Unit- II : Partial Differentiation

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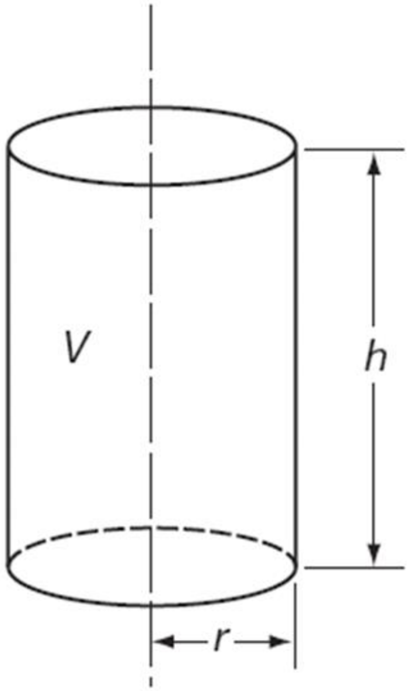
Introduction

The volume V of a cylinder of radius r and height h is given by:

$$V = \pi r^2 h$$



If r is kept constant and h increases then V increases.
We can find the rate of change of V with respect to h by differentiating with respect to h, keeping r constant:

$$\left[\frac{dV}{dh}\right]_{r \text{ constant}} = \pi r^2 \quad \text{we write this as} \quad \frac{\partial V}{\partial h} = \pi r^2$$

This is called the *first partial derivative* of V with respect to h.

Similarly, if *h* is kept constant and *r* increases then *V* increases.

We can then find the rate of change of *V* by differentiating with respect to *r* keeping *h* constant:

$$\left[\frac{dV}{dr}\right]_{h\ \text{constant}} = 2\pi rh \quad \text{we write this as} \quad \frac{\partial V}{\partial r} = 2\pi rh$$

This is called the *first partial derivative* of *V* with respect to *r.*

How to find partial derivatives?

If $z = f(x, y)$ is a function of two real variables x and y, then partial derivative of z w.r.to x is denoted by

$$\frac{\partial z}{\partial x} \text{ or } z_x \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

and **is the ordinary derivative of z w.r.to x by keeping y constant/fixed**.

Similarly partial derivative of z w.r.to y is denoted by

$$\frac{\partial z}{\partial y} \text{ or } z_y \text{ or } \frac{\partial f}{\partial y} \text{ or } f_y$$

and **is the ordinary derivative of z w.r.to y by keeping x constant/fixed.**

Note : (1) $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called as first order partial derivatives of z.

(2) $$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

(3) If $f(x, y, z)$ is a function of three variables x, y and z then partial derivative of f w.r.to any single variable is obtained by treating remaining all variables constant.

(4) All the usual rules for differentiating sums, differences, products, quotients and functions of a function obeys in partial derivatives.

1. Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if:

$$f(x,y) = x^2 + 3xy + y - 1$$

Solution: To find $\frac{\partial f}{\partial x}$, treat y as a constant and differentiate with respect to x

$$\therefore \frac{\partial f}{\partial x} = 2x + 3y$$

To find $\frac{\partial f}{\partial y}$, treat x as a constant and differentiate with respect to y

$$\therefore \frac{\partial f}{\partial y} = 3x + 1$$

$$\Longrightarrow \left(\frac{\partial f}{\partial x}\right)_{(4,-5)} = 2(4) + 3(-5) = -7$$

$$\left(\frac{\partial f}{\partial y}\right)_{(4,-5)} = 3(4) + 1 = 13$$

2. Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x, y) = y\sin(xy)$.

Solution: $f(x, y) = y\sin(xy) \dots\dots\dots (1)$

Differentiate (1) w.r.to x by treating y as a constant

$$\therefore \frac{\partial f}{\partial x} = y\cos(xy) \times y = y^2\cos(xy)$$

Differentiate (1) w.r.to y by treating x as a constant

$$\therefore \frac{\partial f}{\partial y} = \sin(xy) + y\cos(xy) \times x$$

$$= \sin(xy) + xy\cos(xy)$$

3. Find f_x and f_y if $f(x,y) = \dfrac{2y}{y+\cos x}$

Solution: $f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{2y}{y+\cos x}\right)$

$$= \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - (2y)\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(0) - (2y)(0-\sin x)}{(y+\cos x)^2}$$

$$\frac{\partial f}{\partial x} = \frac{0 - (2y)(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right)$$

$$= \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - (2y)\frac{\partial}{\partial y}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(2) - (2y)(1 + 0)}{(y + \cos x)^2}$$

$$= \frac{2(y + \cos x) - 2y}{(y + \cos x)^2}$$

$$= \frac{2y + 2\cos x - 2y}{(y + \cos x)^2}$$

$$= \frac{2\cos x}{(y + \cos x)^2}$$

Second-Order Partial Derivatives

By differentiating a function $z = f(x, y)$ twice,we get its second-order derivatives.

