Chp.7 - Boundary-Value Problems for Ordinary Differential Equations

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The differential equations in Chapter 5 were :
* of first order
* and have one initial condition to satisfy.

These are initial-value problems. In this chapter we show how to approximate the solution to boundary-value problems i.e., differential equations with conditions imposed at different points.

Note : For first-order differential equations, only one condition is specified, so there is no distinction between initial-value and boundary-value problems. We will be considering second-order equations with two boundary values.

Methods Covered:

- The Linear Shooting Method.
- The Shooting Method for Nonlinear Problems.
- Finite-Difference Methods for Linear Problems.
- Finite-Difference Methods for Nonlinear Problems.
- The Rayleigh-Ritz Method.

The Linear Shooting Method

The Shooting method for linear equations is based on the replacement of the linear boundary value problem :

$$y'' = p(x)y' + q(x)y + r(x) \tag{1}$$

, for $a \leq x \leq b$, with $y(a) = \alpha$ and $y(b) = \beta$.

by the two initial-value problems :

$$y'' = p(x)y' + q(x)y + r(x) \tag{2}$$

, for $a \leq x \leq b$, $y(a) = \alpha$, and $y'(a) = 0$.

$$y'' = p(x)y' + q(x)y \tag{3}$$

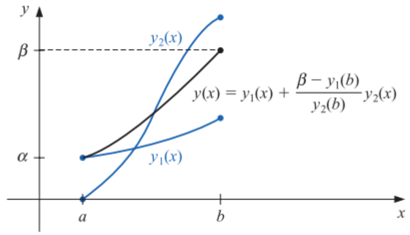
, for $a \leq x \leq b$, $y(a) = 0$, and $y'(a) = 1$.

Numerous methods are available from Chapter 5 for approximating the solutions $y_1(x)$ and $y_2(x)$, and once these approximations are available, the solution to the boundary-value problem is approximated using :

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} \cdot y_2(x)$$

Graphically, the method has the appearance as given below -

Figure 11.1



Example: Apply the Linear Shooting technique with $N = 10$ (number of iterations) to the boundary-value problem

$$y'' = -2xy' + 2x^2y + \frac{\sin(\ln x)}{x^2}, \quad \text{for } 1 \leq x \leq 2,$$

with $y(1) = 1$ and $y(2) = 2$,
and compare the results to those of the exact solution
$y = c_1x + \frac{c_2}{x^2} - \frac{3}{10}sin(lnx) - \frac{1}{10}cos(lnx)$,
where $c_2 \approx -0.03920701320$ and $c_1 \approx 1.1392070132$.

* Say for x=2, y=2.0000

Solution: First we reduce the given problem to the initial-value problems -

$$y_1'' = -\frac{2}{x}y_1' + \frac{2}{x^2}y_1 + \frac{sin(lnx)}{x^2}$$

for $1 \leq x \leq 2$, with $y_1(1) = 1$ and $y_1'(1) = 0$.
and,

$$y_2'' = -\frac{2}{x}y_2' + \frac{2}{x^2}y_2$$

for $1 \leq x \leq 2$, with $y_2(1) = 0$ and $y_2'(1) = 1$.

We can now solve these two equations and find our required solution as -

$$y(x_i) = y_1(x_i) + \frac{2 - y_1(2)}{y_2(2)}y_2(x_i)$$

The results with $N = 10$ and $h = 0.1$, are given in Table :

Table 11.1

x_i	$u_{1,i} \approx y_1(x_i)$	$v_{1,i} \approx y_2(x_i)$	$w_i \approx y(x_i)$	$y(x_i)$	$\lvert y(x_i) - w_i \rvert$
1.0	1.00000000	0.00000000	1.00000000	1.00000000	
1.1	1.00896058	0.09117986	1.09262917	1.09262930	1.43×10^{-7}
1.2	1.03245472	0.16851175	1.18708471	1.18708484	1.34×10^{-7}
1.3	1.06674375	0.23608704	1.28338227	1.28338236	9.78×10^{-8}
1.4	1.10928795	0.29659067	1.38144589	1.38144595	6.02×10^{-8}
1.5	1.15830000	0.35184379	1.48115939	1.48115942	3.06×10^{-8}
1.6	1.21248372	0.40311695	1.58239245	1.58239246	1.08×10^{-8}
1.7	1.27087454	0.45131840	1.68501396	1.68501396	5.43×10^{-10}
1.8	1.33273851	0.49711137	1.78889854	1.78889853	5.05×10^{-9}
1.9	1.39750618	0.54098928	1.89392951	1.89392951	4.41×10^{-9}
2.0	1.46472815	0.58332538	2.00000000	2.00000000	

