Unit 5 : MULTIPLE INTEGRALS

In this chapter, we extend the idea of a definite integral to double and triple integrals of functions of two or three variables.

Contents:

Double Integration

An integral of the form $\iint_R f(x,y)\,dxdy$ is called as double integral of a continuous function $f(x,y)$ defined over a region R of $xy-$plane.

If region R is defined by $a \le x \le b$, $f_1(x) \le y \le f_2(x)$, with $f_1(x)$ and $f_2(x)$ are continuous on $[a, b]$, then

$$\iint_R f(x,y)\,dxdy = \int_{x=a}^{x=b} \left[\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy \right] dx$$

If R is defined by $c \le y \le d,\ g_1(y) \le x \le g_2(y)$ with $g_1(y)$ and $g_2(y)$ are continuous on $[c, d]$, then

$$\iint\limits_R f(x,y)\,dxdy = \int_{y=c}^{y=d} \left[\int_{x=g_1(y)}^{x=g_2(y)} f(x,y)\,dx \right] dy$$

Four types of problems arises on double integration

1. Evaluation of double integral when limits are given.
2. Evaluation of double integral when limits are not given but region of integration R is given
3. Evaluation of double integral by change of order of integration
4. Evaluation of double integral by transforming to polar coordinates

Type-I : Double Integral when limits are given

Case-I :Limits of x are functions of y

$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \int_{x=g_1(y)}^{x=g_2(y)} f(x,y)\,dx\,dy$$

Outer integral *Inner integral*

1.As limits of inner integral are functions of y, integrate integrand $f(x,y)$ w.r.to x first by keeping y constant.

2.Then put upper limit $x = g_2(y)$ and lower limit $x = g_1(y)$ of x.

3. After putting limits of x, integrate the resulting function w.r.to y.

4. Lastly put upper limit $y = d$ and lower limit $y = c$ of y.

Case-II : Limits of y are functions of x

$$\iint_R f(x,y)\,dx\,dy = \underbrace{\int_{x=a}^{x=b}}_{\text{Outer integral}} \underbrace{\int_{y=f_1(x)}^{y=f_2(x)}}_{\text{Inner integral}} f(x,y)\,dy\,dx$$

1. As limits of inner integral are functions of x, integrate integrand $f(x,y)$ w.r.to y first by keeping x constant.
2. Then put upper limit $y = f_2(x)$ and lower limit $y = f_1(x)$ of y.
3. After putting limits of y, integrate the resulting function w.r.to x
4. Lastly put upper limit $x = b$ and lower limit $x = a$ of x.

Example 1. Evaluate $\int_0^1 \int_y^{1+y^2} x^2 y \, dxdy$.

Solution : Limits of inner integral are functions of y. So we need to integrate w.r.to x first by keeping y constant.

$$\text{Let } I = \int_{y=0}^{y=1} \int_{x=y}^{x=1+y^2} x^2 y \, dxdy = \int_{y=0}^{y=1} \left[\int_{x=y}^{x=1+y^2} x^2 y \, dx \right] dy$$

$$= \int_{y=0}^{y=1} y \left[\frac{x^3}{3} \right]_{x=y}^{x=1+y^2} dy = \int_{y=0}^{y=1} y \left[\frac{(1+y^2)^3}{3} - \frac{y^3}{3} \right] dy$$

$$= \frac{1}{3} \int_{y=0}^{y=1} y[(1 + 3y^2 + 3y^4 + y^6) - y^3]dy$$

$$= \frac{1}{3} \int_{y=0}^{y=1} [y + 3y^3 + 3y^5 + y^7 - y^4]dy$$

$$= \frac{1}{3}\left[\frac{y^2}{2} + \frac{3y^4}{4} + \frac{3y^6}{6} + \frac{y^8}{8} - \frac{y^5}{5}\right]_{y=0}^{y=1}$$

$$= \frac{1}{3}\left[\frac{1}{2} + \frac{3}{4} + \frac{3}{6} + \frac{1}{8} - \frac{1}{5}\right] = \frac{67}{120}.$$

Example 2. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \dfrac{dxdy}{1+x^2+y^2}$.

Solution : Limits of inner integral are functions of x. So we need to integrate w.r.to y first by keeping x constant.

$$\text{Let I} = \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2} = \int_{x=0}^{x=1} \left[\int_{y=0}^{y=\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy \right] dx$$

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2 + y^2}\, dy \right] dx$$

$$I = \int_{x=0}^{x=1} \left[\int_{y=0}^{y=k} \frac{1}{k^2 + y^2} \, dy \right] dx \qquad \text{where } \sqrt{1 + x^2} = k$$

$$= \int_{x=0}^{x=1} \left[\frac{1}{k} \tan^{-1} \left(\frac{y}{k} \right) \right]_{y=0}^{y=k} dx$$

$$= \int_{x=0}^{x=1} \frac{1}{k} [\, \tan^{-1}(1) - \tan^{-1}(0)] \, dx$$

$$= \int_{x=0}^{x=1} \frac{1}{k} \left[\frac{\pi}{4} - 0 \right] dx$$

$$I = \frac{\pi}{4}\int_{x=0}^{x=1} \frac{1}{k}\,dx = \frac{\pi}{4}\int_{x=0}^{x=1} \frac{1}{\sqrt{1+x^2}}\,dx \qquad \text{where } k = \sqrt{1+x^2}$$

$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log\left(1+\sqrt{2}\right) - \log 1\right]$$

$$= \frac{\pi}{4}\log\left(1+\sqrt{2}\right)$$

Example 3. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$.

Solution : Limits of inner integral are functions of y. So we need to integrate w.r.to x first by keeping y constant.

$$I = \int_{y=0}^{y=a} \int_{x=0}^{x=\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dxdy = \int_{y=0}^{y=a} \left[\int_{x=0}^{x=\sqrt{a^2-y^2}} \sqrt{a^2-y^2-x^2}\, dx \right] dy$$

$$= \int_{y=0}^{y=a} \left[\int_{x=0}^{x=\sqrt{a^2-y^2}} \sqrt{\left(\sqrt{a^2-y^2}\right)^2 - x^2}\, dx \right] dy$$

$$I = \int_{y=0}^{y=a} \left[\int_{x=0}^{x=k} \sqrt{k^2 - x^2}\, dx \right] dy \qquad \text{where } \sqrt{a^2 - y^2} = k$$

Recall: $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

$$I = \int_{y=0}^{y=a} \left[\frac{x\sqrt{k^2 - x^2}}{2} + \frac{k^2}{2} \sin^{-1}\left(\frac{x}{k}\right) \right]_{x=0}^{x=k} dy$$

$$= \int_{y=0}^{y=a} \left[0 + \frac{k^2}{2} \sin^{-1}(1) - 0 - \frac{k^2}{2} \sin^{-1}(0) \right] dy$$

$$= \int_{y=0}^{y=a} \frac{k^2}{2}\frac{\pi}{2} dy$$

$$= \frac{\pi}{4} \int_{y=0}^{y=a} (a^2 - y^2) dy \quad \text{where } k = \sqrt{a^2 - y^2}$$

$$= \frac{\pi}{4}\left[a^2 y - \frac{y^3}{3}\right]_{y=0}^{y=a}$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi a^3}{6}$$

Case-III: All limits are constant

Region R is bounded by $a \leq x \leq b,\ c \leq y \leq d$

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b} \boxed{\int_{y=c}^{y=d} f(x,y)\,dy}\,dx = \int_{y=c}^{y=d} \boxed{\int_{x=a}^{x=b} f(x,y)\,dx}\,dy$$

Here all limits are constants. Hence the order of integration is immaterial provided the limits of integration are changed accordingly.

Example 4. Evaluate $\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \cos(x+y)\,dx\,dy$.

Solution:Here all limits are constants.Hence the order of integration is immaterial.

Method − 1: Let $\mathrm{I} = \int_0^{\frac{\pi}{2}} \boxed{\int_{\frac{\pi}{2}}^{\pi} \cos(x+y)\,dx}\, dy.$

$$= \int_0^{\frac{\pi}{2}} [\sin(x+y)]_{x=\frac{\pi}{2}}^{x=\pi}\, dy \ = \int_0^{\frac{\pi}{2}} \left[\sin(\pi+y) - \sin\left(\frac{\pi}{2}+y\right)\right] dy$$

$$= \int_{0}^{\frac{\pi}{2}} \left[\sin(\pi + y) - \sin\left(\frac{\pi}{2} + y\right)\right] dy$$

$$= \int_{0}^{\frac{\pi}{2}} [-\sin y - \cos y] dy$$

$$= [\cos y - \sin y]_{0}^{\pi/2}$$

$$= \cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) - \cos 0 + \sin 0$$

$$= -2$$

Method − 2: Let $I = \int_{\frac{\pi}{2}}^{\pi} \boxed{\int_{0}^{\frac{\pi}{2}} \cos(x+y)\,dy}\,dx.$

$$= \int_{\frac{\pi}{2}}^{\pi} [\sin(x+y)]_{y=0}^{y=\pi/2}\,dx = \int_{\frac{\pi}{2}}^{\pi} \left[\sin\left(x+\frac{\pi}{2}\right) - \sin x\right] dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} [\cos x - \sin x]dx = [\sin x + \cos x]_{\pi/2}^{\pi}$$

$$= \sin\pi + \cos\pi - \sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right)$$

$$= 0 - 1 - 1 - 0 = -2$$

Example 5 . Evaluate $\int_3^4 \int_1^2 \frac{dx\, dy}{(x+y)^2}$.

Solution : Limits of both x and y are constants. Hence the order of integration is immaterial provided the limits of integration are changed accordingly.

Method – 1: Let $I = \int_3^4 \left[\int_1^2 \frac{dx}{(x+y)^2} \right] dy$

$$= \int_3^4 \left[\frac{-1}{x+y} \right]_{x=1}^{x=2} dy = -\int_3^4 \left[\frac{1}{2+y} - \frac{1}{1+y} \right] dy$$

$$= -[\log(2+y) - \log(1+y)]_3^4$$

$$= -(\log 6 - \log 5 - \log 5 + \log 4)$$

$$= -\log(6 \times 4) + \log(5 \times 5)$$

$$= \log\left(\frac{25}{24}\right)$$

Method 2: Let I $= \int_1^2 \left[\int_3^4 \frac{dy}{(x+y)^2}\right] dx$

$$= \int_1^2 \left[\frac{-1}{x+y}\right]_{y=3}^{y=4} dx = -\int_1^2 \left[\frac{1}{x+4} - \frac{1}{x+3}\right] dx$$

$$= -[\log(x+4) - \log(x+3)]_{x=1}^{x=2}$$

$$= -(\log 6 - \log 5 - \log 5 + \log 4)$$

$$= \log\left(\frac{25}{24}\right)$$

Some Practice Problems:

1. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$. $\boldsymbol{Ans}: -\frac{1}{8}$

2. Evaluate $\int_0^1 \int_0^{x^2} (x^2 + y^2) \, dx \, dy$. $\boldsymbol{Ans}: \frac{26}{105}$

3. Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy$. $\boldsymbol{Ans}: \frac{a^4}{6}$

4. Evaluate $\int_0^1 \int_{y^2}^{y} (1 + xy^2) \, dx \, dy$. $\boldsymbol{Ans}: \frac{41}{210}$

5. Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} dx\, dy.$ **Ans**: $\dfrac{\pi a^2}{4}$

6. Evaluate $\displaystyle\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)}\, x\, dxdy.$ **Ans**: $\dfrac{\pi}{4}$

7. Evaluate $\displaystyle\int_1^a \int_1^b \frac{dx\, dy}{xy}.$ **Ans**: $\log a \log b$

8. Evaluate $\displaystyle\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}.$ **Ans**: $\dfrac{\pi^2}{4}$

Type-II : Double Integral when limits are not given

Let $I = \iint\limits_R f(x,y)dxdy$ be a given integral to be integrated over a region R.

Step 1 : Sketch region of inegration *R* from given information.

Step 2(a) : If integrand $f(x,y)$ is to be integrated *w.r.to y first* then take a *strip parallel to y-axis* in region *R*.

Curve where *lower end of the strip* lies give ***lower limit for y***.

Curve where *upper end of the strip* lies give ***upper limit for y***.

Move strip from *left to right* in region R.

Left end of the region gives ***lower limit for x.***

Right end of the region gives ***upper limit for x***.

Step 2(b) : If integrand $f(x, y)$ is to be integrated *w.r.to x first* then

take a *strip parallel to x-axis* in region *R*.

Curve where *left end of the strip* lies gives ***lower limit for x.***

Curve where *right end of the strip* lies gives ***upper limit for x.***

Move strip from *bottom to top* in region R.

Bottom end of the region gives ***lower limit for y.***

Top end of the region gives ***upper limit for y.***

Step 3 : After finding limits, evaluate double integral as per problems of type-I.

Very Important :

1.If integrand is easy to integrate w.r.to *y* first than *x* then take strip parallel to *y*-axis

2.If integrand is easy to integrate w.r.to *x* first than *y* then take strip parallel to *x*-axis

3.If integrand is easy to integrate w.r.to both *x* and *y* then choice is yours.

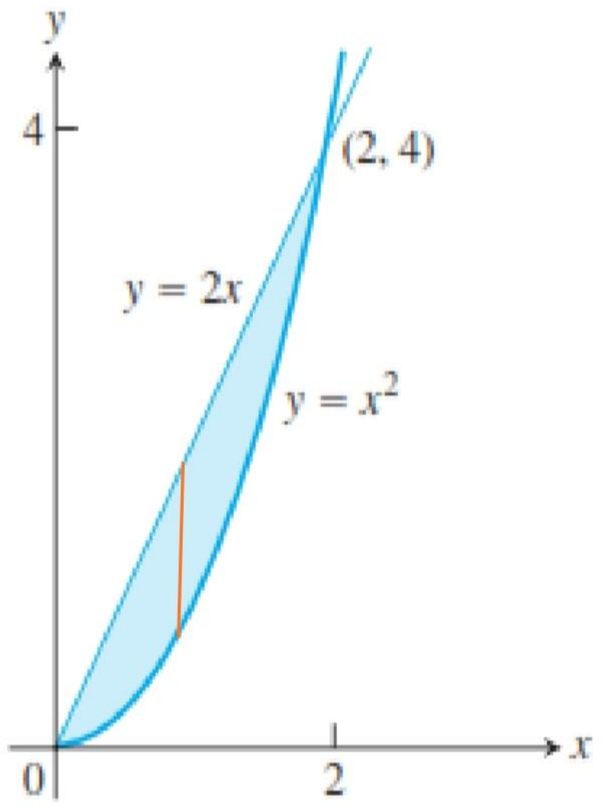
1. Evaluate $\iint\limits_R (4x+2)\,dxdy$ where R is the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution : The region of integration is shown in the figure.