These derivatives are usually denoted by:

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx} \text{ or } \frac{\partial^2 z}{\partial x^2} \text{ or } z_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy} \text{ or } \frac{\partial^2 z}{\partial y^2} \text{ or } z_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{xy} \text{ or } \frac{\partial^2 z}{\partial x \partial y} \text{ or } z_{xy}$$

$$\frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{yx} \text{ or } \frac{\partial^2 z}{\partial y \partial x} \text{ or } z_{yx}$$

The defining equations are:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$\Rightarrow$ Differentiate first with respect to y, then with respect to x

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\Rightarrow$ Differentiate first with respect to x, then with respect to y

Note:

$$
\begin{tikzcd}[column sep=large]
 & & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2} \\
 & \frac{\partial f}{\partial x} \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[r, "\frac{\partial}{\partial y}"'] & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x} \\
f \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & & \\
 & \frac{\partial f}{\partial y} \arrow[r, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y} \\
 & & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}
\end{tikzcd}
$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$$

When both are continuous

1. If $f(x,y) = x\cos y + y\,e^x$, then find

$$\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial y^2},\ \frac{\partial^2 f}{\partial x \partial y},\ \frac{\partial^2 f}{\partial y \partial x}.$$

Solution: $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(x\cos y + y\,e^x) = \cos y + y\,e^x$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + y\,e^x) = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos y + y\,e^x) = 0 + y\,e^x = y\,e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-x\sin y + e^x) = -x\cos y + 0 = -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-x\sin y + e^x) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\cos y + y\, e^x) = -\sin y + e^x$$

2. If $u = \tan^{-1}\left(\frac{y}{x}\right)$ then find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$

Solution: $\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right)$$

$$= -\frac{(x^2+y^2)\frac{\partial}{\partial x}(y) - y\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial y}(x) - x\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right)$$

$$= -\frac{(x^2+y^2)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= -\frac{(x^2+y^2)(1) - y(2y)}{(x^2 + y^2)^2}$$

$$= -\frac{(x^2 + y^2 - 2y^2)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

3. If $u=\log(x^2+y^2)$ verify $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

Solution: $\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2}$

$$\frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right)$$

$$=\frac{(x^2+y^2)\frac{\partial}{\partial x}(2y)-2y\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(0) - 2y(2x)}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial y}(2x) - 2x\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(0) - 2x(2y)}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2}$$

4. If $u = \log(x^3 + y^3 - x^2y - y^2x)$ then show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{4}{(x+y)^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$$

Solution: $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) \dots\dots\dots (1)$$

$$u = \log(x^3 + y^3 - x^2y - y^2x)$$
$$= \log(x^3 - x^2y + y^3 - y^2x)$$

$$u = \log[x^2(x-y) - y^2(x-y)]$$
$$= \log(x^2 - y^2)(x-y)$$
$$= \log(x+y)(x-y)(x-y) = \log(x+y)(x-y)^2$$
$$= \log(x+y) + \log(x-y)^2$$
$$= \log(x+y) + 2\log(x-y)$$

Differentiating u w.r.to x we get,

$$\frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2}{x-y}$$

Differentiating u w.r.to y we get,

$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{1}{x+y}+\frac{2}{x-y}+\frac{1}{x+y}-\frac{2}{x-y}=\frac{2}{x+y}$$

Putting value of $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}$ in (1) we get,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{2}{x+y}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{2}{x+y}\right)+\frac{\partial}{\partial y}\left(\frac{2}{x+y}\right)$$

$$=-\frac{2}{(x+y)^2}-\frac{2}{(x+y)^2}=-\frac{4}{(x+y)^2}$$

5. Find the value of n if $u = r^n(3\cos^2\theta - 1)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{sin\theta}\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial u}{\partial\theta}\right) = 0.$$

Solution: $u = r^n(3\cos^2\theta - 1) \ldots\ldots\ldots (1)$

Diff.(1) w.r.to r, we get

$$\frac{\partial u}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$$

$$\therefore\; r^2\frac{\partial u}{\partial r} = r^2nr^{n-1}(3\cos^2\theta - 1) = nr^{n+1}(3\cos^2\theta - 1)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(nr^{n+1}(3\cos^2\theta - 1)\right)$$

$$= n(n+1)r^n(3\cos^2\theta - 1) \ldots\ldots\ldots (2)$$

Diff.(1) w.r.to θ, we get

$$\frac{\partial u}{\partial \theta} = r^n(-6\cos\theta\sin\theta) = -6\, r^n \sin\theta\cos\theta$$