The Shooting Method for Nonlinear Problems

The Shooting Method for Nonlinear Problems

Now we are going to look at how to deal with Non-Linear problems such as -

$$y'' = f(x, y, y'), \quad \text{for } a \leq x \leq b,$$

with $y(a) = \alpha$ and $y(b) = \beta$.

* Here, We can't separate the equation into two initial value problems.

* Instead, we approximate the solution to the boundary-value problem by using the solutions to a sequence of initial-value problems involving a parameter t.
These problems have the form

$$y'' = f(x, y, y'), \quad \text{for } a \leq x \leq b,$$

with $y(a) = \alpha$ and $y'(a) = t$.

We, choose parameters $t = t_k$ in a manner to ensure that

$$\lim_{k \to \infty} y(b, t_k) = y(b) = \beta$$

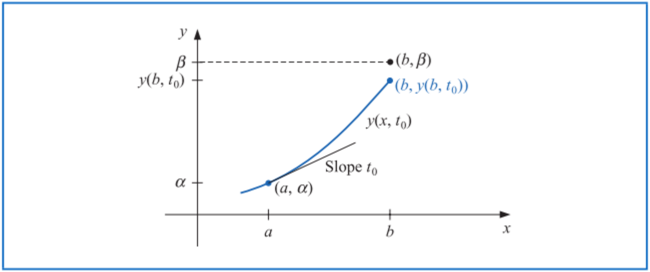
* where, $y(x, t_k)$ denotes the solution to the initial value problem.
with $t = t_k$,
* and $y(x)$ denotes the solution to the boundary value problem.

This technique is called a "shooting" method by analogy to the procedure of firing objects at a stationary target. We start with a parameter t_0 that determines the initial elevation at which the object is fired from the point (a, α) and along the curve described by the solution to the initial-value problem:

$$y'' = f(x, y, y'), \quad \text{for } a \leq x \leq b,$$

with $y(a) = \alpha$ and $y'(a) = t_0$.

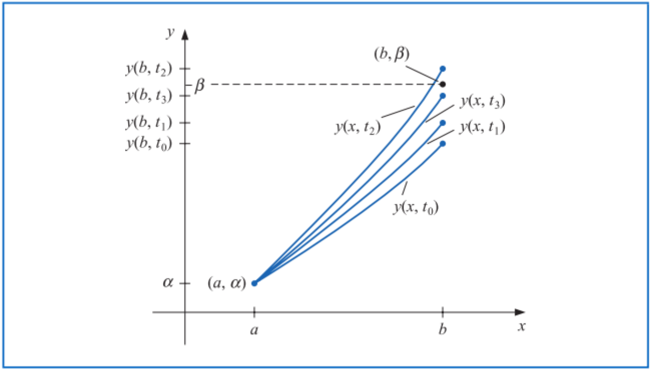
Figure 11.2



If $y(b, t_0)$ is not sufficiently close to β, we correct our approximation by choosing elevations t_1, t_2, and so on, until $y(b, t_k)$ is sufficiently close to "hitting" β.

We next determine t with $y(b, t) - \beta = 0$.

Figure 11.3



y
$y(b, t_2)$
$y(b, t_3)$
$y(b, t_1)$
$y(b, t_0)$
β
(b, β)
$y(x, t_3)$
$y(x, t_2)$
$y(x, t_1)$
$y(x, t_0)$
α
(a, α)
a
b
x

Example 1: Apply the Shooting method with Newton's Method to the boundary-value problem

$$y'' = \frac{1}{8}\left(32 + 2x^3 - yy'\right), \quad \text{for } 1 \leq x \leq 3,$$

with $y(1) = 17$ and $y(3) = \frac{43}{3}$.

Use $N = 20$, $M = 10$, and $\text{TOL} = 10^{-5}$, and compare the results with the exact solution $y(x) = x^2 + \frac{16}{x}$.