$y = x^2$ and $y = 2x \Rightarrow x^2 = 2x \Rightarrow x = 0, 2$

$x = 0 \Rightarrow y = 0$ and $x = 2 \Rightarrow y = 4.$

Thus points of intersection of parabola $y = x^2$ and line $y = 2x$ are $(0, 0)$ and $(2, 4)$.



Part-I : Solution by taking strip parallel to y-axis.

Draw a strip parallel to y-axis as shown in figure.

Lower end of the strip lies on parabola $y = x^2$.

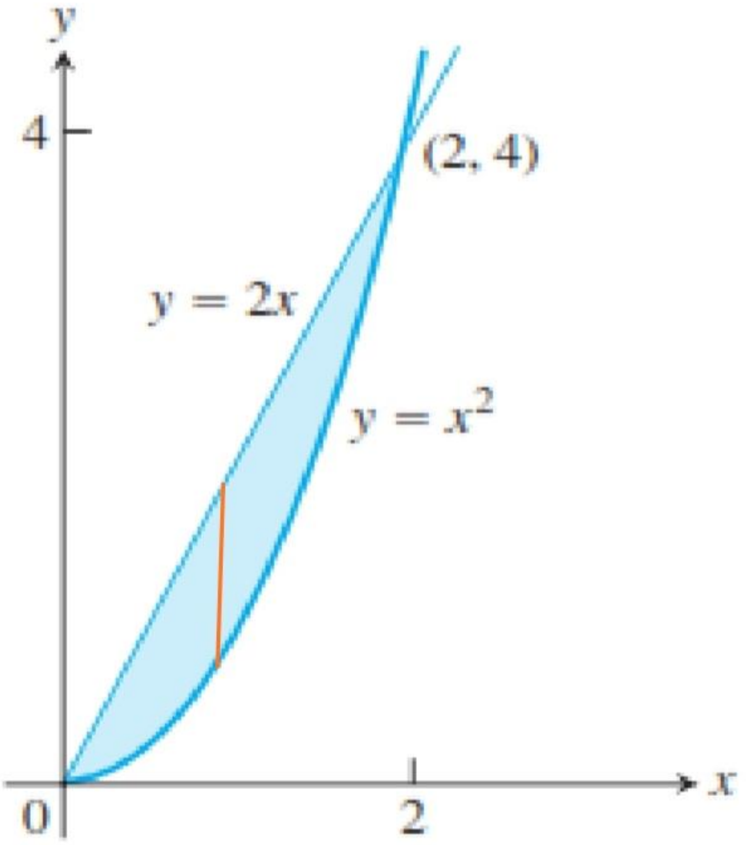
Therefore, lower limit of y is $y = x^2$.

Upper end of the strip lies on line $y = 2x$.

Therefore, upper limit of y is $y = 2x$.

Move strip from left to right in region R.

In this movement x varies from 0 to 2.
Lower limit of x is $x = 0$ and
upper limit of x is $x = 2$.



$$I = \iint_R (4x+2)\, dxdy = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} (4x+2)\, dx\, dy$$

$$= \int_{x=0}^{x=2} \left[\int_{y=x^2}^{y=2x} (4x+2)\, dy \right] dx$$

$$= \int_{x=0}^{x=2} (4x+2)[y]_{y=x^2}^{y=2x}\, dx$$

$$= \int_{x=0}^{x=2} (4x+2)[2x-x^2]dx$$

$$= \int_{x=0}^{x=2} (8x^2 - 4x^3 + 4x - 2x^2)\, dx$$

$$= \int_{x=0}^{x=2} (4x + 6x^2 - 4x^3) dx$$

$$= (2x^2 + 2x^3 - x^4)_{x=0}^{x=2}$$

$$= 8 + 16 - 16$$

$$= 8$$

Part-II: Solution by considering strip parallel to x-axis

Draw a strip parallel to x-axis as shown in figure.

Left end of the strip lies on line $y = 2x$ i.e. $x = \frac{y}{2}$.

Therefore, lower limit of x is $x = \frac{y}{2}$.

Right end of the strip lies on parabola $y = x^2$

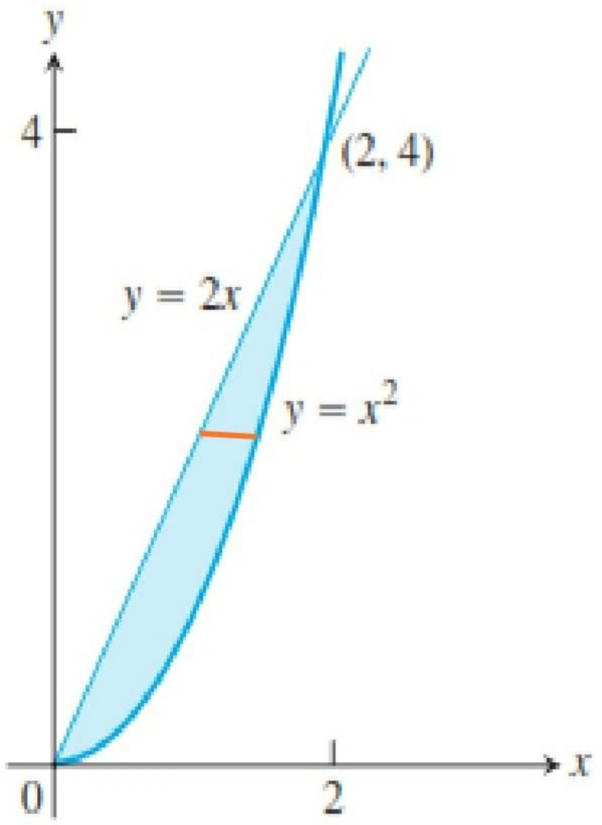
i.e. $x = \sqrt{y}$.

Therefore, upper limit of x is $x = \sqrt{y}$.

Move strip from bottom to top in region R.

In this movement y varies from 0 to 4.

Lower limit of y is y = 0 and upper limit of y is y = 4.



$$I = \iint_R (4x+2)\,dxdy = \int_{y=0}^{y=4} \int_{x=\frac{y}{2}}^{x=\sqrt{y}} (4x+2)\,dx\,dy$$

$$= \int_{y=0}^{y=4} \left[\int_{x=\frac{y}{2}}^{x=\sqrt{y}} (4x+2)\,dx \right] dy = \int_{y=0}^{y=4} \left[2x^2+2x\right]_{x=y/2}^{x=\sqrt{y}} dy$$

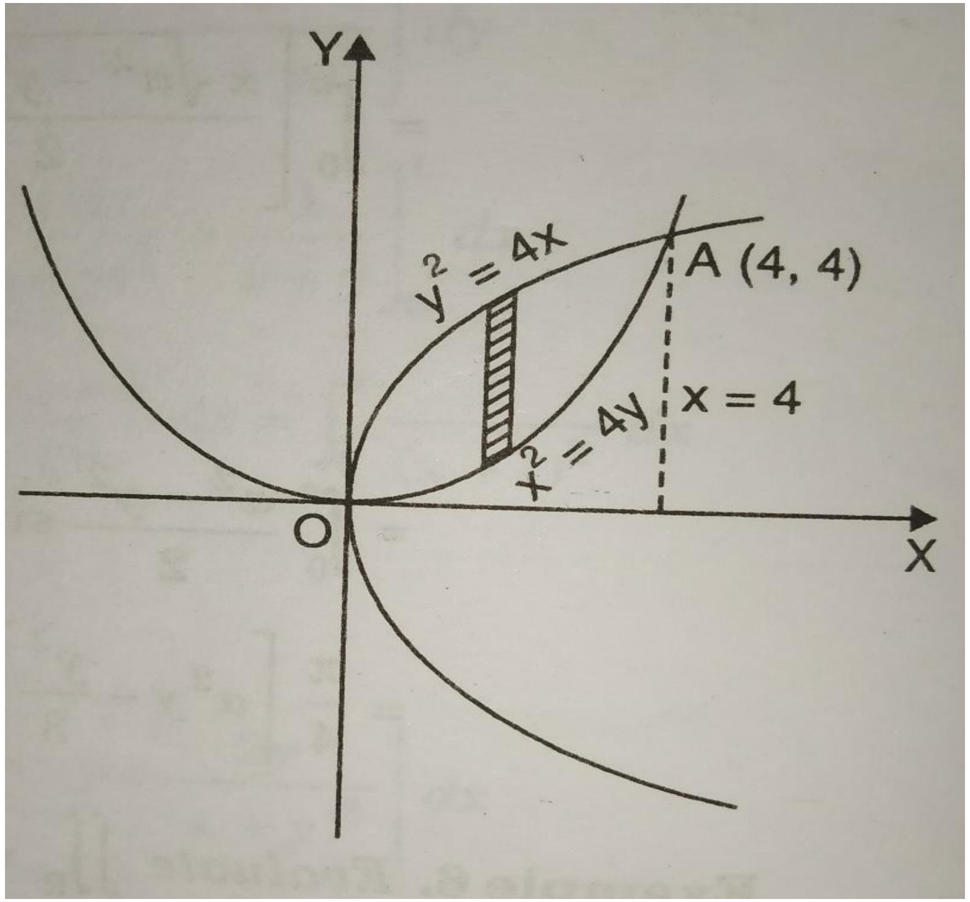
$$= \int_{y=0}^{y=4} \left[2y + 2\sqrt{y} - \frac{y^2}{2} - y\right] dy = \int_{y=0}^{y=4} \left[2\sqrt{y} + y - \frac{y^2}{2}\right] dy$$

$$= \left[2\,\frac{y^{3/2}}{3/2} + \frac{y^2}{2} - \frac{y^3}{6}\right]_{y=0}^{y=4} = \frac{4}{3}\,(4)^{3/2} + 8 - \frac{64}{6} = \frac{32}{3} + 8 - \frac{64}{6} = 8$$

2. Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $x^2 = 4y$ and $y^2 = 4x$

Solution : The region of integration (ROI) is shown in the figure.

Points of intersection of two parabolas $x^2 = 4y$ and $y^2 = 4x$ are $(0,0)$ and $(4,4)$.



Draw a strip parallel to y-axis as shown in figure.

Lower end of the strip lies on parabola $x^2 = 4y$.

Therefore, lower limit of y is $y = \frac{x^2}{4}$.

Upper end of the strip lies on parabola $y^2 = 4x$.

Therefore, upper limit of y is $y = 2\sqrt{x}$.

Move strip from left to right in region R.

In this movement, x varies from 0 to 4.

Lower limit for x is 0 and upper limit is 4.

$$I = \iint_R y\,dx\,dy = \int_{x=0}^{x=4} \int_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} y\,dx\,dy$$

$$= \int_{x=0}^{x=4} \left[\int_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} y\,dy \right] dx = \int_{x=0}^{x=4} \left[\frac{y^2}{2} \right]_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} dx$$

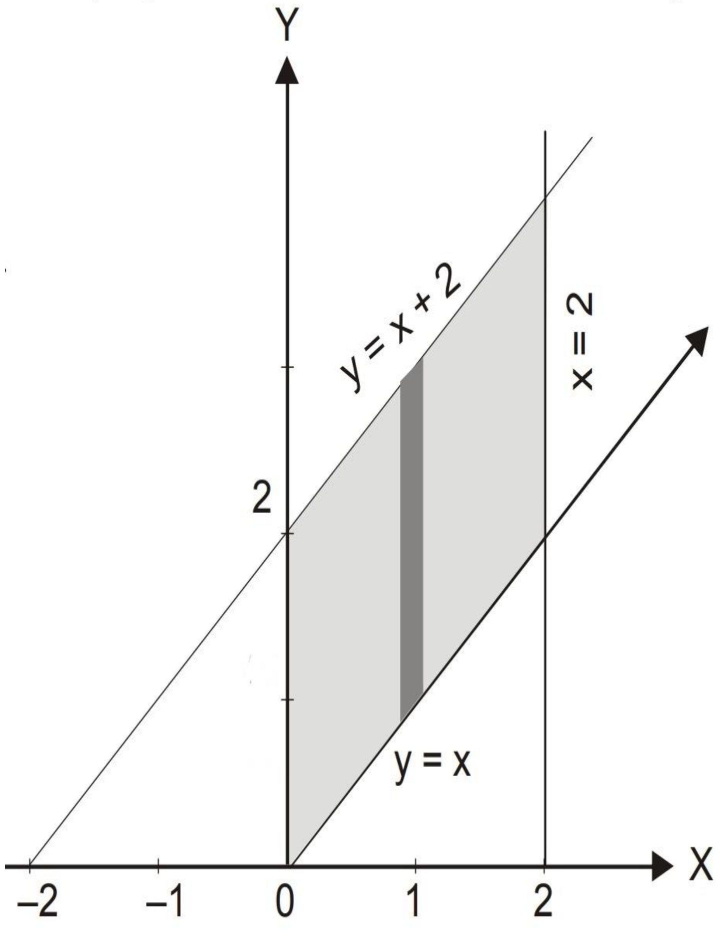
$$= \frac{1}{2} \int_{x=0}^{x=4} \left[4x - \frac{x^4}{16} \right] dx = \frac{1}{2} \left[2x^2 - \frac{x^5}{80} \right]_{x=0}^{x=4}$$

$$= \frac{1}{2} \left[32 - \frac{1024}{80} \right] = \frac{48}{5}$$

H.W*: Try to find answer of by taking strip parallel to x-axis*

Example 3. Evaluate $\iint\limits_R (x+y)dxdy$ where R is the region $in\ \ xy -$ plane bounded by four lines $x=0, x=2, y=x, y=x+2$.

Solution : The region of integration is shown in figure.



Draw a strip parallel to y-axis as shown in figure.

Lower end of the strip lies on line $y = x$.

Therefore, lower limit of y is $y = x$.

Upper end of the strip lies on line $y = x + 2$.

Therefore, upper limit of y is $y = x + 2$.

Move strip from left to right in region R.

In this movement x varies from 0 to 2 .

