Midterm Examination
Math 327/397 Winter 2019

Monday 25 February, 2019, 18:00 – 20:00

McGill University

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Instructions

1. Answer all questions in the exam booklets provided.
2. Questions 1 – 5 are for everyone. Question 6 is for Math 397.
3. All questions carry equal weight.
4. This is a closed book exam. No crib sheets or any other aids are permitted.
5. Only nonprogrammable calculators are permitted.
6. For numeric calculations, use at least 3 significant digits of precision.
7. This exam comprises 1 title page and 2 pages of questions.

Question 1

(a) Define the term *span* for vectors within a vector space.

(b) Let $(V, \mathbb{R}, +, \cdot, < \cdot, \cdot >)$ be an inner product space. State the properties satisfied by the inner product map.

(c) Consider $\mathbb{R}^n$ equipped with the usual vector addition, scalar product and inner product $< u, v >= u^T v$, and let $u, v \in \mathbb{R}^n$, and $W \subset \mathbb{R}^n$ be a subspace. Define orthogonal projection of u onto v. Define orthogonal projection of u onto W.

(d) In what sense is the Modified Gram-Schmidt an improvement over the Gram-Schmidt method for orthogonalization?

(e) State the Full and Reduced QR decompositions for a full column rank matrix $A \in \mathbb{R}^{m \times n}$. Is the reduced QR factorization always unique? What about the full QR factorization? Briefly explain conditions which may ensure uniqueness - if any (no detailed proof required).

Question 2

Consider $\mathbb{R}^3$ with the usual vector addition and scalar multiplication and let:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \qquad v = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

(a)Write the linear combination $a_1 v_1 + a_2 v_2 + a_3 v_3$ as a matrix-vector product. Are the v_1, v_2, v_3 linearly independent? Do they span $\mathbb{R}^3$? Are they a basis for $\mathbb{R}^3$? Justify.

(b) Can you write v as a linear combination of v_1, v_2, v_3? Justify your answer.

Question 3

Consider the full column rank matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 2 \end{bmatrix}.$$

(a) Find the reduced QR factorization of A using Classical Gram-Schmidt.

(b) If you were to find the QR factorization of A using the Modified Gram Schmidt algorithm, which method do you expect to provide more precise answer for this matrix? Briefly justify.

Question 4

(a) For the matrix A in Question 3, state a Given's matrix G such that GA puts the weight of the $(3, 1)$ element into the $(1, 1)$ element of A, and hence creates a zero in the (3,1) position. Multiply this matrix to the left of A and compute the resulting product.

(b) Compute the full QR decomposition of A by starting from the answer in part (a) and continuing the Given's Algorithm. If you were unable to answer part (a), you may, instead, apply the full Given's method to the initial matrix A.

(c) Using the QR decomposition, state a basis for the span of the column space of A and a basis for the span of the orthogonal complement of the column space of A.

Question 5

(a) Let V, W be vector spaces and $T : V \to W$ be a linear transformation. Show that the Null Space $N(T)$ and Image $Im(T)$ are subspaces of V and W respectively.

(b) Let $B = [v_1, v_2, v_3]$ where v_1, v_2, v_3 are defined in Question 2. What are the Null Space, Image, Rank and Nullity of B? For the Null Space and Image, specify them by finding a basis and using span notation.

Question 6 (Math 397 Only)

Consider $\mathbb{R}^n$ equipped with the usual vector addition, scalar product and inner product $< u, v >= u^T v$, and let v, W be a vector and a subspace of $\mathbb{R}^n$ respectively. Give a matrix which represents the orthogonal projector operator onto $span(v)$. How would extend this definition to represent the orthogonal projector operator onto the subspace W? Show that your defined matrix correctly projects any $u \in \mathbb{R}^n$ orthogonally onto W. (Part of this question tests what you understand the concept of orthogonal projection onto a subspace.)