Solution: We need approximate solutions to the initial-value problems

$$y'' = \left(32 + 2x^3 - yy'\right), \quad \text{for } 1 \le x \le 3,$$

$$\text{with } y(1) = 17 \text{ and } y'(1) = t_k,$$

and,

$$z'' = \frac{\partial f}{\partial y} z + \frac{\partial f}{\partial y'} z' = -\frac{1}{8}(yy' + y'y), \quad \text{for } 1 \le x \le 3$$

$$\text{with } z(1) = 0 \text{ and } z'(1) = 1,$$

at each step in the iteration. If the stopping technique in Algorithm 11.2 requires $|w_{1,N}(t_k) - y(3)| \le 10^{-5}$, then we need four iterations and $t_4 \approx -14.000203$. The results obtained for this value of t are shown in Table 11.2.

Table 11.2

x_i	$w_{1,i}$	$y(x_i)$	$\lvert w_{1,i} - y(x_i)\rvert$
1.0	17.000000	17.000000	
1.1	15.755495	15.755455	4.06×10^{-5}
1.2	14.773389	14.773333	5.60×10^{-5}
1.3	13.997752	13.997692	5.94×10^{-5}
1.4	13.388629	13.388571	5.71×10^{-5}
1.5	12.916719	12.916667	5.23×10^{-5}
1.6	12.560046	12.560000	4.64×10^{-5}
1.7	12.301805	12.301765	4.02×10^{-5}
1.8	12.128923	12.128889	3.14×10^{-5}
1.9	12.031081	12.031053	2.84×10^{-5}
2.0	12.000023	12.000000	2.32×10^{-5}
2.1	12.029066	12.029048	1.84×10^{-5}
2.2	12.112741	12.112727	1.40×10^{-5}
2.3	12.246532	12.246522	1.01×10^{-5}
2.4	12.426673	12.426667	6.68×10^{-6}
2.5	12.650004	12.650000	3.61×10^{-6}
2.6	12.913847	12.913845	9.17×10^{-7}
2.7	13.215924	13.215926	1.43×10^{-6}
2.8	13.554282	13.554286	3.46×10^{-6}
2.9	13.927236	13.927241	5.21×10^{-6}
3.0	14.333327	14.333333	6.69×10^{-6}

Finite-Difference Methods for Linear Problems

* The linear and nonlinear Shooting methods for boundary-value problems can present problems of instability. The methods in this section have better stability characteristics, but they generally require more computation to obtain a specified accuracy.

* Methods involving finite differences for solving boundary-value problems replace each of the derivatives in the differential equation with an appropriate difference-quotient approximation (Discussed in Chapter 4).

* The particular difference quotient and Step size h is chosen to maintain a specified order of truncation error. However, h cannot be chosen too small because of the general instability of the derivative approximations.

Discrete Approximation -

* The finite difference method for the linear second-order boundary-value problem , requires that difference-quotient approximations be used to approximate both y' and y''.
* First, we select an integer $N > 0$ and divide the interval $[a, b]$ into $(N + 1)$ equal subintervals whose endpoints are the mesh points $x_i = a + ih$, for $i = 0, 1, \ldots, N + 1$, where $h = \frac{(b-a)}{(N+1)}$.
* At the interior mesh points, x_i, for $i = 1, 2, \ldots, N$, the differential equation to be approximated is $y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i)$.

Expanding y in a third Taylor polynomial about x_i evaluated at x_{i+1} and x_{i-1}, we have, assuming that $y \in C^4[x_{i-1}, x_{i+1}]$:

$$\begin{aligned} y(x_{i+1}) &= y(x_i + h) \\ &= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y''''(\xi_i^+), \end{aligned}$$

for some ξ_i^+ in (x_i, x_{i+1}), and

$$\begin{aligned} y(x_{i-1}) &= y(x_i - h) \\ &= y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y''''(\xi_i^-), \end{aligned}$$

for some ξ_i^- in (x_{i-1}, x_i).

* If these equations are added, we have

$$y(x_{i+1}) + y(x_{i-1}) = 2y(x_i) + h^2 y''(x_i) + \frac{h^4}{24}[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)],$$

and solving for $y''(x_i)$ gives

$$y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^4}{24}[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)].$$

* The Intermediate Value Theorem 1.11 can be used to simplify the error term to give $y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{12}y^{(4)}(\xi_i)$, (11.16) for some ξ_i in (x_{i-1}, x_{i+1}).