(At left end of the region R, $x = 0$ and at right end of the region R, $x = 2$)

Lower limit for x is 0 and upper limit is 2.

$$I = \iint_R (x+y)\,dx\,dy = \int_{x=0}^{x=2} \int_{y=x}^{y=x+2} (x+y)\,dx\,dy$$

$$= \int_{x=0}^{x=2} \left[\int_{y=x}^{y=x+2} (x+y)\,dy \right] dx = \int_{x=0}^{x=2} \left[xy + \frac{y^2}{2} \right]_{y=x}^{y=x+2} dx$$

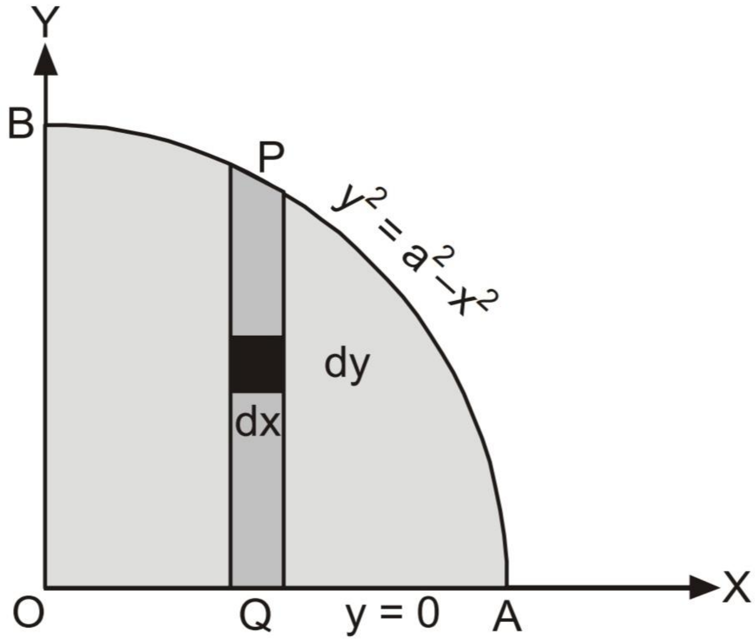
$$= \int_{x=0}^{x=2} \left[x(x+2) + \frac{(x+2)^2}{2} - x^2 - \frac{x^2}{2} \right] dx$$

$$= \int_{x=0}^{x=2} \left[x^2 + 2x + \frac{1}{2}(x^2+4x+4) - x^2 - \frac{x^2}{2} \right] dx = \int_{x=0}^{x=2} [4x+2]\,dx$$

$$= (2x^2+2x)_0^2 = 8 + 4 = 12$$

4. Evaluate $\iint_R xy\, dxdy$ where R is the circle $x^2 + y^2 = a^2$ in first quadrant of xy − plane.

Solution : The region of integration is shown in figure.



Draw a strip parallel to *y*-axis as shown in figure.

Lower end of the strip lies on *x*-axis i.e. *y=0*

Therefore, lower limit of y is $y = 0$.

Upper end of the strip lies on circle

$x^2 + y^2 = a^2$ $i.e.$ $y^2 = a^2 - x^2$.

Therefore,upper limit of y is $y = \sqrt{a^2 - x^2}$.

Move strip from left to right in region *R*.

In this movement x varies from 0 to a .

(At left end of the region *R*, $x = 0$ and at right end of the region *R*, $x = a$)

Lower limit for x is 0 and upper limit is a.

$$I = \iint_R xydxdy = \int_{x=0}^{x=a} \int_{y=0}^{y=\sqrt{a^2-x^2}} xy\,dx\,dy$$

$$= \int_{x=0}^{x=a} \left[\int_{y=0}^{y=\sqrt{a^2-x^2}} xy\,dy \right] dx = \int_{x=0}^{x=a} x \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{x=a} x[a^2 - x^2]dx = \frac{1}{2} \int_{x=0}^{x=a} [a^2x - x^3]dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{a^4}{8}$$

5. Evaluate $\iint\limits_{R} \frac{\sin x}{x}\, dxdy$ where R is the triangle in $xy-$plane bounded by the $x-$axis, the line $y=x$ and the line $x=1$.

Solution : The region of integration is shown in the figure.

As integrand is easy to integrate w.r.to y first than x, take a strip parallel to y-axis as shown in figure.

Lower end of the strip lies on x-axis i.e. $y=0$.

Therefore, lower limit of y is $y=0$.

Upper end of the strip lies on line $y=x$.

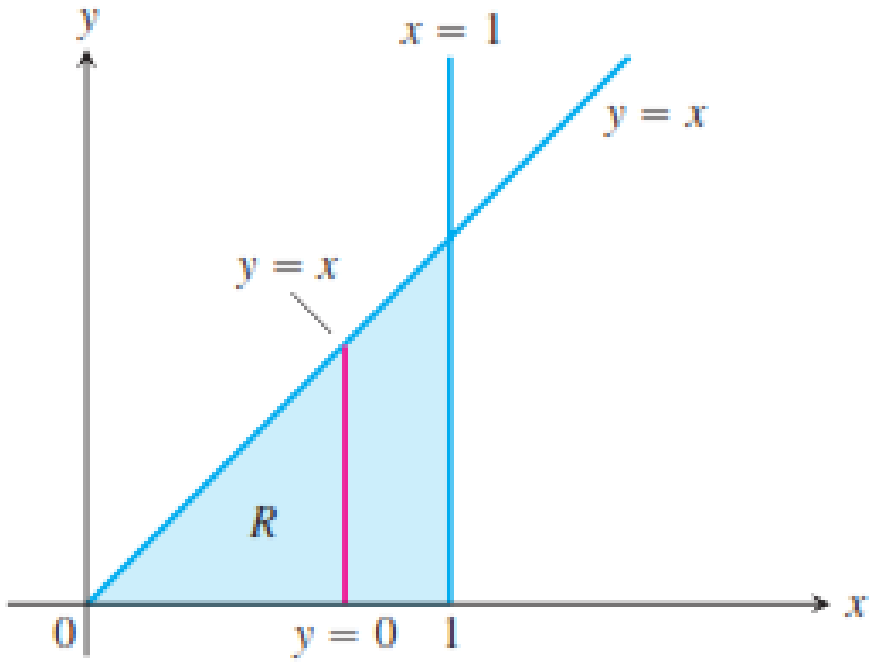
Therefore, upper limit of y is $y=x$.

Move strip from left to right in region R.

In this movement x varies from 0 to 1.

(At left end of the region R, $x=0$ and at right end of the region R, $x=1$)

Lower limit for x is 0 and upper limit is 1.



$$I = \iint_R \frac{\sin x}{x}\, dxdy = \int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{\sin x}{x}\, dx\, dy$$

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=x} \frac{\sin x}{x}\, dy \right] dx = \int_{x=0}^{x=1} \frac{\sin x}{x} [y]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{x=1} \frac{\sin x}{x} [x - 0] dx = \int_{x=0}^{x=1} \sin x\, dx$$

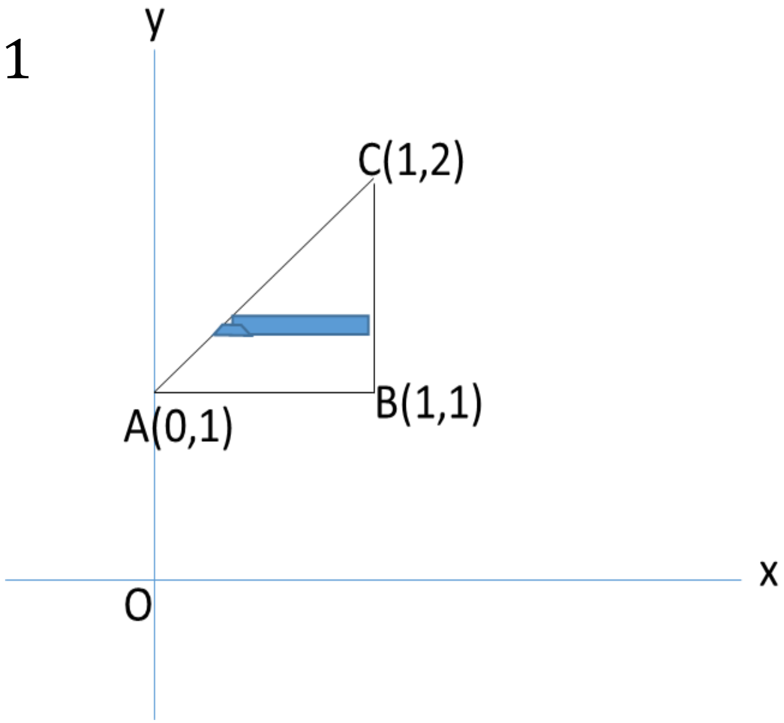
$$= (-\cos x)_{x=0}^{x=1} = -\cos 1 + \cos 0$$

$$= 1 - \cos 1$$

6. Evaluate $\iint (x^2 + y^2)\, dxdy$ over area of the triangle whose vertices are (0,1), (1,1) and (1,2).

Solution : The region of integration (ROI) is shown in following figure.

Take a strip parallel to x-axis as shown in figure.



Left end of the strip lies on side AC whose equation is $y = x + 1$

i.e. $x = y - 1$

Therefore, lower limit of x is $x = y - 1$.

Right end of the strip lies on side BC whose equation is $x = 1$.

Therefore, upper limit of x is $x = 1$.

Move strip from bottom to top in ROI.

In this movement y varies from 1 to 2.

(At bottom end of the region , $y = 1$ and at

top end of the region, $y = 2$)

Lower limit of y is $y = 1$ and upper limit of y is $y = 2$.

$$I = \iint (x^2 + y^2)\, dxdy = \int_{y=1}^{y=2} \int_{x=y-1}^{x=1} (x^2 + y^2)\, dxdy$$

$$= \int_{y=1}^{y=2} \left[\int_{x=y-1}^{x=1} (x^2 + y^2)\, dx \right] dy = \int_{y=1}^{y=2} \left[\frac{x^3}{3} + y^2 x \right]_{x=y-1}^{x=1} dy$$

$$= \int_{y=1}^{y=2} \left[\frac{1}{3} + y^2 - \frac{(y-1)^3}{3} - y^2(y-1) \right] dy$$

$$= \int_{y=1}^{y=2} \left[\frac{1}{3} + 2y^2 - y^3 - \frac{(y-1)^3}{3} \right] dy = \left[\frac{y}{3} + \frac{2y^3}{3} - \frac{y^4}{4} - \frac{(y-1)^4}{12} \right]_{y=1}^{y=2}$$

$$= \left[\left(\frac{2}{3} + \frac{16}{3} - \frac{16}{4} - \frac{1}{12} \right) - \left(\frac{1}{3} + \frac{2}{3} - \frac{1}{4} - 0 \right) \right] = 6 - 5 - \frac{1}{12} + \frac{1}{4} = {}^{7}/_{6}$$

Some Practice Problems

1. Evaluate $\iint\limits_R \sqrt{4x^2 - y^2}\,dxdy$ where R is the region in bounded by lines $y = 0, y = x$ and $x = 1$. **Ans**: $\frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$

2. Evaluate $\iint\limits_R x^2 dxdy$ where R is the region $xy -$ plane bounded by the parabola $y = x^2$ and the line $y = x$. **Ans**: $\frac{1}{20}$

3. Evaluate $\iint\limits_R xy\,dxdy$ where R is the region in $xy -$ plane bounded by the parabolas $x^2 = y$ and $y^2 = -x$. **Ans**: $-\frac{1}{12}$

4. Evaluate $\iint\limits_R \frac{1}{x^4 + y^2}\,dxdy$ where R is the region in $xy -$ plane formed by $y \geq x^2$ and $x \geq 1$ **Ans**: $\frac{\pi}{4}$.

5. Evaluate $\iint_R \frac{xy}{\sqrt{1-y^2}}\, dxdy$ where R is the circle $x^2 + y^2 = 1$ in positive quadrant of $xy -$ plane **Ans**: $\frac{1}{6}$

6.Evaluate $\iint_R \, ydxdy$ where R is the region in $xy -$ plane bounded by lines $x = 0, x + y = 2$ and the parabola $y = x^2$ in first quadrant.

Ans: $\frac{16}{15}$

Type-III : Double integral in polar co-ordinates

Let $I = \iint\limits_R f(x,y)dxdy$ be the given double integral in cartesian form where $f(x,y)$ is a continuous function on region R in xy − plane.

In many cases it is convenient to transform the above integral into polar coordinates.

In which cases?

To integrate $\iint\limits_R sin(x^2 + y^2)dxdy$ in cartesian coordinates, is not possible.

When integrand $f(x,y)$ contains term like $x^2 + y^2$ and region of integration is circle or ellipse then change varaibles from cartesian to polar coordinates.

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To evaluate double integral by changing variables from cartesian to polar coordinates we need to go through the following steps.

Step 1: Sketch the region of integration (ROI)

Step 2: Put $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$

$$\therefore dxdy = rdrd\theta$$

Step 3: Draw a *radial strip* (strip along radius vector) in ROI.

Inner end of the radial strip gives *lower limit of* r.

Outer end of the radial strip gives *upper limit of* r.

Step 4: Rotate this radial strip in positive (anticlockwise) direction

in ROI and find lower & upper limits of θ.

Step 5: Using limits of r and θ, evaluate double integral.

Important Note :

When region of integration (ROI) is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then put $x = ar\cos\theta$ and $y = br\sin\theta$

$$\therefore dxdy = abrdrd\theta$$

Limits of r : Lower limit : $r = 0$.

Upper limit : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow r^2 = 1 \Rightarrow r = 1$.

Limits of θ : For complete ellipse, θ varies from 0 to 2π.

For half ellipse, θ varies from 0 to π.

For ellipse in first quadrant, θ varies from 0 to $\frac{\pi}{2}$

$$\iint_R f(x,y)\,dxdy = \underbrace{\int_{\theta=\theta_1}^{\theta=\theta_2}}_{\textit{Outer integral}} \underbrace{\int_{r=r_1}^{r=r_2}}_{\textit{Inner integral}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Here limits of r are either constants or functions of θ.

1.If limits of inner integral are functions of θ,then integrate w.r.to r first by keeping θ constant.

2. If all limits are constants then order of integration is immaterial provided the limits of integration are changed accordingly.

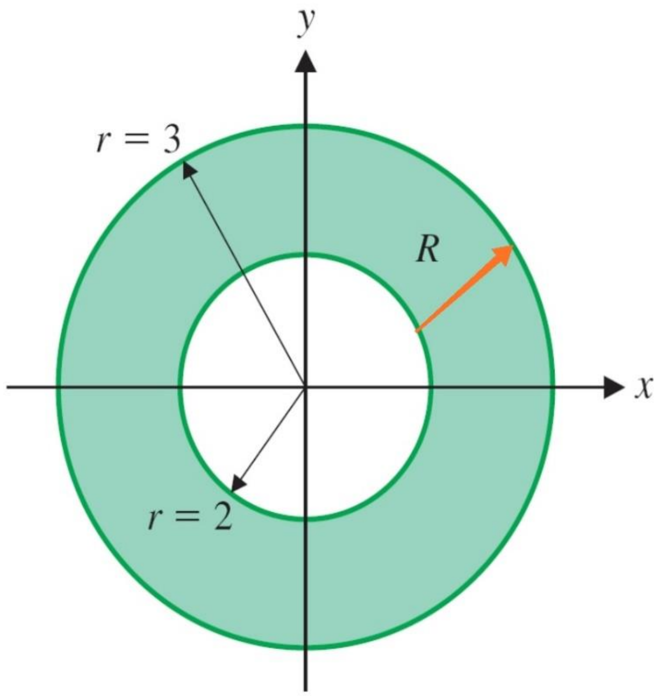
1. Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\,dxdy$ where R is an annulus between two circles $x^2+y^2=4$ and $x^2+y^2=9$.