$$\therefore \sin\theta\frac{\partial u}{\partial \theta} = -6r^n \sin^2\theta\cos\theta$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial\theta}(-6r^n\sin^2\theta\cos\theta)$$

$$= -6r^n\left[\cos\theta\frac{\partial}{\partial\theta}(\sin^2\theta) + \sin^2\theta\frac{\partial}{\partial\theta}(\cos\theta)\right]$$

$$= -6r^n[\cos\theta\,(2\sin\theta\,\cos\theta) + \sin^2\theta(-\sin\theta)]$$

$$= -6r^n[2\sin\theta\cos^2\theta - \sin^3\theta]$$

$$= -6r^n\sin\theta\,[2\cos^2\theta - \sin^2\theta]$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) = -6r^n \sin\theta\,[2\cos^2\theta - \sin^2\theta]$$

$$= -6r^n \sin\theta\,[2\cos^2\theta - (1 - \cos^2\theta)]$$

$$= -6r^n \sin\theta\,[3\cos^2\theta - 1]$$

$$\therefore \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) = \frac{1}{\sin\theta}\left[-6r^n \sin\theta\,[3\cos^2\theta - 1]\right]$$

$$= -6r^n[3\cos^2\theta - 1] \ldots\ldots\ldots (3)$$

Now $u = r^n(3\cos^2\theta - 1)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{sin\theta}\ \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial u}{\partial\theta}\right) = 0.$$

From eq.(2) and (3),

$$\therefore n(n+1)r^n(3\cos^2\theta - 1) - 6r^n[3\cos^2\theta - 1] = 0$$

$$n(n+1)r^n(3\cos^2\theta - 1) = 6r^n[3\cos^2\theta - 1]$$

$$\Longrightarrow n(n+1) = 6$$

$$\Longrightarrow n^2 + n - 6 = 0$$

$$\Longrightarrow (n+3)(n-2) = 0 \Longrightarrow n = -3, 2$$

6. Find the value of n if $\theta = t^n e^{\frac{-r^2}{4t}}$ satisfies the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}.$$

Solution: $\theta = t^n e^{\frac{-r^2}{4t}}$

$$\log\theta = \log\left(t^n e^{\frac{-r^2}{4t}}\right) = \log t^n + \log e^{\frac{-r^2}{4t}}$$

$$\log\theta = n\log t - \frac{r^2}{4t} \dots\dots\dots (1)$$

Diff.(1) w.r.to r, we get

$$\frac{1}{\theta}\frac{\partial \theta}{\partial r} = 0 - \frac{2r}{4t} = -\frac{r}{2t}$$

$$\therefore \frac{\partial \theta}{\partial r} = -\frac{r\theta}{2t}$$

$$\Longrightarrow r^2 \frac{\partial \theta}{\partial r} = -\frac{r^3\theta}{2t}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{r^3\theta}{2t}\right) = -\frac{1}{2t}\left[r^3 \frac{\partial \theta}{\partial r} + \theta \frac{\partial}{\partial r}(r^3)\right]$$

$$= -\frac{1}{2t}\left[r^3\left(-\frac{r\theta}{2t}\right) + 3r^2\theta\right]$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{r^4\theta}{4t^2}-\frac{3r^2\theta}{2t}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{r^4\theta}{4t^2}-\frac{3r^2\theta}{2t}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{1}{r^2}\left[\frac{r^4\theta}{4t^2}-\frac{3r^2\theta}{2t}\right]$$

$$=\frac{r^2\theta}{4t^2}-\frac{3\theta}{2t}\dots\dots\dots(2)$$

Now Diff.(1) w.r.to t,

$$\therefore \ \frac{1}{\theta}\frac{\partial\theta}{\partial t} = \frac{n}{t} + \frac{r^2}{4t^2}$$

$$\Longrightarrow \ \frac{\partial\theta}{\partial t} = \frac{n\theta}{t} + \frac{r^2\theta}{4t^2} \ldots\ldots\ldots (3)$$

$\theta = t^n e^{\frac{-r^2}{4t}}$ satisfies the equation $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$.

From eq.(2) and (3),

$$\frac{r^2\theta}{4t^2} - \frac{3\theta}{2t} = \frac{n\theta}{t} + \frac{r^2\theta}{4t^2} \Longrightarrow \ -\frac{3\theta}{2t} = \frac{n\theta}{t} \Longrightarrow n = -\frac{3}{2}$$

7. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Solution: $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \dots\dots\dots (1)$$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}$$

$$x^3+y^3+z^3-3xyz=(x^2+y^2+z^2-xy-yz-xz)(x+y+z)$$

$$\frac{(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}=\frac{1}{x+y+z}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$$

Putting value of $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$ in (1) we get,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$=-\frac{9}{(x+y+z)^2}$$