* This is called the centered-difference formula for $y''(x_i)$. A centered-difference formula for $y'(x_i)$ is obtained in a similar manner (the details were considered in Section 4.1), resulting in...
$y'(x_i) = \frac{1}{2h}[y(x_{i+1}) - y(x_{i-1})] - \frac{h^2}{6}y'''(\eta_i)$
for some η_i in (x_{i-1}, x_{i+1}).

* The use of these centered-difference formulas results in the equation

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = p(x_i)\frac{y(x_{i+1}) - y(x_{i-1})}{2h} + r(x_i) - \frac{h^2}{12} + q(x_i)y(x_i) + 2p(x_i)y''(\eta_i) - y^{(4)}(\xi_i).$$

* A Finite-Difference method with truncation error of order $O(h^2)$ results by using this equation together with the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ to define the system of linear equations

$$w_0 = \alpha,$$
$$w_{N+1} = \beta,$$
$$\text{and} \quad -w_{i+1} + 2w_i - w_{i-1} + h^2 p(x_i)(w_{i+1} - w_{i-1}) + q(x_i)w_i = -h^2 r(x_i),$$

for each $i = 1, 2, \ldots, N$.
* In the form we will consider it rewritten as

$$-1 + p(x_i)w_{i-1} + 2 + h^2 q(x_i)w_i - p(x_i)w_{i+1} = -h^2 r(x_i).$$

and the resulting system of equations is expressed in the tridiagonal $N \times N$ matrix form $Aw = b$, where

$$A = \begin{bmatrix} 2+h^2q(x_1) & -1+p(x_1) & 0 & \dots & 0 \\ h^2 & 2+h^2q(x_2) & -1+p(x_2) & \dots & 0 \\ 0 & h^2 & 2+h^2q(x_3) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & h^2 & 2+h^2q(x_N) \end{bmatrix},$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix},$$

$$b = \begin{bmatrix} -h^2r(x_1) + 1 + p(x_1)w_0 \\ -h^2r(x_2) \\ \vdots \\ -h^2r(x_N) + 1 - p(x_N)w_{N+1} \end{bmatrix}$$

Example 1: Use Finite Difference Method, with $N = 9$ to approximate the solution to the linear boundary-value problem

$$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin(\ln x)}{x^2}, \quad \text{for } 1 \le x \le 2,$$

$$\text{with } y(1) = 1 \text{ and } y(2) = 2,$$

Solution: For this example, we will use $N = 9$, so $h = 0.1$, and we have the same spacing as in Example 2 of Section 11.1. The complete results are listed in Table 11.3.

Table 11.3

x_i	w_i	$y(x_i)$	$\lvert w_i - y(x_i)\rvert$
1.0	1.00000000	1.00000000	
1.1	1.09260052	1.09262930	2.88×10^{-5}
1.2	1.18704313	1.18708484	4.17×10^{-5}
1.3	1.28333687	1.28338236	4.55×10^{-5}
1.4	1.38140205	1.38144595	4.39×10^{-5}
1.5	1.48112026	1.48115942	3.92×10^{-5}
1.6	1.58235990	1.58239246	3.26×10^{-5}
1.7	1.68498902	1.68501396	2.49×10^{-5}
1.8	1.78888175	1.78889853	1.68×10^{-5}
1.9	1.89392110	1.89392951	8.41×10^{-6}
2.0	2.00000000	2.00000000	

Finite-Difference Methods for Nonlinear Problems

Here, however, the system of equations will not be linear, so an iterative process is required to solve it. For the development of the procedure, we assume throughout that f satisfies the following conditions:

- f and the partial derivatives f_y and f_{yy} are all continuous on $D = \{(x, y, y') \mid a \leq x \leq b, -\infty < y < \infty, -\infty < y' < \infty\}$;
- $f_y(x, y, y') \geq \delta$ on D, for some $\delta > 0$;
- Constants k and L exist, with $k = \max_{(x,y,y')\in D} |f_y(x, y, y')|$ and $L = \max_{(x,y,y')\in D} |f_{yy}(x, y, y')|$.

This ensures, by Theorem 11.1, that a unique solution exists.