Solution : The region of integration is shown in the figure.

Put $x=r\cos\theta, y=r\sin\theta, x^2+y^2=r^2$ and $dxdy=rdrd\theta$.

$x^2+y^2=4 \Rightarrow r^2=4 \Rightarrow r=2$

$x^2+y^2=9 \Rightarrow r^2=9 \Rightarrow r=3$



Draw a radial strip as shown in figure.

Inner end of the strip lies on circle $r=2$ and

Outer end of the strip lies on circle $r=3$.

Therefore, lower limit of r is $r=2$.

Therefore,upper limit of r is $r=3$.

Rotate this strip in ROI in positive direction.

In this rotation, θ varies from 0 to 2π.

So, lower limit of θ is $\theta=0$ and upper limit for θ is $\theta=2\pi$.

$$\mathrm{I} = \iint_R \frac{x^2y^2}{x^2+y^2}\, dxdy = \int_{\theta=0}^{\theta=2\pi} \int_{r=2}^{r=3} \frac{r^2cos^2\theta\ \ r^2sin^2\theta}{r^2}\ rdrd\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\int_{r=2}^{r=3} r^3dr \right] cos^2\theta\ sin^2\theta\ d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\frac{r^4}{4} \right]_{r=2}^{r=3} cos^2\theta\ sin^2\theta\ d\theta$$

$$= \left(\frac{81-16}{4}\right)\int_{\theta=0}^{\theta=2\pi} cos^2\theta \; sin^2\theta \; d\theta$$

$$= \frac{65}{4} \times 4 \int_{\theta=0}^{\theta=\frac{\pi}{2}} sin^2\theta \; cos^2\theta \; d\theta$$

$$= 65 \times \frac{1 \times 1}{4 \times 2}\frac{\pi}{2}$$

$$= \frac{65\pi}{16}$$

2. Evaluate $\iint\limits_{R} e^{-x^2-y^2}\,dxdy$ over area of the circle $x^2+y^2=4$.

Solution : The region of integration (ROI) is shown in following figure.

Put $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

and $dxdy = rdrd\theta$.

$x^2+y^2=4 \Rightarrow r^2=4 \Rightarrow r=2$

Draw a radial strip as shown in figure.

Inner end of the strip is at pole (origin).

Therefore, lower limit of r is $r=0$.

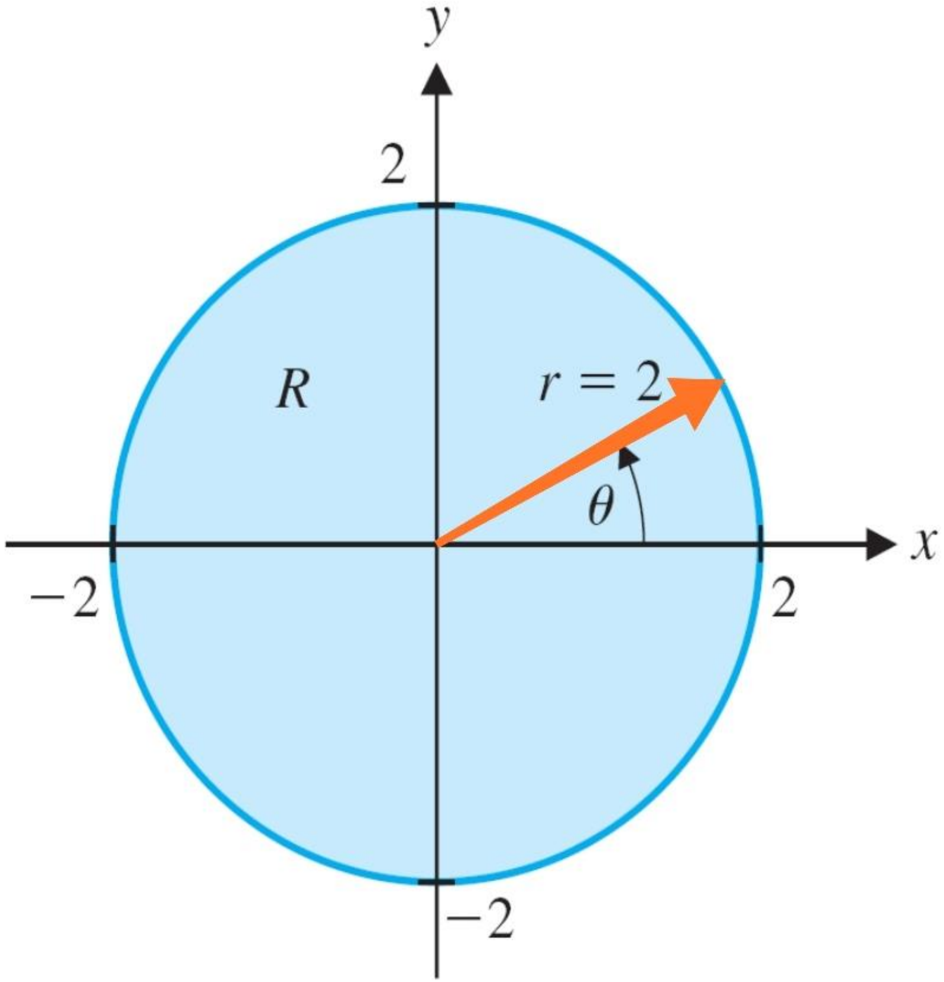
Outer end of the strip lies on circle $r=2$.

Therefore,upper limit of r is $r=2$.

Rotate this strip in ROI in positive direction.

In this rotation, θ varies from 0 to 2π.

So, lower limit of θ is 0 and upper limit is 2π.



$$\mathrm{I} = \iint_R e^{-x^2-y^2}\,dxdy = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} e^{-r^2}\, rdrd\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} \left[\int_{r=0}^{r=2} e^{-r^2}\,(2r)dr\right] d\theta$$

$$\therefore\ \mathrm{I} = \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} \left[-e^{-r^2}\right]_{r=0}^{r=2} d\theta$$

Recall: $\int \boldsymbol{e^{-f(x)} f'(x) dx = -e^{-f(x)}}$

$$= \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} (-e^{-4} + 1)d\theta$$

$$= \frac{1}{2}(1 - e^{-4})(2\pi) = \pi(1 - e^{-4})$$

3. Evaluate $\iint x^2y^2\, dxdy$ over the area of the circle $x^2+y^2=2y$.

Solution : The region of integration (ROI) is shown in following figure.

Put $x=r\cos\theta, y=r\sin\theta, x^2+y^2=r^2$ and $dxdy=r\,dr\,d\theta$.

$x^2+y^2=2y \Rightarrow r^2=2r\sin\theta \Rightarrow r=2\sin\theta$

Draw a radial strip as shown in figure.

Inner end of the strip is at pole (origin).

Therefore, lower limit of r is $r=0$.

Outer end of strip lies on circle $r=2\sin\theta$

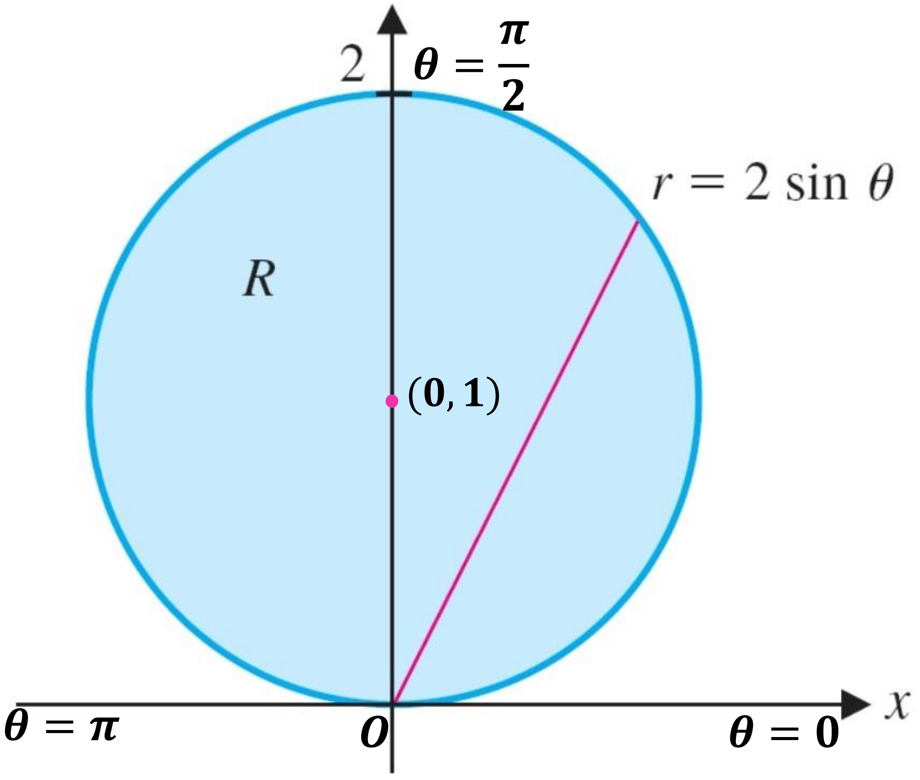
Therefore, upper limit of r is $r=2\sin\theta$

Rotate this strip in ROI in positive direction.

In this rotation, θ varies from 0 to π.

So, lower limit of θ is $\theta=0$ and

upper limit of θ is $\theta=\pi$.



$$\mathrm{I} = \iint x^2y^2dxdy = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2sin\theta} r^2cos^2\theta\ \ r^2sin^2\theta\ rdrd\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\left[\int_{r=0}^{r=2sin\theta} r^5dr\right]sin^2\theta\ cos^2\theta d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\left[\frac{r^6}{6}\right]_{r=0}^{r=2sin\theta} sin^2\theta\ cos^2\theta d\theta = \int_{\theta=0}^{\theta=\pi}\frac{64\ sin^6\theta}{6}sin^2\theta\ cos^2\theta d\theta$$

$$= \frac{64}{6}\times 2\int_{\theta=0}^{\theta=\frac{\pi}{2}} sin^8\theta\ cos^2\theta d\theta$$

$$= \frac{64}{3}\times\frac{(7\times 5\times 3\times 1)\times 1}{10\times 8\times 6\times 4\times 2}\ \frac{\pi}{2} = \frac{7\pi}{24}$$

4. Evaluate $\iint (x^2 + y^2)dxdy$ over the area bounded by an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

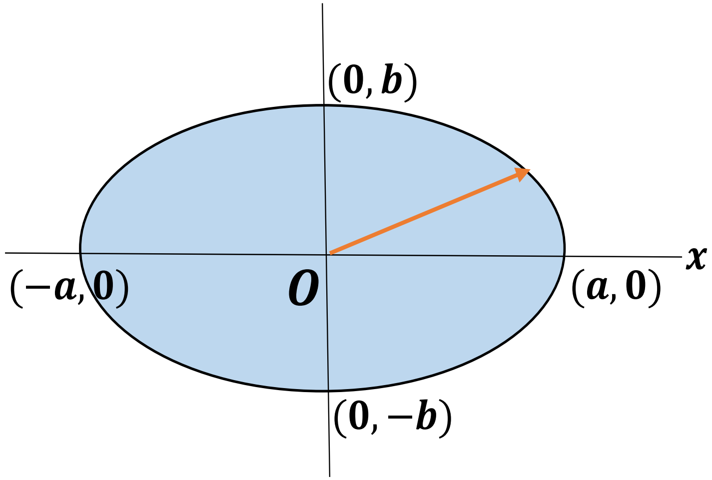
Solution : The region of integration (ROI) is shown in following figure.
Put $x = ar\cos\theta, y = brsin\theta$

and $dxdy = abrdrd\theta.$

Draw a radial strip as shown in figure.

Lower limit of r is $r = 0.$

Upper limit of r is $r = 1$



Rotate this strip in ROI in positive direction.

In this rotation, θ varies from 0 to 2π.

So, lower limit of θ is $\theta = 0$ and

upper limit of θ is $\theta = 2\pi.$

$$\mathrm{I} = \iint (x^2+y^2)dxdy = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (a^2r^2cos^2\theta + b^2\, r^2sin^2\theta)\, abrdrd\theta$$

$$= ab \int_{\theta=0}^{\theta=2\pi} \left[\int_{r=0}^{r=1} r^3dr \right] (a^2cos^2\theta + b^2sin^2\theta)d\theta = ab \int_{\theta=0}^{\theta=2\pi} \left[\frac{r^4}{4}\right]_{r=0}^{r=1} (a^2cos^2\theta + b^2sin^2\theta)d\theta$$

$$= \frac{ab}{4} \int_{\theta=0}^{\theta=2\pi} (a^2cos^2\theta + b^2sin^2\theta)\, d\theta$$

$$= \frac{ab}{4} \times \left\{ \left[a^2 \int_{\theta=0}^{\theta=2\pi} cos^2\theta d\theta \right] + \left[b^2 \int_{\theta=0}^{\theta=2\pi} sin^2\theta d\theta \right] \right\}$$

$$= \frac{ab}{4} \times \left[\left(a^2 \times 4 \times \frac{1}{2} \times \frac{\pi}{2} \right) + \left(b^2 \times 4 \times \frac{1}{2} \times \frac{\pi}{2} \right) \right] = \frac{\pi ab}{4} (a^2 + b^2)$$

5. Evaluate $\iint y^2 dxdy$ over area outside the circle $x^2+y^2=ax$ & inside the circle $x^2+y^2=2ax$.

Solution : The region of integration (ROI) is shown in following figure.

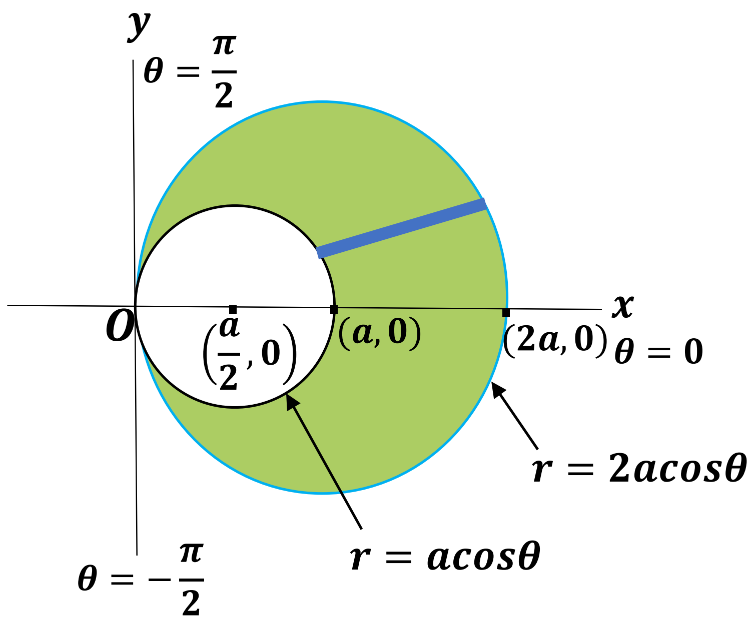
$x^2+y^2=ax \Rightarrow \left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$, circle with centre at $\left(\frac{a}{2},0\right)$ & radius $\frac{a}{2}$.