Exercise-1

1. If $u = x^3y - xy^3$ then find the value of $\dfrac{1}{\frac{\partial u}{\partial x}} + \dfrac{1}{\frac{\partial u}{\partial y}}$ at $(1, 2)$. **Ans**: $-\dfrac{13}{22}$

2. If $u = \log(x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)$ then show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

3. If $u = \log(e^x + e^y)$ then show that $\dfrac{\partial^2 u}{\partial x^2} \cdot \dfrac{\partial^2 u}{\partial y^2} = \left(\dfrac{\partial^2 u}{\partial x \partial y}\right)^2.$

4. If $z^3 - zx - y = 0$ then find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ and show that $\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{(3z^2+x)}{(3z^2-x)^3}.$

5. If $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ then show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

6. If $z = \tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$ then show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

7. If $z = \tan(y + ax) - (y - ax)^{\frac{3}{2}}$ then show that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}.$

8. Find the value of n if $u = Ae^{-gx}\sin(nt - gx)$ satisfies the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ where g and A are constants.

Variables to be treated as Constant

Consider the equations $x = r\cos\theta$ and $y = r\sin\theta$.

To find $\dfrac{\partial r}{\partial x}$ we need a relation between r and x.

$$x = r\cos\theta \Longrightarrow r = x\sec\theta$$

$$\Longrightarrow \frac{\partial r}{\partial x} = \sec\theta \;\ldots\ldots\ldots (1)$$

Now $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$$\therefore\; r^2 = x^2 + y^2 \;\ldots\ldots\ldots (2)$$

Differentiating (2) w.r.to x keeping y constant we get,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\Longrightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \;\ldots\ldots\ldots (3)$$

From (1), $\dfrac{\partial r}{\partial x} = \sec\theta$ and from (3), $\dfrac{\partial r}{\partial x} = \cos\theta$

These two values of $\frac{\partial r}{\partial x}$ make confusion. To avoid the confusion we use the following notations:

Notations:

1. $\left(\dfrac{\partial r}{\partial x}\right)_\theta$ means the partial derivative of r w. r. to x keeping θ constant in a relation expressing r as a function of x and θ.

From $r = x\sec\theta$, $\left(\dfrac{\partial r}{\partial x}\right)_\theta = \sec\theta$

2. $\left(\dfrac{\partial r}{\partial x}\right)_y$ means the partial derivative of r w. r. to x keeping y constant in a relation expressing r as a function of x and y.

From $r^2 = x^2 + y^2$, $\left(\dfrac{\partial r}{\partial x}\right)_y = \dfrac{x}{r} = \cos\theta$

1. If $x = r\cos\theta$ & $y = r\sin\theta$ then find $\left(\frac{\partial r}{\partial x}\right)_y, \left(\frac{\partial r}{\partial y}\right)_x, \left(\frac{\partial \theta}{\partial x}\right)_y$ and $\left(\frac{\partial \theta}{\partial y}\right)_x$

Solution:

To find $\left(\frac{\partial r}{\partial x}\right)_y$ & $\left(\frac{\partial r}{\partial y}\right)_x$, express r in terms of x and y.

$$\therefore\; x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \Rightarrow r^2 = x^2 + y^2$$

$$\Rightarrow\; 2r\left(\frac{\partial r}{\partial x}\right)_y = 2x \Rightarrow \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \cos\theta$$

Also $2r\left(\frac{\partial r}{\partial y}\right)_x = 2y \Rightarrow \left(\frac{\partial r}{\partial y}\right)_x = \frac{y}{r} = \sin\theta$

To find $\left(\frac{\partial\theta}{\partial x}\right)_y$ and $\left(\frac{\partial\theta}{\partial y}\right)_x$, express θ in terms of x and y.

$x = r\cos\theta$ and $y = r\sin\theta$

$$\Longrightarrow \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

$$\Longrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\left(\frac{\partial\theta}{\partial x}\right)_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\left(\frac{\partial\theta}{\partial y}\right)_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

2. If $x^2 = au + bv, y^2 = au - bv$ then show that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

Solution:

To find $\left(\frac{\partial u}{\partial x}\right)_y$, express u in terms of x and y.

$$x^2 = au + bv \;\;\&\;\; y^2 = au - bv$$

$$\Longrightarrow x^2 + y^2 = au + bv + au - bv = 2au$$

$$\Longrightarrow u = \frac{x^2 + y^2}{2a}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = \frac{2x}{2a} = \frac{x}{a} \ldots\ldots\ldots(1)$$

To find $\left(\frac{\partial x}{\partial u}\right)_v$, express x in terms of u and v.

$$x^2 = au + bv \Longrightarrow x = \sqrt{au + bv}$$

$$\therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2\sqrt{au + bv}} = \frac{a}{2x} \dots \dots \dots (2)$$

To find $\left(\frac{\partial v}{\partial y}\right)_x$, express v in terms of x and y.

$$x^2 = au + bv \;\; \& \;\; y^2 = au - bv$$

$$\Longrightarrow x^2 - y^2 = au + bv - (au - bv) = 2bv$$

$$\Longrightarrow v = \frac{x^2 - y^2}{2b}$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_x = \frac{-2y}{2b} = \frac{-y}{b} \dots \dots \dots (3)$$

To find $\left(\frac{\partial y}{\partial v}\right)_u$, express y in terms of u and v.