* As in the linear case, we divide $[a, b]$ into $(N+1)$ equal subintervals whose endpoints are at $x_i = a + ih$, for $i = 0, 1, \ldots, N+1$. Assuming that the exact solution has a bounded fourth derivative allows us to replace $y''(x_i)$ and $y'(x_i)$ in each of the equations:

$$y''(x_i) = f(x_i, y(x_i), y'(x_i))$$

* This gives, for each $i = 1, 2, \ldots, N$,

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = f(x_i, y(x_i), \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6} y''(\eta_i)) + \frac{h^2}{12} y^{(4)}(\xi_i),$$

for some ξ_i and η_i in the interval (x_{i-1}, x_{i+1}).

* As in the linear case, the difference method results from deleting the error terms and employing the boundary conditions: $w_0 = \alpha$, $w_{N+1} = \beta$, and

$$-\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + f(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}) = 0,$$

for each $i = 1, 2, \ldots, N$.

The $N \times N$ nonlinear system obtained from this method,

$$
\begin{aligned}
2w_1 - w_2 + h^2 f(x_1, w_1, \frac{w_2 - \alpha}{2h}) - \alpha &= 0 \\
-w_1 + 2w_2 - w_3 + h^2 f(x_2, w_2, \frac{w_3 - w_1}{2h}) &= 0 \\
&\vdots \\
-w_{N-2} + 2w_{N-1} - w_N + h^2 f(x_{N-1}, w_{N-1}, \frac{w_N - w_{N-2}}{2h}) &= 0 \\
-w_{N-1} + 2w_N + h^2 f(x_N, w_N, \frac{\beta - w_{N-1}}{2h}) - \beta &= 0
\end{aligned}
$$

has a unique solution provided that $h < \frac{2}{L}$.

Example 1: Apply Algorithm 11.4, with $h = 0.1$, to the nonlinear boundary-value problem

$$y'' = \frac{1}{8}(32 + 2x^3 - yy''), \quad \text{for } 1 \leq x \leq 3,$$

$$\text{with } y(1) = 17 \text{ and } y(3) = \frac{43}{3},$$

Solution: The stopping procedure used in Algorithm 11.4 was to iterate until values of successive iterates differed by less than 10^{-8}. This was accomplished with four iterations. The results are shown in Table 11.5. They are less accurate than those obtained using the nonlinear shooting method, which gave results in the middle of the table accurate on the order of 10^{-5}.

Table 11.5

x_i	w_i	$y(x_i)$	$\lvert w_i - y(x_i)\rvert$
1.0	17.000000	17.000000	
1.1	15.754503	15.755455	9.520×10^{-4}
1.2	14.771740	14.773333	1.594×10^{-3}
1.3	13.995677	13.997692	2.015×10^{-3}
1.4	13.386297	13.388571	2.275×10^{-3}
1.5	12.914252	12.916667	2.414×10^{-3}
1.6	12.557538	12.560000	2.462×10^{-3}
1.7	12.299326	12.301765	2.438×10^{-3}
1.8	12.126529	12.128889	2.360×10^{-3}
1.9	12.028814	12.031053	2.239×10^{-3}
2.0	11.997915	12.000000	2.085×10^{-3}
2.1	12.027142	12.029048	1.905×10^{-3}
2.2	12.111020	12.112727	1.707×10^{-3}
2.3	12.245025	12.246522	1.497×10^{-3}
2.4	12.425388	12.426667	1.278×10^{-3}
2.5	12.648944	12.650000	1.056×10^{-3}
2.6	12.913013	12.913846	8.335×10^{-4}
2.7	13.215312	13.215926	6.142×10^{-4}
2.8	13.553885	13.554286	4.006×10^{-4}
2.9	13.927046	13.927241	1.953×10^{-4}
3.0	14.333333	14.333333	

The Rayleigh-Ritz Method

The Rayleigh-Ritz Method

* The Shooting method for approximating the solution to a boundary-value problem replaced the boundary-value problem with pair of initial-value problems.

* The finite-difference approach replaces the continuous operation of differentiation with the discrete operation of finite differences.

* The Rayleigh-Ritz method is a variational technique that attacks the problem from a third approach.

* To describe the Rayleigh-Ritz method, we consider approximating the solution to a linear two-point boundary-value problem from beam-stress analysis. This boundary-value problem is described by the differential equation

$$-\frac{d}{dx}(p(x)\frac{dy}{dx}) + q(x)y = f(x), \text{for } 0 \leq x \leq 1$$

with the boundary conditions

$$y(0) = y(1) = 0$$

* We assume that $p \in C^1[0,1]$ and $q, f \in C[0,1]$.