$x^2+y^2=2ax \Rightarrow (x-a)^2+y^2=a^2$, circle with centre at $(a,0)$ & radius a.

Put $x=r\cos\theta, y=r\sin\theta, x^2+y^2=r^2$ and $dxdy=rdrd\theta$.

$x^2+y^2=ax \Rightarrow r^2=ar\cos\theta \Rightarrow r=a\cos\theta$

$x^2+y^2=2ax \Rightarrow r^2=2ar\cos\theta \Rightarrow r=2a\cos\theta$



Draw a radial strip as shown in figure.

Inner end of strip lies on circle $r=a\cos\theta$

Therefore,upper limit of r is $r=a\cos\theta$

Outer end of strip lies on circle $r=2a\cos\theta$

Therefore,upper limit of r is $r=2a\cos\theta$

Rotate this strip in ROI in positive direction.

In this rotation, θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$$\mathrm{I} = \iint y^2 dxdy = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=acos\theta}^{r=2acos\theta} r^2 sin^2\theta \, rdrd\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left[\int_{r=acos\theta}^{r=2acos\theta} r^3 dr \right] sin^2\theta \; d\theta = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{r=acos\theta}^{r=2acos\theta} sin^2\theta d\theta$$

$$= \frac{1}{4} \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} (16a^4cos^4\theta - a^4cos^4\theta) \, sin^2\theta \; d\theta$$

$$= \frac{15a^4}{4} \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} sin^2\theta \; cos^4\theta \; d\theta = \frac{15a^4}{4} \times 2 \times \frac{(1) \times (3 \times 1)}{6 \times 4 \times 2} \; \frac{\pi}{2} = \frac{15a^4\pi}{64}$$

Example 6. Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\ dxdy$.

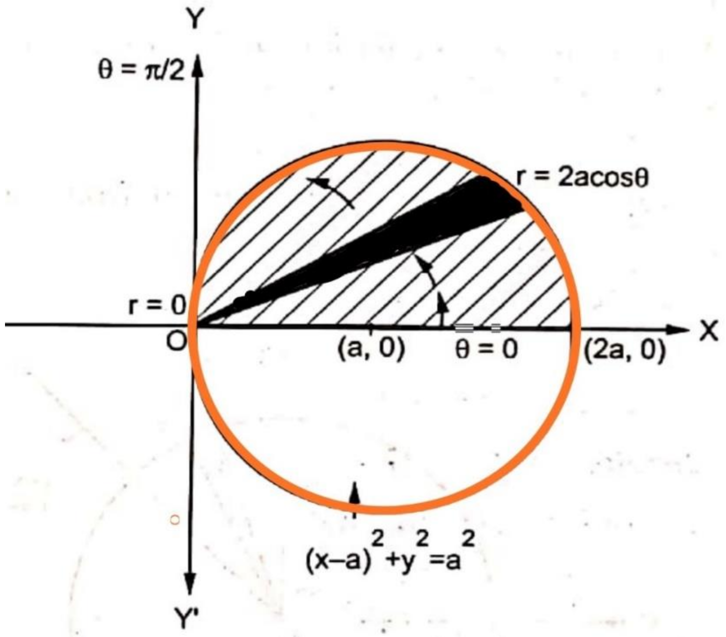
Solution : Here lower limit of y is $y = 0$ and upper limit of y is $y = \sqrt{2ax - x^2}$

Lower limit of x is $x = 0$ and upper limit of x is $x = 2a$.

$y = \sqrt{2ax - x^2} \Rightarrow y^2 = 2ax - x^2 \Rightarrow x^2 + y^2 = 2ax \Rightarrow (x-a)^2 + y^2 = a^2$, circle with centre at $(a, 0)$ & radius a.

Since lower limit of y is $y = 0$, region of integration (ROI) is the semi-circle as shown in figure.

Put $x = r\cos\theta, y = r sin\theta, x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$.
$x^2 + y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta \Rightarrow r = 2a\cos\theta$



Draw a radial strip as shown in figure.

Inner end of the strip is at pole (origin).

Outer end of the strip lies on circle $r = 2a\cos\theta$

Lower limit of r is 0 & upper limit is $2a\cos\theta$

Rotate this strip in ROI in positive direction.

In this rotation, θ varies from 0 to $\frac{\pi}{2}$. **So, lower limit of θ is 0 and upper limit is $\frac{\pi}{2}$.**

$$\therefore I = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\ dxdy = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2a\cos\theta} r^2\ rdrd\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\int_{r=0}^{r=2a\cos\theta} r^3 dr \right] d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_{r=0}^{r=2a\cos\theta} d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} 16a^4 cos^4\theta\ d\theta = 4a^4 \frac{3 \times 1}{4 \times 2} \frac{\pi}{2} = \frac{3\pi a^4}{4}$$

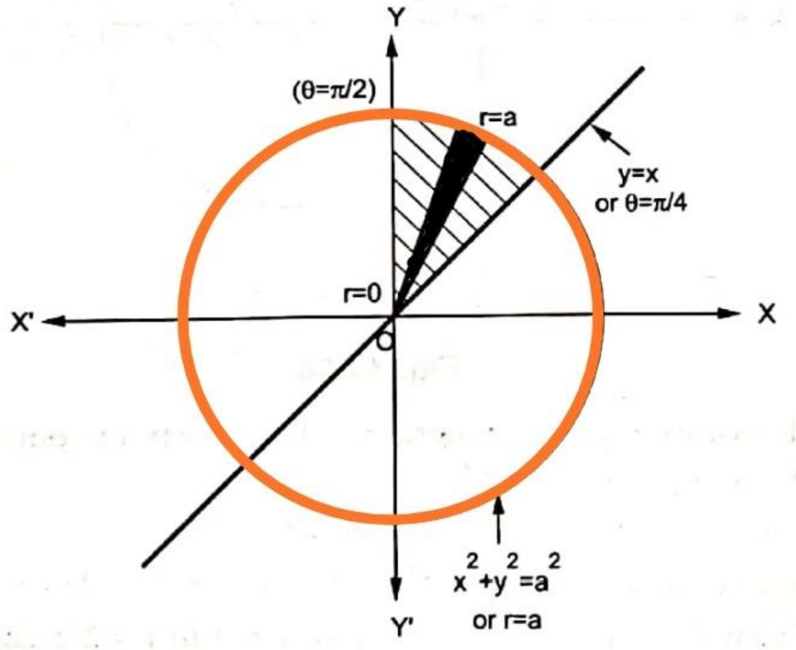
Example 7. Evaluate $\int_{0}^{\frac{a}{\sqrt{2}}} \int_{x}^{\sqrt{a^2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dxdy$.

Solution : Here lower limit of y is $y = x$ and upper limit of y is $y = \sqrt{a^2 - x^2}$

Lower limit of x is $x = 0$ and upper limit of x is $x = \frac{a}{\sqrt{2}}$.

$y = \sqrt{a^2 - x^2} \Rightarrow x^2 + y^2 = a^2$

Since lower limit of y is $y = x$ and upper limit of x is $x = \frac{a}{\sqrt{2}}$, region of integration (ROI) is the shaded region as shown in following figure.



Put $x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$.

$x^2 + y^2 = a^2 \Rightarrow r = a$

Draw a radial strip as shown in figure.

Inner end of the strip is at pole (origin).

Therefore, lower limit of r is $r = 0$.

Outer end of the strip lies on circle $r = a$

Therefore,upper limit of r is $r = a$.

Rotate this strip in ROI in positive direction.In this rotation, θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$.

So, lower limit of θ is $\theta = \frac{\pi}{4}$ and upper limit of θ is $\theta = \frac{\pi}{2}$.

$$\therefore I = \int_{0}^{\frac{a}{\sqrt{2}}} \int_{x}^{\sqrt{a^2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dxdy = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=a} \frac{r\cos\theta}{\sqrt{r^2}}\, rdrd\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \left[\int_{r=0}^{r=a} rdr\right] \cos\theta\, d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{r=0}^{r=a} \cos\theta\, d\theta = \frac{a^2}{2} \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cos\theta\, d\theta = \frac{a^2}{2}[sin\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

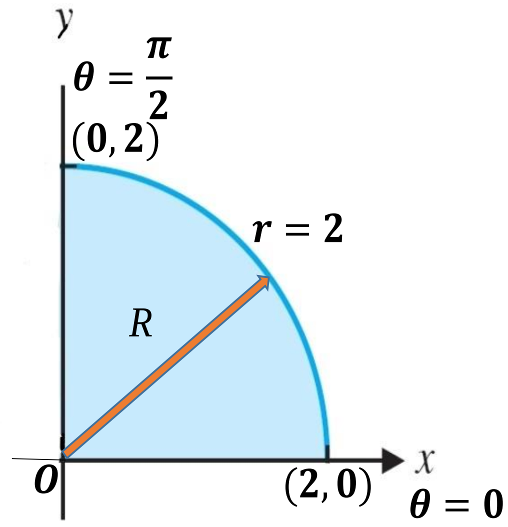
Example 8. Evaluate $\int_0^2 \int_0^{\sqrt{4-y^2}} \frac{y}{\sqrt{(4-x^2)(x^2+y^2)}}\, dxdy$.

Solution : Here lower limit of x is $x = 0$ and upper limit of x is $x = \sqrt{4-y^2}$

Lower limit of y is $y = 0$ and upper limit of y is $y = 2$.

Region of integration (ROI) is positive quadrant of the circle $x^2 + y^2 = 4$.

The region of integration (ROI) is shown in following figure.



Put $x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$.

$x^2 + y^2 = 4 \Rightarrow r = 4$

Draw a radial strip as shown in figure.

Inner end of the strip is at pole (origin).

Therefore, lower limit of r is $r = 0$.

Outer end of the strip lies on circle $r = 2$.

Therefore,upper limit of r is $r = 2$.

Rotate this strip in ROI in positive direction. In this rotation, θ varies from 0 to $\frac{\pi}{2}$.

So, lower limit of $\boldsymbol{\theta}$ is $\boldsymbol{\theta = 0}$ and upper limit of $\boldsymbol{\theta}$ is $\boldsymbol{\theta = \frac{\pi}{2}}$.

$$\mathrm{I} = \int_{0}^{2} \int_{0}^{\sqrt{4-y^2}} \frac{y}{\sqrt{(4-x^2)(x^2+y^2)}}\, dxdy = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2} \frac{r\sin\theta}{\sqrt{(4-r^2cos^2\theta)(r^2)}}\, rdrd\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\int_{r=0}^{r=a} rdr \right] \cos\theta\, d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \left[\frac{r^2}{2} \right]_{r=0}^{r=a} \cos\theta\, d\theta$$

$$= \frac{a^2}{2} \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cos\theta\, d\theta = \frac{a^2}{2} [sin\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{a^2}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

Type-IV : Change of Order of Integration

Suppose $\mathrm{I} = \displaystyle\int_{x=a}^{x=b} \int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy\,dx$ is a given double integral.

As limits of inner integral are functions of x, we need to integrate w.r.to y first.

But in many cases, it happens that integrand $f(x,y)$ in above integral is difficult or even impossible to integrate w.r.to y first.

However it is easy to integrate w.r.to x first.

In this case it is rquired to change the order of integration.

For example, in integral $\displaystyle\int_0^1 \int_x^1 \frac{siny}{y}\, dydx$, limits of inner integral are functions of x means we have integrate w.r.to y first. But integrand is difficult to integrate w.r.to y first. But it is easy to integrate w.r.to x first. So in order to integrate w.r.to x first we have to change the order of integration.

Also suppose $I = \displaystyle\int_{y=c}^{y=d} \int_{x=g_1(y)}^{x=g_2(y)} f(x,y)\,dx\,dy$, is a given double integral.

As limits of inner integral are functions of y, we need to integrate w.r.to x first.But in many cases, it happens that integrand $f(x,y)$ in above integral is difficult or even impossible to integrate w.r.to x first.

However it is easy to integrate w.r.to y first.

In this case it is rquired to change the order of integration.

For example, in integral $\int_0^1 \int_{4y}^4 e^{x^2}\, dxdy$, limits of inner integral are functions of y means we have to integrate w.r.to x first.

But integrand is difficult to integrate w.r.to x first.

But it is easy to integrate w.r.to y first.

So in order to integrate w.r.to x first we have to change the order of integration.

Now how to change the Order of Integration?

Case-I : Limits of inner integral are functions of x and integrand is difficult to integrate w.r.to y first.

Step 1: Sketch the region of integration from given limits.
Step 2: Draw a strip parallel to x −axis in ROI.
Step 3: Find limits of x from left and right ends of the strip.
Step 4: Find limits of y by moving strip from bottom to top in ROI.
Step 5: Using these limits, integrate w.r.to x first and then resulting integrand w.r.to y.

Note that in such a integral, strip is considered as parallel to y −axis. Hence we change/reverse the strip as per given in step 2.

Case-II : Limits of inner integral are functions of y and integrand is difficult to integrate w.r.to x first.

Step 1: Sketch the region of integration from given limits.
Step 2: Draw a strip parallel to y −axis in ROI.
Step 3: Find limits of y, from lower and upper ends of the strip.
Step 4: Find limits of x by moving strip from left to right in ROI.
Step 5: Using these limits, integrate w.r.to y first and then resulting integrand w.r.to x.

Note that in such a integral, strip is considered as parallel to x−axis. Hence we change/reverse the strip as per given in step 2.

Example 1. Evaluate $\int_0^1 \int_y^1 \frac{sinx}{x}\, dxdy$

Solution : As limits of inner integral are functions of y, we need to integrate w.r.to x first. But integrand is difficult to integrate w.r.to x first.However it is easy to integrate w.r.to y first.

Thus it is rquired to change the order of integration.

Given Limits : $x = y$, $x = 1$, $y = 0$ and $y = 1$.

Thus region of integration ROI is as shown in figure.

In order to change the order of integration,

draw a strip parrallel to y-axis as shown in figure.