$y^2 = au - bv \Longrightarrow y = \sqrt{au - bv}$

$$\therefore \left(\frac{\partial y}{\partial v}\right)_u = \frac{-b}{2\sqrt{au - bv}} = \frac{-b}{2y} \dots\dots\dots (4)$$

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a} \times \frac{a}{2x} = \frac{1}{2}$$

$$\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \frac{-y}{b} \times \frac{-b}{2y} = \frac{1}{2}$$

3. If $x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right)$ and $y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$ then show that $\left(\frac{\partial x}{\partial r}\right)_{\theta} = \left(\frac{\partial r}{\partial x}\right)_{y}$.

Solution: To find $\left(\frac{\partial x}{\partial r}\right)_{\theta}$, express x in terms of r and θ.

$$x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right) \Longrightarrow \left(\frac{\partial x}{\partial r}\right)_{\theta} = \frac{e^{\theta} + e^{-\theta}}{2} \dots\dots\dots (1)$$

To find $\left(\frac{\partial r}{\partial x}\right)_{y}$, express r in terms of x and y.

$$x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right) \text{ and } y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$$

$$\Longrightarrow x^2 - y^2 = \frac{r^2}{4}\left(e^{\theta} + e^{-\theta}\right)^2 - \frac{r^2}{4}\left(e^{\theta} - e^{-\theta}\right)^2$$

$$\Rightarrow x^2 - y^2 = \frac{r^2}{4}\left(e^{2\theta} + e^{-2\theta} + 2e^{\theta}e^{-\theta}\right) - \frac{r^2}{4}(e^{2\theta} + e^{-2\theta} - 2e^{\theta}e^{-\theta})$$

$$\Rightarrow x^2 - y^2 = \frac{r^2}{4}\left(e^{2\theta} + e^{-2\theta} + 2 - e^{2\theta} - e^{-2\theta} + 2\right)$$

$$\Rightarrow x^2 - y^2 = \frac{r^2}{4}(4) = r^2$$

$$\Rightarrow r = \sqrt{x^2 - y^2}$$

$$\therefore \left(\frac{\partial r}{\partial x}\right)_y = \frac{2x}{2\sqrt{x^2 - y^2}} = \frac{x}{\sqrt{x^2 - y^2}} = \frac{x}{r} = \frac{e^{\theta} + e^{-\theta}}{2} \ldots\ldots\ldots (2)$$

From (1) & (2), $\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y$

4. If $ux + vy = 0$ and $\frac{u}{x} + \frac{v}{y} = 1$ show that $\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = 0.$

Solution:

To find $\left(\frac{\partial x}{\partial u}\right)_v$, express x in terms of u and v.

$$ux + vy = 0 \Longrightarrow y = -\frac{ux}{v} \dots\dots\dots (1)$$

$$\frac{u}{x} + \frac{v}{y} = 1 \Longrightarrow \frac{v}{y} = 1 - \frac{u}{x} = \frac{x-u}{x}$$

$$\frac{y}{v} = \frac{x}{x-u} \Longrightarrow y = \frac{vx}{x-u} \dots\dots\dots (2)$$

From (1) and (2), $-\frac{ux}{v} = \frac{vx}{x-u}$

$$-\frac{u}{v} = \frac{v}{x-u} \implies -u(x-u) = v^2$$

$$\implies -ux + u^2 = v^2 \implies x = \frac{u^2 - v^2}{u}$$

$$y = -\frac{ux}{v} = -\frac{u}{v}\left(\frac{u^2 - v^2}{u}\right) = \frac{v^2 - u^2}{v}$$

$$x = \frac{u^2 - v^2}{u} \implies \left(\frac{\partial x}{\partial u}\right)_v = \frac{u(2u) - (u^2 - v^2)(1)}{u^2}$$

$$\implies \left(\frac{\partial x}{\partial u}\right)_v = \frac{2u^2 - u^2 + v^2}{u^2} = \frac{u^2 + v^2}{u^2}$$

$$y = \frac{v^2 - u^2}{v} \Rightarrow \left(\frac{\partial y}{\partial v}\right)_u = \frac{v(2v) - (v^2 - u^2)(1)}{v^2}$$

