Assignment 4: Singular Value Decomposition
Math 327/397 Winter 2023

Due: Tuesday, March 28, 2023

Instructions

Submit a copy on myCourses by midnight on the due date or on paper in class. Include a printout of your Matlab m-files as well as output. Questions 1 – 5 are for everyone. Question 6 is for Math 397, and this time also gives +5 points bonus for Math 327.

Question 1: Geometry of the SVD

(a)(3pts) Let $\Sigma = diag(3, 2, 0) \in \mathbb{R}^{3\times 3}$, and let the unit ball in the domain be described by $\{x \in \mathbb{R}^3 : \|x\|_2 \leq 1\}$. Using set notation as well as in words, describe the image of this unit ball under action by Σ. Which subset of the surface of the unit ball maps to the surface of the image object?

(a)(7pts) Here we generalize part (a). A k-dimensional ellipse, surface and interior, with axes along the standard coordinates is algebraically defined as the set of points $z = (z_1, z_2, \ldots, z_k)^T$ satisfying $\left(\frac{z_1}{\alpha_1}\right)^2 + \ldots + \left(\frac{z_k}{\alpha_k}\right)^2 \leq 1$. Note that we can have a k-dimensional ellipse embedded inside $\mathbb{R}^n$ even in the case $n > k$ by allowing some of the z_j to be identically zero. Using these definitions, show that the matrix $\Sigma = diag(\sigma_1, \ldots, \sigma_{\min(m,n)}) \in \mathbb{R}^{m\times n}$, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(m,n)} \geq 0$, maps the unit sphere $\{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$, surface and interior, to an ellipse embedded in $\mathbb{R}^m$. Under what conditions is the surface of the unit sphere mapped to the surface of the ellipse? (Suggestion: it may be handy to consider the cases $m \geq n$ and $n > m$ separately. Also, some of the axes of the ellipse may be zero, so it may be convenient to introduce $r \leq \min(m, n)$ such that $\sigma_1 \geq \ldots \geq \sigma_r > 0$.). In case the surface of the unit ball in the domain doesn't fully map to the surface of the image ellipse, write down into a partition of two sets, the parts of the unit ball which do map to the surface of the ellipse in the image, versus the parts which map to the interior.

Question 2: Geometry of the SVD with Matlab

Write a matlab function *plotMatrixImage(A,N,n)* which does the following:

- partitions the unit 2-D unit circle into N equally spaced angles, computes the x and y values for these and plots the circle.
- plots the image of that circle under multiplication by the 2-by-2 matrix A.
- generates n random points drawn from the normal distribution using *randn(2,n)*, plots them and also plots their image under multiplication by A.

Show the output for *plotMatrixImage(A,20,100)*, *plotMatrixImage(A,20,1000)*, and *plotMatrixImage(A,20,10000)* for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. This should give you three different figures where each time the function is run you plot all elements on a single figure, using the same scaling for the x and y axes.

Question 3: SVD as an Analytic Tool

Consider $A \in \mathbb{R}^{m \times n}$ and its (full) SVD $A = U\Sigma V^T$. Justifying your answers and using this SVD answer the following questions:

(a)(4pts) Express $Null(A), Range(A), Range(A)^{\perp}$. What is the dimension of each of these spaces? Here $Range(A)^{\perp}$ is the subspace of $\mathbb{R}^m$ orthogonal to the column space of A.

(b)(2pts) show that dim colspace A = dim rowspace A, i.e. the column and row ranks are the same. This justifies the single notion of *rank* for A.

(c)(2pts) What are the 2 and Frobenius norms of A?

(d)(2pts) What are the eigenvalues of A^TA? What about AA^T?

Question 4: SVD as a Computational Tool

Let $A \in \mathbb{R}^{5\times 4}$ have full SVD

$$A = U\Sigma V^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a)(2pts) What are $\|A\|_2$, $\|A\|_F$, $\|U\|_2$ and rank(A)?

(b)(2pts) State the dimension and a basis (if nontrivial) for each of range(A) and null(A).

(c)(2pts) State the reduced SVD of A.

(d)(2pts) State A^TA as a product of three 4×4 matrices. What are the eigenvalues and singular values of A^TA?

(e)(2pts) Let A_2 be the best rank-2 approximation to A in the 2-norm. What is $\|A - A_2\|_2$? State A_2. You do not need to compute the matrix, either state it as a sum of appropriative terms that you define or as a product of matrices.

Question 5: Compression with SVD - Application

Download the 256×256 image matrix *A.mat* from MyCourses, and read it into Matlab using *load A*. To see that it is an image execute (on the same line) *imagesc*(A); *colormap*(*gray*); We'll use the SVD to compute low rank approximations to A.

(a)(3pts) Using Matlab's command $[U, S, V] = svd(A)$, find the SVD of A and then compute the best (in the 2-norm) rank k approximations A_k to A for $k = 1, 2, 4, 16, 64$. Plot them and compare visually with the original image.

(b)(3pts) Using Matlab's command *norm* compute the error of A_k relative to A, i.e, $E_k := \|A - A_k\|_2 / \|A\|_2$, and plot a graph of E_k against k. Also plot the ratio σ_i/σ_1 of A against i for $i = 1, \ldots 256$. Comment on the two graphs. How large should k be to ensure that $E_k = \|A_k - A\|_2 / \|A\|_2 \leq 0.05$?

(c)(4pts) Let $A = \sum_{j=1}^{\min(m,n)} \sigma_j u_j v_j^T \in \mathbb{R}^{m \times n}$ represent the sum-of-rank-one matrices SVD of A, where the u_j are the left singular vectors, v_j are the right singular vectors and σ_j are ranked in decreasing order. Assuming that $rank(A) \geq k$, show that the best rank k approximation of A in the matrix 2-norm is given by $A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$.

Question 6 (Math 397), +5 bonus (Math 327): Least Squares for Non-Full Column Rank Matrices

If $A \in \mathbb{R}^{m \times n}$ is non-full column rank and $y \in \mathbb{R}^m$, using the SVD describe the solution set to $\min_{x \in \mathbb{R}^n} \|y - Ax\|_2$? Justify.