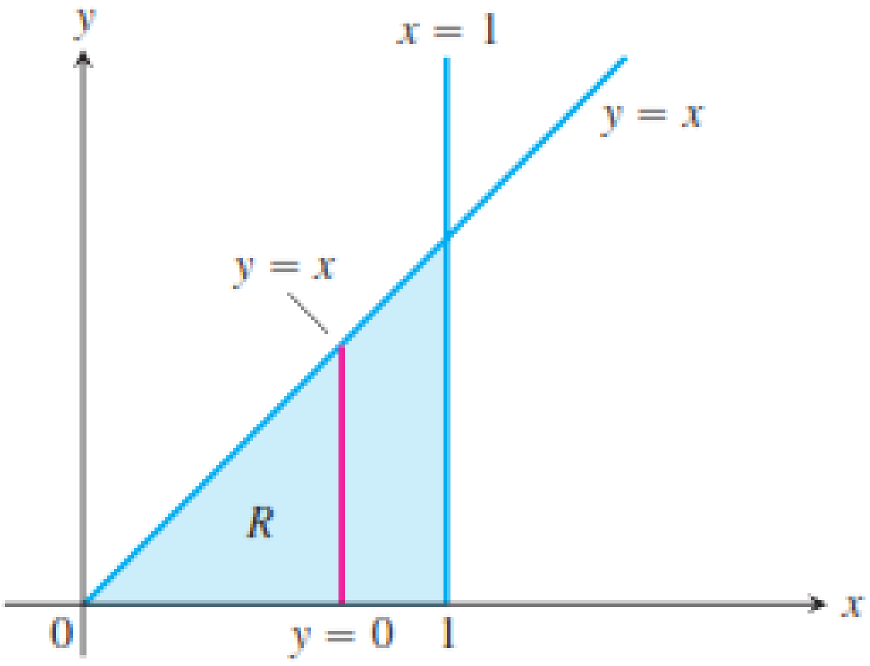
Lower end of the strip lies on x-axis i.e. y=0

Upper end of the strip lies on line $y = x$

Therefore, lower limit of y is 0 & upper limit x.

Move strip from left to right in region R.

In this movement x varies from 0 to 1.



Lower limit for *x* is *x* = 0 and upper limit for *x* is *x* =1.

$$\therefore \int_{y=0}^{y=1} \int_{x=y}^{x=1} \frac{\sin x}{x}\, dx\, dy = \int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{\sin x}{x}\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=x} \frac{\sin x}{x}\, dy \right] dx = \int_{x=0}^{x=1} \frac{\sin x}{x} [y]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{x=1} \frac{\sin x}{x} [x - 0] dx = \int_{x=0}^{x=1} \sin x\, dx$$

$$= (-\cos x)_{x=0}^{x=1}$$

$$= -\cos 1 + \cos 0 = 1 - \cos 1$$

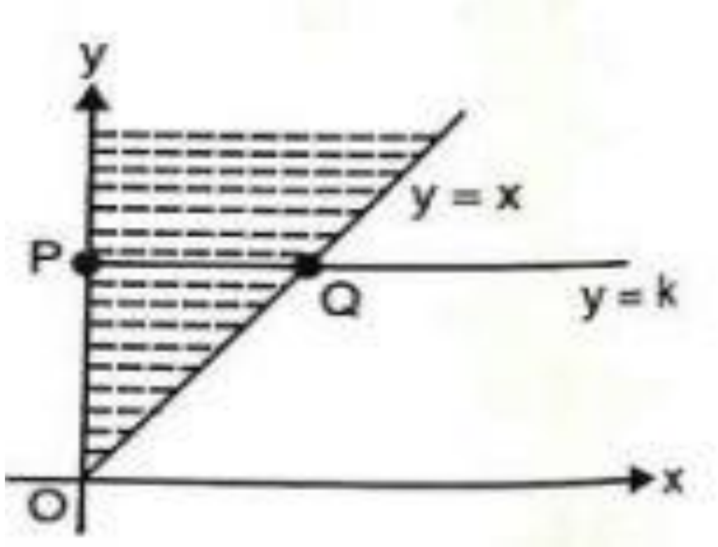
Example 2. Evaluate $\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}dxdy$

Solution : As limits of inner integral are functions of x, we need to integrate w.r.to y first. But integrand is difficult to integrate w.r.to y first.However it is easy to integrate w.r.to x first.

Thus it is rquired to change the order of integration.

Given Limits : $\boldsymbol{y = x\ , y = \infty,\ \ x = 0}$ **and** $\boldsymbol{x = \infty}$**.**

Thus region of integration ROI is as shown in figure.



In order to change the order of integration,

draw a strip parrallel to x-axis in ROI as shown in figure.

Left end of the strip lies on y-axis i.e. x=0.

Therefore, lower limit of *x* is *x*= 0.

Right end of the strip lies on line x = y.

Therefore, upper limit of *x* is x = y.

Move strip from bottom to top in region R.In this movement y varies from 0 to ∞.

Lower limit for y is $y = 0$ and upper limit for y is ∞.

$$\therefore \int_{x=0}^{x=\infty} \int_{y=x}^{y=\infty} \frac{e^{-y}}{y}\, dydx = \int_{y=0}^{y=\infty} \int_{x=0}^{x=y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_{y=0}^{y=\infty} \left[\int_{x=0}^{x=y} \frac{e^{-y}}{y}\, dx \right] dy = \int_{y=0}^{y=\infty} \frac{e^{-y}}{y} [x]_{x=0}^{x=y} dy$$

$$= \int_{y=0}^{y=\infty} \frac{e^{-y}}{y} (y - 0)\, dy = \int_{y=0}^{y=\infty} e^{-y}\, dy$$

$$= (-e^{-y})_{y=0}^{y=\infty} = -e^{-\infty} + e^{0} = 1$$

3. *Evaluate* $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ *by changing the order of integration.*

Solution:

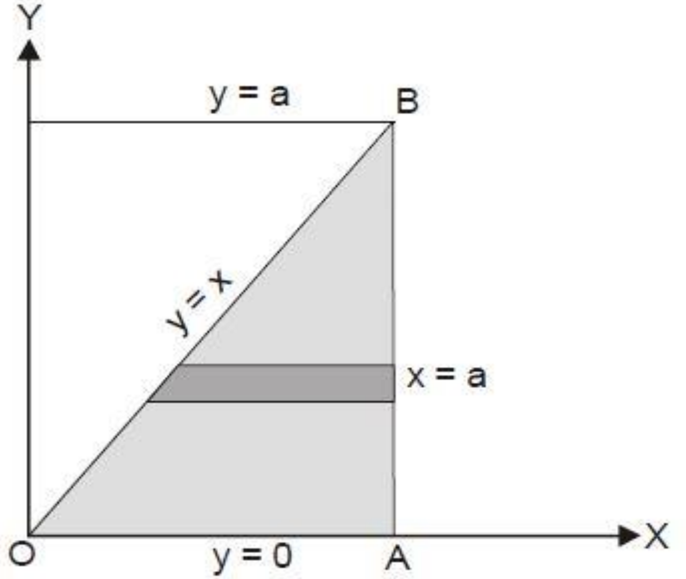
$$I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$$

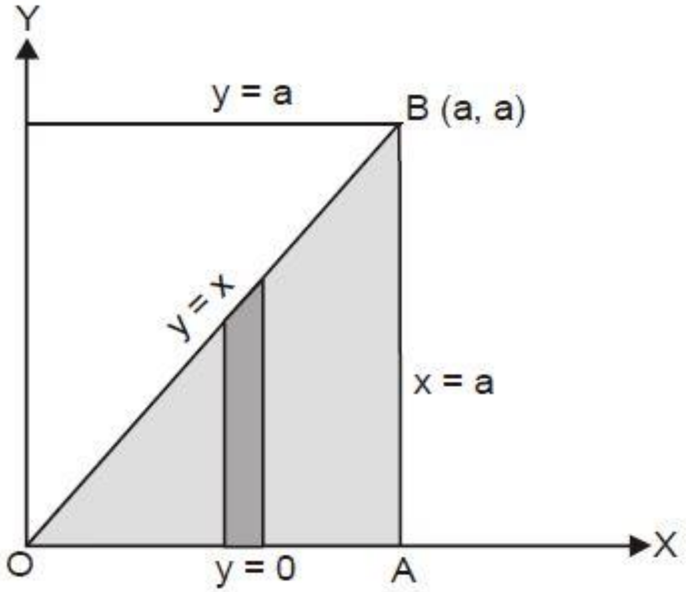
Here $x = a, x = y, y = 0$ and $y = a$

The area of integration is OAB.

On changing the order of integration Lower limit of $y = 0$ and upper limit is $y = x$.

Lower limit of $x = 0$ and upper limit is $x = a$.





$$I = \int_0^a x\,dx \int_0^{y=x} \frac{1}{x^2+y^2}\,dy$$

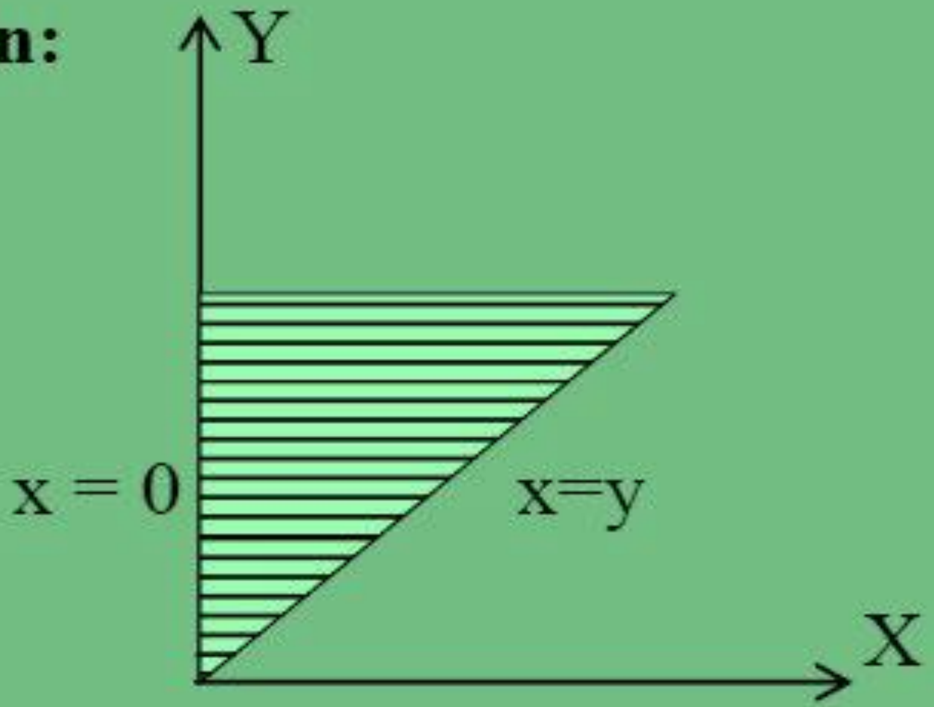
$$= \int_0^a x\,dx \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^{y=x}$$

$$= \int_0^a \frac{x}{x}\,dx\left(\tan^{-1}\frac{x}{x} - \tan^{-1} 0\right)$$

$$= \int_0^a dx\left(\frac{\pi}{4}\right) \quad = \frac{\pi}{4}[x]_0^a = \frac{a\pi}{4}$$

Evaluate $\int_0^\infty \int_0^y y e^{-\frac{y^2}{x}} dxdy$ by changing the order of integration.

Solution:



Given x=0, x = y, y = 0, y = ∞ .

By changing the order of integration y: x to ∞, x : 0 to ∞

$$\int_0^\infty \int_0^y ye^{-\frac{y^2}{x}} dxdy = \int_0^\infty \int_x^\infty ye^{-\frac{y^2}{x}} dydx$$

$$= \int_0^\infty \int_x^\infty ye^{-\frac{y^2}{x}} d\left(\frac{y^2}{2}\right) dx$$

$$= \frac{1}{2}\int_0^\infty \left[\frac{e^{-\frac{y^2}{x}}}{-1/x}\right]_x^\infty dx \;\; = \frac{1}{2}\int_0^\infty xe^{-x} dx$$

$Take\; u = x, dv = e^{-x}dx \quad implies\; du = dx, v = -e^{-x},$

by integration by parts,

$$= \frac{1}{2}\left[x\left(\frac{e^{-x}}{-1}\right) - e^{-x}\right]_0^\infty = \frac{1}{2}$$

Triple Integration

The triple integration is an extension of the double integration into third dimension. Triple integration is related to the volume of solid.

An integral of the form $\iiint\limits_V f(x, y, z)\, dxdydz$

is called as triple integral of a continuous function $f(x, y, z)$ defined over a finite region V of three dimensional space.

Three types of problems arises in triple integration

1. Evaluation of triple integral when limits are given
2. Evaluation of triple integral by transforming to spherical polar coordinates
3. Evaluation of triple integral by Dirichlet's Theorem

Type-I : Direct Evaluation of Triple Integral

Let $f(x,y,z)$ be continuous function of three variables x, y & z defined on a finite region V of three dimensional space.

Case-I : If V is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, $f_1(x,y) \le z \le f_2(x,y)$, then

$$\iiint\limits_V f(x,y,z)\, dV = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\, dxdydz$$

Outer integral *Middle integral* *Inner integral*

Here integrand $f(x,y,z)$ is to be integrated w.r.to z first.

After putting limits of z, resulting integrand is to be integrated w.r.to y.Lastly after putting limits of y, resulting integrand is to be integrated w.r.to x.

Note that If limits of inner integral are functions of any two variables, then integrate integrand $f(x,y,z)$ w.r.to third variable first by keeping remaining two variables constant.

Case-II : If all limits are constants i.e.if V is defined by $a \leq x \leq b$, $c \leq y \leq d$ and $p \leq z \leq q$ where a, b, c, d, p and q all are constants then the order of integration is immaterial provided the limits of integration are changed accordingly.

$$\iiint_V f(x,y,z)\,dV = \boxed{\int_{x=a}^{x=b} \boxed{\int_{y=c}^{y=d} \boxed{\int_{z=p}^{z=q} f(x,y,z)\,dz}\,dy}\,dx}$$

Example 1. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dxdydz$.

Solution : Limits of inner integral are functions of *x* and *y*.

Limits of middle integral are functions of *x*.

So we need to integrate w.r.to *z* first by keeping *x* and *y* constant.

Then we need to integrate resulting integrand w.r.to *y* by keeping *x* constant.

$$\text{Let I} = \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dxdydz = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \left[\int_{z=0}^{z=x+y} e^z dz \right] dydx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} [e^z]_{z=0}^{z=x+y} \, dydx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (e^{x+y} - 1)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left[e^{x+y} - y\right]_{y=0}^{y=1-x} dx$$

$$= \int_{x=0}^{x=1} \left[e^{x+(1-x)} - (1-x) - e^{x} + 0\right] dx$$

$$= \int_{x=0}^{x=1} \left[e - 1 + x - e^{x}\right] dx$$

$$= \left[ex - x + \frac{x^2}{2} - e^x\right]_{x=0}^{x=1}$$

$$= \left\{\left[e - 1 + \frac{1}{2} - e\right] - [0 - 0 + 0 - 1]\right\} = \frac{1}{2}$$

Example 2. Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}\, dzdydx.$

Solution : Limits of inner integral are functions of *x* and *y*.

Limits of middle integral are functions of *x*.

So we need to integrate w.r.to *z* first by keeping *x* and *y* constant.