* Further, we assume that there exists a constant $\delta > 0$ such that $p(x) \geq \delta$, and that $q(x) \geq 0$, for each x in $[0,1]$.

* These assumptions are sufficient to guarantee that the boundary-value problem given, has a unique solution.

Illustration: Consider the boundary-value problem $-y'' + \pi^2 y = 2\pi^2 \sin(\pi x)$, for $0 \leq x \leq 1$, with $y(0) = y(1) = 0$. Let $h_i = h = 0.1$, so that $x_i = 0.1i$, for each $i = 0, 1, \ldots, 9$. The integrals are:

$$Q_{1,i} = 100 \int_{0.1i}^{0.1i+0.1} (0.1i + 0.1 - x)(x - 0.1i)\pi^2 dx = \frac{\pi^2}{60},$$

$$Q_{2,i} = 100 \int_{0.1i-0.1}^{0.1i} (x - 0.1i + 0.1)^2 \, \pi^2 \, dx = \frac{\pi^2}{30},$$

$$Q_{3,i} = 100 \int_{0.1i}^{0.1i+0.1} (0.1i + 0.1 - x)^2 \pi^2 dx = \frac{\pi^2}{30},$$

$$Q_{4,i} = 100 \int_{0.1i-0.1}^{0.1i} dx = 10,$$

$$\begin{aligned} Q_{5,i} &= 10 \int_{0.1i-0.1}^{0.1i} (x - 0.1i + 0.1) 2\pi^2 sin \pi x dx \\ &= -2\pi \cos(0.1\pi i) + 20 \left[\sin(0.1\pi i) - \sin((0.1i - 0.1)\pi) \right], \end{aligned}$$

$$\begin{aligned} Q_{6,i} &= 10 \int_{0.1i}^{0.1i+0.1} (0.1i + 0.1 - x) 2\pi^2 sin \pi x dx \\ &= 2\pi \cos(0.1\pi i) - 20 \left[\sin((0.1i + 0.1)\pi) - \sin(0.1\pi i) \right]. \end{aligned}$$

The linear system $Ac = b$ has

$$a_{i,j} = 20 + \frac{\pi^2}{15}$$

for each $i = 1, 2, \ldots, 9$.

$$a_{i,j+1} = -10 + \frac{\pi^2}{60}$$

for each $i = 1, 2, \ldots, 8$.

$$a_{i,j-1} = -10 + \frac{\pi^2}{60}$$

for each $i = 2, 3, \ldots, 9$.
The values of b_i for each i are given by:

$$b_i = 40 \sin(0.1\pi i)(1 - \cos(0.1\pi)).$$

for each $i = 1, 2, \ldots, 9$.
The solution to the tridiagonal linear system is given by:

$$\begin{aligned} c_9 &= 0.3102866742, & c_6 &= 0.9549641893, & c_3 &= 0.8123410598, \\ c_8 &= 0.5902003271, & c_5 &= 1.004108771, & c_2 &= 0.5902003271, \\ c_7 &= 0.8123410598, & c_4 &= 0.9549641893, & c_1 &= 0.3102866742. \end{aligned}$$

The piecewise-linear approximation is

$$\phi(x) = \sum_{i=1}^{9} c_i \phi_i(x)$$

and the actual solution to the boundary-value problem is $y(x) = \sin(\pi x)$.
Table below lists the error in the approximation at x_i, for each $i = 1, \ldots, 9$

Table 11.7

i	x_i	$\phi(x_i)$	$y(x_i)$	$\lvert\phi(x_i) - y(x_i)\rvert$
1	0.1	0.3102866742	0.3090169943	0.00127
2	0.2	0.5902003271	0.5877852522	0.00241
3	0.3	0.8123410598	0.8090169943	0.00332
4	0.4	0.9549641896	0.9510565162	0.00390
5	0.5	1.0041087710	1.0000000000	0.00411
6	0.6	0.9549641893	0.9510565162	0.00390
7	0.7	0.8123410598	0.8090169943	0.00332
8	0.8	0.5902003271	0.5877852522	0.00241
9	0.9	0.3102866742	0.3090169943	0.00127

Thank You !