0.1197
         0         0    0.8615
R = 3×3
    1.4142    1.4142    2.8284
         0    3.0000   -0.6667
         0         0    4.6428
```

Q4

a)

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

Step 1:

We modify on first and third row to start off.

- $r^2 = 1^2 + 1^2 \implies r = 1.414$
- $c = \frac{1}{1.414} = 0.707$
- $s = -\frac{1}{1.414} = -0.707$

$$G_1^T = \begin{bmatrix} c & 0 & -s & 0 \\ 0 & 1 & 0 & 0 \\ s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.707 & 0 & 0.707 & 0 \\ 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = G_1^T A = \begin{bmatrix} 0.707 & 0 & 0.707 & 0 \\ 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1.414 & 1.414 & 2.828 \\ 0 & 1 & 2 \\ 0 & -2.828 & 1.414 \\ 0 & 0 & 4 \end{bmatrix}$$

Step 2:

We modify on second and third row

- $r^2 = 1 + (-2.828)^2 \implies r = 3$
- $c = \frac{1}{3} = 0.333$
- $s = -\frac{-2.828}{3} = 0.943$

$$G_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.333 & -0.943 & 0 \\ 0 & 0.943 & 0.333 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = G_2^T G_1^T A = \begin{bmatrix} 1.414 & 1.414 & 2.828 \\ 0 & 3 & -0.6674 \\ 0 & 0 & 2.3569 \\ 0 & 0 & 4 \end{bmatrix}$$

Step 3:

We modify on third and fourth row

- $r^2 = 2.3569^2 + 4^2 \implies r = 4.643$
- $c = 2.3569/4.643 = 0.508$
- $s = -4/4.643 = -0.862$

$$G_3^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.508 & 0.862 \\ 0 & 0 & -0.862 & 0.508 \end{bmatrix}$$

$$\widetilde{R} = A_3 = G_3^T G_2^T G_1^T A = \begin{bmatrix} 1.414 & 1.414 & 2.828 \\ 0 & 3 & -0.667 \\ 0 & 0 & 4.645 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{Q} = G_1 G_2 G_3 = \begin{bmatrix} 0.7070 & 0.6667 & -0.1196 & 0.2029 \\ 0 & 0.3330 & 0.4790 & -0.8129 \\ 0.7070 & -0.6667 & 0.1196 & -0.2029 \\ 0 & 0 & 0.8620 & 0.5080 \end{bmatrix}$$

b)

- **Input:** $A = [a_1, \dots a_n] \in \mathbb{R}^{m \times n}, m \geq n,$ full column rank
- **Algorithm:**

  $R = A$

  $Q = I_m$

  for $j = 1 \dots n$

      for $i = 0 \dots m - 1 - j$

          $GT = I_m$

          $x = R_{m-i-1,j}$

          $y = R_{m-i,j}$

          $r = \sqrt{x^2 + y^2}$

          $c = x/r$

          $s = -y/r$

          $GT_{m-i,m-i} = c$

          $GT_{m-i-1,m-i-1} = c$

          $GT_{m-i-1,m-i} = s$

          $GT_{m-i,m-i-1} = -s$

          $R = GT \cdot R$

          $Q = Q \cdot GT^T$

      end

  end

- **Output:** $\widetilde{R}$ and $\widetilde{Q} = [q_1, \dots q_n, q_{n+1}, \dots, q_m]$

c)

```
function [Q,R] = myGivens(A)
[m,n] = size(A);
R = A;
Q = eye(m);
for j = 1:n
    for i = 0: m-1-j
        GT = eye(m);
        x = R(m-i-1,j);
        y = R(m-i,j);
        r = sqrt(x^2 + y^2);
        c = x/r;
        s = -y/r;
        GT(m-i, m-i) = c;
        GT(m-i-1, m-i-1) = c;
        GT(m-i-1, m-i) = -s;
        GT(m-i, m-i-1) = s;
        R = GT * R;
        Q = Q * GT';
    end
end
```

d)