Then we need to integrate resulting integrand w.r.to *y* by keeping *x* constant.

$$\text{Let I} = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} dzdydx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \left[\int_{z=0}^{z=\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{\mathbf{1-x^2-y^2}-z^2}} \, dz \right] dydx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \left[sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_{z=0}^{z=\sqrt{1-x^2-y^2}} dydx$$

$(\because 1-x^2-y^2 \text{is a constant w.r.to } z)$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} [sin^{-1}1 - sin^{-1}0]\, dydx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \frac{\pi}{2}\, dydx$$

$$= \frac{\pi}{2} \int_{x=0}^{x=1} [y]_{y=0}^{y=\sqrt{1-x^2}}\, dx = \frac{\pi}{2} \int_{x=0}^{x=1} \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2} sin^{-1}x \right]_{x=0}^{x=1}$$

$$= \frac{\pi}{2} sin^{-1}1 = \frac{\pi}{2}\frac{\pi}{4} = \frac{\pi^2}{8}$$

Example 3. Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dzdxdy$.

Solution : Limits of inner integral are functions of x.

Limits of middle integral are functions of y.

So we need to integrate w.r.to z first by keeping x and y constant.

Then we need to integrate resulting integrand w.r.to x by keeping y constant.

$$\text{Let I} = \int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dzdxdy = \int_{y=0}^{y=1} \int_{x=y^2}^{x=1} \left[\int_{z=0}^{z=1-x} xdz \right] dxdy$$

$$= \int_{y=0}^{y=1} \int_{x=y^2}^{x=1} x[z]_{z=0}^{z=1-x}\, dxdy$$

$$= \int_{y=0}^{y=1} \int_{x=y^2}^{x=1} x(1-x)dxdy = \int_{y=0}^{y=1} \int_{x=y^2}^{x=1} (x - x^2)\, dxdy$$

$$= \int_{y=0}^{y=1} \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=y^2}^{x=1} dy$$

$$= \int_{y=0}^{y=1} \left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right)\right] dy$$

$$= \int_{y=0}^{y=1} \left[\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right] dy$$

$$= \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21}\right]_{y=0}^{y=1} = \frac{1}{6} - \frac{1}{10} + \frac{1}{21} = \frac{4}{35}$$

Example 4. Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{cos\theta} \int_0^{\sqrt{1-r^2}} r\, dzdrd\theta$.

Solution:Limits of inner integral are functions of r. Limits of middle integral are functions of θ.

So we need to integrate w.r.to z first by keeping r and θ constant.

Then we need to integrate resulting integrand w.r.to r by keeping θ constant.

$$\text{Let I} = \int_0^{\frac{\pi}{2}} \int_0^{cos\theta} \int_0^{\sqrt{1-r^2}} r\, dzdrd\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=cos\theta} \left[\int_{z=0}^{z=\sqrt{1-r^2}} dz\right] rdrd\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=cos\theta} [z]_{z=0}^{z=\sqrt{1-r^2}} \, rdrd\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=cos\theta} \sqrt{1-r^2} \; rdrd\theta = -\frac{1}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\int_{r=0}^{r=cos\theta} (1-r^2)^{\frac{1}{2}} \; (-2r)dr \right] d\theta$$

We know that $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$

$$\therefore \mathrm{I} = -\frac{1}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\frac{(1-r^2)^{\frac{3}{2}}}{3/2} \right]_{r=0}^{r=cos\theta} d\theta$$

$$= -\frac{1}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[(1 - cos^2\theta)^{\frac{3}{2}} - 1\right] d\theta$$

$$= -\frac{1}{3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} [sin^3\theta \ - 1] d\theta$$

$$= -\frac{1}{3}\left[\frac{2}{3} \times 1 - \frac{\pi}{2}\right] = \frac{\pi}{6} - \frac{2}{9}$$

Example 5. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dzdydx.$

Solution : Limits of inner integral are functions of *x* and *y*.

Limits of middle integral are functions of *x*.

So we need to integrate w.r.to *z* first by keeping *x* and *y* constant.

Then we need to integrate resulting integrand w.r.to *y* by keeping *x* constant.

$$\text{Let I} = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dzdydx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} xy \left[\int_{z=0}^{z=\sqrt{1-x^2-y^2}} z\, dz \right] dydx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{1-x^2-y^2}} xy\, dydx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} (1 - x^2 - y^2)xy\, dydx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} x \left[\int_{y=0}^{y=\sqrt{1-x^2}} (y - x^2 y - y^3)\, dy\right] dx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} x \left[\frac{y^2}{2} - x^2 \frac{y^2}{2} - \frac{y^4}{4}\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} x \left[\frac{1-x^2}{2} - x^2 \frac{(1-x^2)}{2} - \frac{(1-x^2)^2}{4} \right] dx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} x(1-x^2) \left[\frac{1}{2} - \frac{x^2}{2} - \frac{1-x^2}{4} \right] dx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} x(1-x^2) \left[\frac{2-2x^2-1+x^2}{4} \right] dx \; = \frac{1}{8} \int_{x=0}^{x=1} x(1-x^2)^2 dx$$

Put $x = sin\theta \Longrightarrow dx = cos\theta d\theta$. Limits of θ are 0 to $\frac{\pi}{2}$.

$$\therefore \mathrm{I} = \frac{1}{8} \int_{\theta=0}^{\theta=\frac{\pi}{2}} sin\theta (1-sin^2\theta)^2 cos\theta d\theta = \frac{1}{8} \int_{\theta=0}^{\theta=\frac{\pi}{2}} sin\theta cos^5\theta \; d\theta$$

$$= \frac{1}{8} \frac{1 \times (4 \times 2)}{6 \times 4 \times 2} = \frac{1}{48}$$

Example 6. Evaluate $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dxdydz$.

Solution : Here all limits are constants.Thus order of integration is immaterial provided the limits of integration are changed accordingly.

$$\text{Let I} = \int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dxdydz = \int_{z=0}^{z=1} \int_{y=0}^{y=1} \left[\int_{x=0}^{x=1} e^{x+y+z} dx \right] dydz$$

$$= \int_{z=0}^{z=1} \int_{y=0}^{y=1} [e^{x+y+z}]_{x=0}^{x=1}\ dydz$$

$$= \int_{z=0}^{z=1} \int_{y=0}^{y=1} (e^{1+y+z} - e^{y+z})\ dydz$$

$$= \int_{z=0}^{z=1} \left[e^{1+y+z} - e^{y+z}\right]_{y=0}^{y=1} dz$$

$$= \int_{z=0}^{z=1} \left[e^{2+z} - e^{1+z} - e^{1+z} + e^{z}\right] dz$$

$$= \left[e^{2+z} - 2e^{1+z} + e^{z}\right]_{z=0}^{z=1}$$

$$= e^3 - 2e^2 + e - e^2 + 2e - 1$$
$$= e^3 - 3e^2 + 3e - 1$$
$$= (e-1)^3$$

$$\boldsymbol{Note}: \boldsymbol{I} = \int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dxdydz = \int_0^1 \int_0^1 \int_0^1 e^x e^y e^z\, dxdydz..$$

Here all limits are constants.Also we can write integrand as product of function of each variable separately

$$\mathrm{I} = \int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dxdydz = \left[\int_{x=0}^{x=1} e^x dx\right]\left[\int_{y=0}^{y=1} e^y dy\right]\left[\int_{z=0}^{z=1} e^z dz\right]$$

$$= (e-1)^3 = (e-1)(e-1)(e-1)$$

Some Practice Problems

1. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\, dxdydz.$ $\quad$ ***Ans***: $\frac{1}{720}$

2. Evaluate $\int_0^{log2} \int_0^{x} \int_0^{x+y} e^{x+y+z}\, dxdydz.$ $\quad$ ***Ans***: $\frac{5}{8}$

3. Evaluate $\int_0^{a} \int_0^{x} \int_0^{\sqrt{x+y}} z\, dxdydz.$ $\quad$ ***Ans***: $\frac{a^3}{4}$

4. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dxdydz}{(1+x+y+z)^3}.$ $\quad$ ***Ans***: $\frac{1}{8}\left(log2 - \frac{5}{8}\right)$

5. Evaluate $\int_0^2 \int_0^{y} \int_{x-y}^{x+y} (x+y+z)dxdydz\, .$ $\quad$ ***Ans***: 16

6. Evaluate $\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^2 xyz\, dzdydx.$ **Ans**: $\frac{3}{8}$

7. Evaluate $\int_1^2 \int_0^1 \int_{-1}^1 (x^2+y^2+z^2) dxdydz\,.$ **Ans**: 6

8. Evaluate $\int_0^1 \int_0^2 \int_1^2 x^2yz\, dzdydx.$ **Ans**: 1

9. Evaluate $\int_0^2 \int_1^3 \int_1^2 y dxdydz\,.$ **Ans**: 8

10. Evaluate $\int_0^\infty \int_0^\infty \int_0^\infty \frac{dxdydz}{(1+x^2+y^2+z^2)^2}.$ **Ans**: $\frac{\pi^2}{8}$

Type-II : Triple integration by spherical polar coordinates

- Spherical polar coordinate system is very useful coordinate system in three dimensions .It simplifies the evaluation of triple integrals over regions bounded by **spheres** or **cones**.
- When integrand $f(x,y,z)$ contains term like $x^2+y^2+z^2$ and region of integration is sphere or ellipsoid then change variables from cartesian to spherical polar coordinates.
- To change variables from cartesian to spherical polar coordinates,

put $x = r\,sin\theta\cos\phi\,,\ \ y = r\,sin\theta\,\sin\phi,\ \ z = rcos\theta$

$\Longrightarrow x^2+y^2+z^2 = r^2$ and $dxdydz = r^2 sin\theta\,dr\,d\theta\,d\phi$

$$\iiint_V f(x,y,z)\,dxdydz = \iiint_V f(rsin\theta\cos\phi, rsin\theta\sin\phi, rcos\theta\,)\,r^2 sin\theta dr d\theta d\phi$$

Some standard limits

1.Complete sphere $x^2 + y^2 + z^2 = a^2$

r varies from 0 to a, θ varies from 0 to π, ϕ varies from 0 to 2π.

2.Hemi-sphere $x^2 + y^2 + z^2 = a^2$ $(z \geq 0)$

r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$, ϕ varies from 0 to 2π.

3. Sphere $x^2 + y^2 + z^2 = a^2$ in positive octant

r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$, ϕ varies from 0 to $\frac{\pi}{2}$.

Note : When region of integration is an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ then

put $x = ar\ sin\theta\cos\phi\ , y = br\ sin\theta\ \sin\phi,\ \ z = crcos\theta,$

$\Longrightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=r^2$ and $dxdydz = abc\, r^2 sin\theta\, dr\, d\theta\, d\phi$

Limits for Complete ellipsoid:

r varies from 0 to $1,$ θ varies from 0 to $\pi,$ ϕ varies from 0 to 2π.

Limits for half ellipsoid:

r varies from 0 to $1,$ θ varies from 0 to $\frac{\pi}{2},$ ϕ varies from 0 to 2π.

Limits for ellipsoid in positive octant:

r varies from 0 to $1,$ θ varies from 0 to $\frac{\pi}{2},$ ϕ varies from 0 to $\frac{\pi}{2}$.

Example 1. $\iiint\limits_V \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}}$ where V is an annulus between two spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ where $a<b$

Solution: Put $x = r\,sin\theta \cos\phi,\ \ y = r\,sin\theta \sin\phi,\ \ z = rcos\theta$

$x^2+y^2+z^2 = r^2$ and $dxdydz = r^2 sin\theta\, dr\, d\theta\, d\phi$

$x^2+y^2+z^2 = a^2 \Longrightarrow r^2 = a^2 \Longrightarrow r = a$

$x^2+y^2+z^2 = b^2 \Longrightarrow r^2 = b^2 \Longrightarrow r = b$

Limits : r varies from a to b, θ varies from 0 to π, ϕ varies from 0 to 2π.

$$\therefore \mathrm{I} = \iiint\limits_V \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}} = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=a}^{r=b} \frac{r^2 sin\theta\, dr\, d\theta\, d\phi}{(r^2)^{3/2}}$$

$$= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=a}^{r=b} \frac{\sin\theta}{r} dr\, d\theta\, d\phi$$

$$= \left[\int_{\phi=0}^{\phi=2\pi} d\phi\right]\left[\int_{\theta=0}^{\theta=\pi} sin\theta\, d\theta\right]\left[\int_{r=a}^{r=b} \frac{1}{r} dr\right]$$

$$= (2\pi) \times (-cos\theta)_0^{\pi} \times [\log r]_{r=a}^{r=b}$$

$$= (2\pi) \times [-\cos\pi + \cos 0] \times [\log b - \log a]$$

$$= 4\pi \log\left(\frac{b}{a}\right)$$

Example 2. Evaluate $\iiint\limits_V \frac{dxdydz}{(1+x^2+y^2+z^2)^2}$ over entire positive octant of the space.

Solution: Put $x = r\,sin\theta\cos\phi,\;\; y = r\,sin\theta\,\sin\phi,\;\; z = rcos\theta$
$x^2+y^2+z^2 = r^2$ and $dxdydz = r^2 sin\theta\, dr\, d\theta\, d\phi$

Over the positive octant of the space : r varies from 0 to ∞.

θ varies from 0 to $\frac{\pi}{2}$.

ϕ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \mathrm{I} = \iiint\limits_V \frac{dxdydz}{(1+x^2+y^2+z^2)^2} = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=\infty} \frac{r^2 sin\theta\, dr\, d\theta\, d\phi}{(1+r^2)^2}$$

$$= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\int_{r=0}^{r=\infty} \frac{r^2}{(1+r^2)^2}\, dr \right] sin\theta\, d\theta\, d\phi = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} [I_1]\, sin\theta\, d\theta\, d\phi$$

where $I_1 = \int_{r=0}^{r=\infty} \frac{r^2}{(1+r^2)^2}\, dr$

Put $r^2 = tan^2 t$ in $I_1 \Longrightarrow r = tant \Longrightarrow dr = sec^2 t\, dt$ and t varies from 0 to $\frac{\pi}{2}$

$$\therefore I_1 = \int_{t=0}^{t=\frac{\pi}{2}} \frac{tan^2 t}{(1+tan^2 t)^2} sec^2 t\, dt = \int_{t=0}^{t=\frac{\pi}{2}} cos^2 t\, tan^2 t\, dt$$

$$= \int_{t=0}^{t=\frac{\pi}{2}} sin^2 t\, dt = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$$

$$\therefore I = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\frac{\pi}{4}\right] sin\theta \; d\theta \; d\phi$$

$$= \frac{\pi}{4}\left[\int_{\phi=0}^{\phi=\frac{\pi}{2}} d\phi\right]\left[\int_{\theta=0}^{\theta=\frac{\pi}{2}} sin\theta \; d\theta\right]$$

$$= \frac{\pi}{4}\left(\frac{\pi}{2}\right)(-cos\theta)_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}\left(\frac{\pi}{2}\right)\left(-cos\frac{\pi}{2} + cos0\right)$$

$$= \frac{\pi^2}{8}$$

Example 3. Evaluate $\iiint\limits_V \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}}$ over the volume of the sphere $x^2+y^2+z^2=a^2$ in the positive octant.