$$\Rightarrow \left(\frac{\partial y}{\partial v}\right)_u = \frac{2v^2 - v^2 + u^2}{v^2} = \frac{u^2 + v^2}{v^2}$$

$$\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = \frac{u}{x}\left(\frac{u^2 + v^2}{u^2}\right) + \frac{v}{y}\left(\frac{u^2 + v^2}{v^2}\right)$$

$$= \frac{u^2 + v^2}{ux} + \frac{u^2 + v^2}{vy} = (u^2 + v^2)\left(\frac{1}{ux} + \frac{1}{vy}\right)$$

$$= (u^2 + v^2)\left(\frac{vy + ux}{uxvy}\right)$$

$$= (u^2 + v^2)\left(\frac{0}{uxvy}\right) = 0$$

Exercise 2

1) If $u.x + v.y = 0,\ \dfrac{u}{x} + \dfrac{v}{y} = 1,$ prove that $\left(\dfrac{\partial u}{\partial x}\right)_y + \left(\dfrac{\partial v}{\partial y}\right)_x = \dfrac{x^2 + y^2}{y^2 - x^2}$

2) If $x = r\cos\theta, y = r\sin\theta,$ Prove that a) $\left(\dfrac{\partial r}{\partial x}\right)_y = \left(\dfrac{\partial x}{\partial y}\right)_\theta$, b) $\dfrac{1}{r}\left(\dfrac{\partial x}{\partial \theta}\right)_r = r\left(\dfrac{\partial \theta}{\partial x}\right)_y$, c) $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0$

.

3) If $x = \dfrac{\cos\theta}{u},\ y = \dfrac{\sin\theta}{u},$ evaluate $\left(\dfrac{\partial x}{\partial u}\right)_\theta \left(\dfrac{\partial u}{\partial x}\right)_y + \left(\dfrac{\partial y}{\partial u}\right)_\theta \left(\dfrac{\partial u}{\partial y}\right)_x$.

4) If $x = u\tan v,\ y = u\sec v$, prove that $\left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial v}{\partial x}\right)_y = \left(\dfrac{\partial u}{\partial y}\right)_x \left(\dfrac{\partial v}{\partial y}\right)_x$.

Euler's Theorem on Homogenous Functions

Homogenous function of degree *n* means?

A function $f(x, y)$ of two variables x and y is said to homogeneous function of degree n if

$$f(x, y) = x^n \varphi\left(\frac{y}{x}\right) \quad \text{or} \quad f(x, y) = y^n \psi\left(\frac{x}{y}\right)$$

Alternately, function $f(x, y)$ of two variables x and y is said to homogeneous function of degree n if

$$f(tx, ty) = t^n f(x, y) \text{ where } t \text{ is a parameter.}$$

1) $f(x,y) = x^2 + y^2$ is a homogeneous function of degree 2

$$f(tx,ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 f(x,y)$$

2) $f(x,y) = \frac{x^2y^3}{x-y}$ is a homogeneous function of degree 4

$$f(tx,ty) = \frac{(tx)^2(ty)^3}{tx - ty} = \frac{t^5x^2y^3}{t(x-y)} = t^4 f(x,y)$$

3) $f(x,y) = \log(x^2 + y^2)$ is not a homogeneous function

$$f(tx,ty) = \log\big((tx)^2 + (ty)^2\big) = \log t^2(x^2 + y^2) \neq t^2 \log(x^2 + y^2)$$

4) $f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$ is a homogeneous function of degree 0.

$$f(tx,ty) = \tan^{-1}\left(\frac{tx}{ty}\right) = \tan^{-1}\left(\frac{x}{y}\right) = t^0 f(x,y)$$

Euler's Theorem

If u is a homogeneous function of degree n in x and y then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof: u is a homogeneous function of degree n in x and y.

$$\Rightarrow u = x^n f\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x} = -yx^{n-1} f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) \ldots\ldots\ldots (1)$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$y\frac{\partial u}{\partial y} = yx^{n-1} f'\left(\frac{y}{x}\right) \ldots\ldots\ldots \ (2)$$

Adding (1) and (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -yx^{n-1} f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) + yx^{n-1} f'\left(\frac{y}{x}\right)$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \ldots\ldots\ldots \text{Hence the proof}$$

Euler's Theorem for homogeneous function of three variables:

If u is a homogeneous function of degree n in three variables x, y and z then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Deduction from Euler's Theorem:

If u is a homogeneous function of degree n in x and y then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

1. Verify Euler′s Theorem for $u = ax^2 + 2bxy + cy^2$.

Solution: $u = f(x, y) = ax^2 + 2bxy + cy^2$

Replacing x by tx and y by ty in $u = f(x, y)$,

$f(tx, ty) = a(tx)^2 + 2btxty + c(ty)^2 = t^2(ax^2 + 2bxy + cy^2)$

$f(tx, ty) = t^2 f(x, y)$

Thus, $u = ax^2 + 2bxy + cy^2$ is a homogeneous function of degree $n = 2$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u = 2(ax^2 + 2bxy + cy^2) \ldots\ldots\ldots (1)$$