```
% Q4
A = [1 3 1; 0 1 2; 1 -1 3; 0 0 4]
```

```
A = 4×3
     1     3     1
     0     1     2
     1    -1     3
     0     0     4
```

```
[Q,R] = myGivens(A)
```

```
Q = 4×4
    0.7071    0.6667   -0.1197    0.2031
         0    0.3333    0.4786   -0.8123
    0.7071   -0.6667    0.1197   -0.2031
         0         0    0.8615    0.5077
R = 4×3
    1.4142    1.4142    2.8284
         0    3.0000   -0.6667
         0    0.0000    4.6428
         0   -0.0000         0
```

Q5

a)

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

<u>Step 1:</u>

$$u = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_1 = \bar{v}_1 = u - ||u||e_1 = \begin{bmatrix} -0.4142 \\ 0 \\ 1.0000 \\ 0 \end{bmatrix}$$

$$H_{v_1} = I_4 - \frac{2v_1v_1^T}{v_1^Tv_1} = \begin{bmatrix} 0.7071 & 1 & 0.7071 & 1 \\ 0 & 0 & 0 & 0 \\ 0.7071 & 0 & -0.7071 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_1 = H_{v_1}A = \begin{bmatrix} 1.4142 & 1.4142 & 2.8284 \\ 0 & 1 & 2 \\ 0 & 2.8284 & -1.4142 \\ 0 & 0 & 4 \end{bmatrix}$$

<u>Step 2:</u>

$$u = \begin{bmatrix} 1 \\ 2.8284 \\ 0 \end{bmatrix}, \bar{v}_2 = u - ||u||e_2 = \begin{bmatrix} -2 \\ 2.8284 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ 2.8284 \\ 0 \end{bmatrix}$$

$$H_{v_2} = I_4 - \frac{2v_2v_2^T}{v_2^Tv_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.3333 & 0.9428 & 0 \\ 0 & 0.9428 & -0.3333 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = H_{v_2}H_{v_1}A = \begin{bmatrix} 1.4142 & 1.4142 & 2.8284 \\ 0 & 3 & -0.6667 \\ -0 & -0 & 2.3570 \\ 0 & 0 & 4 \end{bmatrix}$$

Step 3:

$$u = \begin{bmatrix} 2.357 \\ 4 \end{bmatrix}, \bar{v}_3 = u - ||u||e_3 = \begin{bmatrix} -2.2858 \\ 4 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ -2.2858 \\ 4 \end{bmatrix}$$

$$H_{v_3} = I_4 - \frac{2v_3v_3^T}{v_3^Tv_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5077 & 0.8615 \\ 0 & 0 & 0.8615 & -0.5077 \end{bmatrix}$$

$$\widetilde{R} = A_3 = H_{v_3}H_{v_2}H_{v_1}A = \begin{bmatrix} 1.4142 & 1.4142 & 2.8284 \\ 0 & 3 & -0.6667 \\ -0 & -0 & 4.6428 \\ -0 & -0 & 0 \end{bmatrix}$$

$$\widetilde{Q} = H_{v_1}H_{v_2}H_{v_3} = \begin{bmatrix} 0.7071 & 0.6667 & -0.1197 & -0.2031 \\ 0 & 0.3333 & 0.4786 & 0.8123 \\ 0.7071 & -0.6667 & 0.1197 & 0.2031 \\ 0 & 0 & 0.8615 & -0.5077 \end{bmatrix}$$

b)

- **Input:** $A = [a_1, \ldots a_n] \in \mathbb{R}^{m \times n}, m \geq n,$ full column rank
- **Algorithm:**

  $R = A$

  $Q = I_m$

  for $i = 1 \ldots n$

  $u = R_{i:m,i}$

  $\bar{v} = u - ||u||e_i$

  $$v = \begin{pmatrix} 0_1 \\ \vdots \\ 0_m \end{pmatrix}$$

  $v_{i:m,1} = \bar{v}$

  $H_v = I_m - 2\frac{vv^T}{v^T v}$

  $R = H_v R$

  $Q = Q H_v$

  end
- **Output:** $\widetilde{R}$ and $\widetilde{Q} = [q_1, \ldots q_n, q_{n+1}, \ldots, q_m]$

c)

```
function [Q,R] = myHouseHolder(A)
[m,n] = size(A);
Q = eye(m);
R = A;
for i = 1:n
    u = R(i:m, i);
    vb = u - (norm(u,2) * eye(m-i+1, 1));
    v = zeros(m,1);
    v(i:m , 1) = vb;
    Hv = eye(m) - 2 * (v * v') / (v' * v);
    R = Hv * R;
    Q = Q * Hv;
end
```

d)

```
% Q5
A = [1 3 1; 0 1 2; 1 -1 3; 0 0 4]
```

```
A = 4×3
     1     3     1
     0     1     2
     1    -1     3
     0     0     4
```

```
[Q,R] = myHouseHolder(A)
```

```
Q = 4×4
    0.7071    0.6667   -0.1197   -0.2031
         0    0.3333    0.4786    0.8123
    0.7071   -0.6667    0.1197    0.2031
         0         0    0.8615   -0.5077
R = 4×3
    1.4142    1.4142    2.8284
    0.0000    3.0000   -0.6667
   -0.0000   -0.0000    4.6428
   -0.0000   -0.0000    0.0000
```