Solution: Put $x = r\, sin\theta \cos\phi,\ \ y = r\, sin\theta\, \sin\phi,\ z = rcos\theta$
$x^2+y^2+z^2=r^2$ and $dxdydz = r^2 sin\theta\, dr\, d\theta\, d\phi$
$x^2+y^2+z^2=a^2 \Rightarrow r^2=a^2 \Rightarrow r=a$

For positive octant of the sphere : r varies from 0 to a.

θ varies from 0 to $\frac{\pi}{2}$.

ϕ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \mathrm{I} = \iiint\limits_V \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}} = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=a} \frac{r^2 sin\theta\, dr\, d\theta\, d\phi}{\sqrt{a^2-r^2}}$$

$$= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\int_{r=0}^{r=a} \frac{r^2}{\sqrt{a^2 - r^2}} \, dr \right] sin\theta \, d\theta \, d\phi = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} [I_1] \, sin\theta \, d\theta \, d\phi$$

where $I_1 = \int_{r=0}^{r=a} \frac{r^2}{\sqrt{a^2 - r^2}} \, dr$

Put $r = asint$ in $I_1 \Longrightarrow dr = acost \, dt$ and t varies from 0 to $\frac{\pi}{2}$

$$\therefore I_1 = \int_{t=0}^{t=\frac{\pi}{2}} \frac{a^2 sin^2 t}{\sqrt{a^2 - a^2 sin^2 t}} acost \, dt = a^2 \int_{t=0}^{t=\frac{\pi}{2}} sin^2 t \, dt = a^2 \frac{1}{2} \frac{\pi}{2} = \frac{\pi a^2}{4}$$

$$\therefore I = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\frac{\pi a^2}{4} \right] sin\theta \, d\theta \, d\phi$$

$$= \frac{\pi a^2}{4}\left[\int_{\phi=0}^{\phi=\frac{\pi}{2}} d\phi\right]\left[\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta \, d\theta\right]$$

$$= \frac{\pi a^2}{4}\left(\frac{\pi}{2}\right)(-\cos\theta)_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^2}{4}\left(\frac{\pi}{2}\right)\left(-\cos\frac{\pi}{2} + \cos 0\right)$$

$$= \frac{\pi^2 a^2}{8}$$

H. W. Evaluate $\iiint_V \frac{dxdydz}{\sqrt{1 - x^2 - y^2 - z^2}}$ over the volume of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant.

Example 4. Evaluate $\iiint\limits_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\; dxdydz$ through out the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution: Put $x = ar\,sin\theta \cos\phi,\;\; y = br\,sin\theta\,\sin\phi,\; z = crcos\theta$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=r^2$ and $dxdydz = abc\,r^2 sin\theta\,dr\,d\theta\,d\phi$

For complete ellipsoid: r varies from 0 to 1,

θ varies from 0 to π and ϕ varies from 0 to 2π.

$$\mathrm{I} = \iiint\limits_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\; dxdydz = \int\limits_{\phi=0}^{\phi=2\pi}\int\limits_{\theta=0}^{\theta=\pi}\int\limits_{r=0}^{r=1}\sqrt{1-r^2}\; abc\,r^2 sin\theta\,dr\,d\theta\,d\phi$$

$$= abc \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \left[\int_{r=0}^{r=1} r^2\sqrt{1-r^2}\ dr \right] sin\theta\ d\theta\ d\phi = abc \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} [I_1]\ sin\theta\ d\theta\ d\phi$$

where $I_1 = \int_{r=0}^{r=1} r^2\sqrt{1-r^2}\ dr$

Put $r = sint$ in $I_1 \Longrightarrow dr = cost\ dt$ and t varies from 0 to $\frac{\pi}{2}$

$$\therefore I_1 = \int_{t=0}^{t=\frac{\pi}{2}} sin^2t\,\sqrt{1-sin^2t}\ cost\ dt = \int_{t=0}^{t=\frac{\pi}{2}} sin^2t\,cos^2tdt = \frac{1\times 1}{4\times 2}\frac{\pi}{2} = \frac{\pi}{16}$$

$$\therefore I = abc \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \left[\frac{\pi}{16}\right] sin\theta\ d\theta\ d\phi$$

$$= \frac{\pi abc}{16}\left[\int_{\phi=0}^{\phi=2\pi} d\phi\right]\left[\int_{\theta=0}^{\theta=\pi} sin\theta\, d\theta\right]$$

$$= \frac{\pi abc}{16}(2\pi)(-cos\theta)_0^{\pi}$$

$$= \frac{\pi abc}{16}(2\pi)(2)$$

$$= \frac{\pi^2 abc}{4}$$

Example 5. $\iiint_V \frac{z^2 dxdydz}{x^2+y^2+z^2}$ over the volume of the sphere $x^2+y^2+z^2=2$.

Solution: Put $x = r\, sin\theta \cos\phi,\quad y = r\, sin\theta \sin\phi,\quad z = rcos\theta$

$x^2+y^2+z^2 = r^2$ and $dxdydz = r^2 sin\theta\, dr\, d\theta\, d\phi$

$x^2+y^2+z^2 = 2 \Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$

Limits : r varies from 0 to $\sqrt{2}$.

θ varies from 0 to π.

ϕ varies from 0 to 2π.

$$\therefore \mathrm{I} = \iiint_V \frac{z^2 dxdydz}{x^2+y^2+z^2} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=\sqrt{2}} \frac{r^2 cos^2\theta\, r^2 sin\theta\, dr\, d\theta\, d\phi}{r^2}$$

$$= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \left[\int_{r=0}^{r=\sqrt{2}} r^2 \, dr \right] sin\theta cos^2\theta \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \left[\frac{r^3}{3} \right]_{r=0}^{r=\sqrt{2}} \; sin\theta cos^2\theta \, d\theta \, d\phi$$

$$= \frac{2\sqrt{2}}{3} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \; sin\theta cos^2\theta \, d\theta \, d\phi$$

$$= \frac{2\sqrt{2}}{3} \left[\int_{\phi=0}^{\phi=2\pi} d\phi \right] \left[\int_{\theta=0}^{\theta=\pi} \; sin\theta cos^2\theta \, d\theta \right]$$

$$= \frac{2\sqrt{2}}{3} (2\pi) \left(2 \times \frac{1}{3} \right) \; = \frac{8\sqrt{2}\pi}{9}$$

1) Evaluate $\iiint z(x^2+y^2)\,dxdydz$ over the volume of the cylinder $x^2+y^2=1$ intercepted by the planes $z=2$ and $z=3$.

2) Evaluate $\iiint\limits_V \sqrt{x^2+y^2}\,dxdydz$ where V is bounded by the surface $x^2+y^2=z^2, z\geq 0$ and the plane $z=1$.

Type-III : Triple integration by Dirichlet's Theorem

Dirichlet's Theorem for three variables

If V is the region defined by $x \geq 0, y \geq 0, z \geq 0$ and $x + y + z \leq 1$ then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dxdydz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

$$e.g. \iiint_V x^3 y^2 z^5\, dxdydz = \iiint_V x^{4-1} y^{3-1} z^{6-1}\, dxdydz = \frac{\Gamma(4)\,\Gamma(3)\,\Gamma(6)}{\Gamma(4+3+6+1)}$$

$$= \frac{3!\,2!\,5!}{13!}$$

1. Evaluate $\iiint xyz\, dxdydz$ over a tetrahedron formed by $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

Solution: By Dirichelt's theorem,

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}\, dxdydz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

$$\therefore I = \iiint xyz\, dxdydz = \iiint x^{2-1}y^{2-1}z^{2-1}\, dxdydz$$

$$= \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)} = \frac{1!\,1!\,1!}{\Gamma(7)} = \frac{1}{6!} = \frac{1}{720}$$

2. Evaluate $\iiint x^2yz\,dxdydz$ over a tetrahedron formed by $x=0, y=0, z=0$ and $x+y+z=1$.

Solution: By Dirichelt's theorem,

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}\,dxdydz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

$$\therefore I = \iiint x^2yz\,dxdydz = \iiint x^{3-1}y^{2-1}z^{2-1}\,dxdydz$$

$$= \frac{\Gamma(3)\,\Gamma(2)\,\Gamma(2)}{\Gamma(3+2+2+1)} = \frac{2!\,1!\,1!}{\Gamma(8)} = \frac{2}{7!} = \frac{2}{5040} = \frac{1}{2520}$$

3. Evaluate $\iiint xyz \, dxdydz$ over a tetrahedron formed by

$x = 0, y = 0, z = 0$ and $x + y + z = a$.

Solution: $x + y + z = a \Longrightarrow \frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1$

Put $\frac{x}{a} = u, \frac{y}{a} = v, \frac{z}{a} = w \Longrightarrow dx = adu, dy = adv, dz = adw$

$$\therefore \mathrm{I} = \iiint xyz \, dxdydz = \iiint au \, av \, aw \, adu \, adv \, adw = a^6 \iiint uvw \, dudvdw$$

As $u + v + w = 1$, by Dirichelt's theorem,

$$\iiint_V u^{l-1} v^{m-1} w^{n-1} \, du \, dv \, d\, w = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

$$\therefore I = a^6 \iiint uvw \, dudvdw = a^6 \iiint u^{2-1} v^{2-1} w^{2-1} \, dudvdw$$

$$= a^6 \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)}$$

$$= a^6 \frac{1!\,1!\,1!}{\Gamma(7)} = \frac{a^6}{6!} = \frac{a^6}{720}$$

4. Evaluate $\iiint (x+y+z)\, dxdydz$ over a tetrahedron formed by $x=0, y=0, z=0$ and $x+y+z=1$.

Solution: By Dirichelt's theorem,

$$\iiint\limits_V x^{l-1}y^{m-1}z^{n-1}\, dxdydz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

$$\therefore \mathrm{I} = \iiint (x+y+z)\, dxdydz = \iiint xdxdydz + \iiint ydxdydz + \iiint zdxdydz$$

4. Evaluate $\iiint (x+y+z)\, dxdydz$ over a tetrahedron formed by $x=0, y=0, z=0$ and $x+y+z=1$.

Solution: By Dirichelt's theorem,

$$\iiint\limits_V x^{l-1}y^{m-1}z^{n-1}\, dxdydz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

$$\therefore \mathrm{I} = \iiint (x+y+z)\, dxdydz = \iiint xdxdydz + \iiint ydxdydz + \iiint zdxdydz$$

$$= \iiint x^{2-1}y^{1-1}z^{1-1}\, dxdydz + \iiint x^{1-1}y^{2-1}z^{1-1}\, dxdydz + \iiint x^{1-1}y^{1-1}z^{2-1}\, dxdydz$$

$$= \frac{\Gamma(2)\,\Gamma(1)\,\Gamma(1)}{\Gamma(2+1+1+1)} + \frac{\Gamma(1)\,\Gamma(2)\,\Gamma(1)}{\Gamma(1+1+n+1)} + \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(2)}{\Gamma(l+m+n+1)}$$

$$= \frac{3}{\lceil(5)} = \frac{3}{4!} = \frac{1}{8}$$

5 . Evaluate $\iiint xyz\,dxdydz$ over first octant of the sphere $x^2+y^2+z^2=a^2$.

Solution: $x^2+y^2+z^2=a^2 \Rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}+\dfrac{z^2}{a^2}=1$

Put $\dfrac{x^2}{a^2}=u, \dfrac{y^2}{a^2}=v, \dfrac{z^2}{a^2}=w \Rightarrow x=a\sqrt{u}, y=a\sqrt{v}, z=a\sqrt{w}$

$$\Rightarrow dx=\frac{a}{2\sqrt{u}}du, \qquad dy=\frac{a}{2\sqrt{v}}dv, \qquad dz=\frac{a}{2\sqrt{w}}dw$$

$$\therefore \mathrm{I}=\iiint xyz\,dxdydz=\iiint a\sqrt{u}\,a\sqrt{v}\,a\sqrt{w}\,\frac{a}{2\sqrt{u}}du\,\frac{a}{2\sqrt{v}}dv\,\frac{a}{2\sqrt{w}}dw$$

$$I = \frac{a^6}{8}\iiint \, dudvdw$$

As $u + v + w = 1$, by Dirichelt's theorem,

$$\therefore \mathrm{I} = \frac{a^6}{8}\iiint \, dudvdw = \frac{a^6}{8}\iiint u^{1-1}v^{1-1}w^{1-1}\, dudvdw$$

$$= \frac{a^6}{8}\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{a^6}{8}\frac{1}{\Gamma(4)}$$

$$= \frac{a^6}{8}\frac{1}{3!} = \frac{a^6}{48}$$

5. Evaluate $\iiint (x^2y^2 + y^2z^2 + z^2x^2)dxdydz$ through out the volume of the sphere $x^2 + y^2 + z^2 = a^2$

Solution: $x^2 + y^2 + z^2 = a^2 \Longrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} = 1$

Put $\frac{x^2}{a^2} = u, \frac{y^2}{a^2} = v, \frac{z^2}{a^2} = w \Longrightarrow x = a\sqrt{u}, y = a\sqrt{v}, z = a\sqrt{w}$

$$\Longrightarrow dx = \frac{a}{2\sqrt{u}}du, \qquad dy = \frac{a}{2\sqrt{v}}dv, \qquad dz = \frac{a}{2\sqrt{w}}dw$$

$$\mathrm{I} = \iiint (x^2y^2 + y^2z^2 + z^2x^2)dxdydz = 8\iiint_V (x^2y^2 + y^2z^2 + z^2x^2)dxdydz$$

where V is the positive octant of the sphere

$$I = 8\left[\iiint x^2y^2dxdydz + \iiint y^2z^2dxdydz + \iiint z^2x^2dxdydz\right]$$

$$= 8\left[\iiint a^2u\, a^2v \frac{a}{2\sqrt{u}}du \frac{a}{2\sqrt{v}}dv \frac{a}{2\sqrt{w}}dw + \iiint a^2v\, a^2w \frac{a}{2\sqrt{u}}du \frac{a}{2\sqrt{v}}dv \frac{a}{2\sqrt{w}}dw\right.$$

$$= a^7 \left[\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}+\frac{1}{2}+1\right)} + \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}+\frac{3}{2}+1\right)} + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}+\frac{3}{2}+1\right)} \right]$$

$$= a^7 \left[3 \times \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)} \right]$$

$$= 3a^7 \left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \times \Gamma\left(\frac{1}{2}\right)}{\frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \right]$$

$$= \frac{4a^7}{35}\left[\sqrt{\pi} \times \sqrt{\pi}\right] = \frac{4\pi a^7}{35} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Thank You