Verification:

$$u = ax^2 + 2bxy + cy^2$$

$$\frac{\partial u}{\partial x} = 2ax + 2by$$

$$\frac{\partial u}{\partial y} = 2bx + 2cy$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(2ax + 2by) + y(2bx + 2cy)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2ax^2 + 2bxy + 2bxy + 2cy^2$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2(ax^2 + 2bxy + cy^2) \dots \dots \dots (2)$$

From (1) and (2), Euler's Theorem is verified

2. Verify Euler's Theorem for $u = (\sqrt{x} + \sqrt{y})(x^n + y^n)$.

Solution: $u = f(x, y) = (\sqrt{x} + \sqrt{y})(x^n + y^n)$

Replacing x by tx and y by ty in $u = f(x, y)$,

$f(tx, ty) = \left(\sqrt{tx} + \sqrt{ty}\right)(t^n x^n + t^n y^n) = \sqrt{t}\, t^n (\sqrt{x} + \sqrt{y})(x^n + y^n)$

$f(tx, ty) = t^{n+\frac{1}{2}} f(x, y)$

Thus, $u = ax^2 + 2bxy + cy^2$ is a homogeneous function of degree $n + \dfrac{1}{2}$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)(\sqrt{x} + \sqrt{y})(x^n + y^n) \ldots\ldots\ldots (1)$$

Verification:

$$u = (\sqrt{x} + \sqrt{y})(x^n + y^n)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}}(x^n + y^n) + nx^{n-1}(\sqrt{x} + \sqrt{y})$$

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}}(x^n + y^n) + ny^{n-1}(\sqrt{x} + \sqrt{y})$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x\left[\frac{1}{2\sqrt{x}}(x^n + y^n) + nx^{n-1}(\sqrt{x} + \sqrt{y})\right]$$

$$+ y\left[\frac{1}{2\sqrt{y}}(x^n + y^n) + ny^{n-1}(\sqrt{x} + \sqrt{y})\right]$$

$$= \frac{\sqrt{x}}{2}(x^n + y^n) + nx^n(\sqrt{x} + \sqrt{y}) + \frac{\sqrt{y}}{2}(x^n + y^n) + ny^n(\sqrt{x} + \sqrt{y})$$

$$= \frac{1}{2}(\sqrt{x} + \sqrt{y})(x^n + y^n) + n(x^n + y^n)$$

$$= \left(n + \frac{1}{2}\right)(\sqrt{x} + \sqrt{y})(x^n + y^n) \ldots \ldots \ldots (2)$$

From (1) and (2), Euler’s Theorem is verified

3. If $u = \frac{1}{x^2} + \frac{1}{y^2} + \frac{\log x - \log y}{x^2 + y^2}$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u = 0$.

Solution: $u = \frac{1}{x^2} + \frac{1}{y^2} + \frac{\log x - \log y}{x^2 + y^2} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{\log\left(\frac{x}{y}\right)}{x^2 + y^2}$

Replacing x by tx and y by ty in $u = f(x, y)$,

$$f(tx, ty) = \frac{1}{(tx)^2} + \frac{1}{(ty)^2} + \frac{\log\left(\frac{tx}{ty}\right)}{(tx)^2 + (ty)^2}$$

$$= \frac{1}{t^2x^2} + \frac{1}{t^2y^2} + \frac{\log\left(\frac{x}{y}\right)}{t^2x^2 + t^2y^2}$$

$$= \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{\log\left(\frac{x}{y}\right)}{x^2 + y^2}\right]$$

$$f(tx,ty) = \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{\log\left(\frac{x}{y}\right)}{x^2+y^2}\right]$$

$$= \frac{1}{t^2} f(x,y)$$

$$f(tx,ty) = t^{-2} f(x,y)$$

Therfore, $u = f(x,y) = \dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{\log x - \log y}{x^2+y^2}$ is a homogeneous function of degree $n = -2$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u \Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u = 0$$

4. If $u = \dfrac{x^3y^3}{x^3+y^3}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$

and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 6u$.

Solution: $u = f(x,y) = \dfrac{x^3y^3}{x^3+y^3}$

Replacing x by tx and y by ty in $u = f(x,y)$,

$$f(tx,ty) = \frac{(tx)^3(ty)^3}{(tx)^3+(ty)^3} = \frac{t^6x^3y^3}{t^3(x^3+y^3)} = t^3\frac{x^3y^3}{(x^3+y^3)}$$

$$f(tx,ty) = t^3 f(x,y)$$

Therfore, $u = \dfrac{x^3y^3}{x^3+y^3}$ is a homogeneous function of degree $n = 3$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$$

$$\text{Also } x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 3(3-1)u = 6u$$

5. If $u = x^2 e^{\frac{y}{x}} + y^2 \tan^{-1}\left(\frac{x}{y}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u.$

Solution: $u = f(x,y) = x^2 e^{\frac{y}{x}} + y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Replacing x by tx and y by ty in $u = f(x,y)$,

$$f(tx,ty) = (tx)^2 e^{\frac{ty}{tx}} + (ty)^2 \tan^{-1}\left(\frac{tx}{ty}\right) = t^2x^2 e^{\frac{y}{x}} + t^2y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

$$f(tx,ty) = t^2\left[x^2 e^{\frac{y}{x}} + y^2 \tan^{-1}\left(\frac{x}{y}\right)\right] = t^2 f(x,y)$$

Thus, $u = x^2 e^{\frac{y}{x}} + y^2 \tan^{-1}\left(\frac{x}{y}\right)$ is a homogeneous function of degree $n = 2.$

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

$$\text{Also } x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u$$

Deduction from Euler's Theorem:

If u is not a homogeneous function of x and y but

$z = f(u)$ is a homogeneous function of degree n in x and y then

1) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{n\, f(u)}{f'(u)}$

2) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$ where $g(u) = \dfrac{n\, f(u)}{f'(u)}$

Note: If u is not a homogeneous function of x, y and z but

$w = f(u)$ is a homogeneous function of degree n in x, y and z then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{n\, f(u)}{f'(u)}$$

1. If $u = \log\left[\frac{x^3+y^3}{x^2+y^2}\right]$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -1.$$

Solution: Replacing x by tx and y by ty in u,

$$\log\left[\frac{(tx)^3+(ty)^3}{(tx)^2+(ty)^2}\right] = \log\left[\frac{t^3}{t^2}\left(\frac{x^3+y^3}{x^2+y^2}\right)\right] = \log\left[t\left(\frac{x^3+y^3}{x^2+y^2}\right)\right] \neq t\log\left[\left(\frac{x^3+y^3}{x^2+y^2}\right)\right]$$

$\therefore u = \log\left[\frac{x^3+y^3}{x^2+y^2}\right]$ is not a homogeneous function of x and y.

$$u = \log\left[\frac{x^3+y^3}{x^2+y^2}\right] \Longrightarrow e^u = \frac{x^3+y^3}{x^2+y^2}$$

Let $f(u) = e^u$

Therfore, $f(u)$ is a homogeneous function of degree $n = 1$.

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{nf(u)}{f'(u)}$$

$$\Longrightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1e^u}{e^u}=1$$

Also $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$

where $g(u)=\frac{n\,f(u)}{f'(u)}=\frac{1e^u}{e^u}=1$

$\Longrightarrow g'\ (u)=0$

$\therefore\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=1[0-1]=-1$

2. If $x = e^u \tan v$ and $y = e^u \sec v$ then find the value of
$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right).$$

Solution: $x = e^u \tan v$ and $y = e^u \sec v$

$\therefore\ y^2 - x^2 = e^{2u}\sec^2 v - e^{2u}\tan^2 v = e^{2u}(\sec^2 v - \tan^2 v) = e^{2u}$

$e^{2u} = y^2 - x^2$

Let $f(u) = e^{2u}$

Therfore, $f(u)$ is a homogeneous function of degree $n = 2.$

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2e^{2u}}{2e^{2u}} = 1$$

Now $x = e^u \tan v$ and $y = e^u \sec v$

$\therefore \dfrac{x}{y} = \dfrac{e^u \tan v}{e^u \sec v} = \cos v \tan v = \sin v$

$\implies v = \sin^{-1}\left(\dfrac{x}{y}\right)$

$\implies v$ is a homogeneous function of of x and y with degree $n = 0$

By Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0.v = 0$$

$$\therefore \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 1 \times 0 = 0$$

3. If $u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$ and

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{4}\tan^3 u-\frac{1}{4}\tan u$$

Solution: Replacing x by tx and y by ty in u,

$$\sin^{-1}\left[\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}\right]=\sin^{-1}\left[\frac{t}{\sqrt{t}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right]=\sin^{-1}\left[t^{1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right]$$

$$\neq t^{1/2}\sin^{-1}\left[\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right]$$

$\therefore u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ is not a homogeneous function of x and y.

$$u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]\Longrightarrow \sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $f(u) = \sin u$

Therfore, $f(u)$ is a homogeneous function of degree $n = \frac{1}{2}$.

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u$$

Also $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where $g(u) = \frac{n\,f(u)}{f'(u)} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u$

$\Rightarrow g'\,(u) = \frac{1}{2}\sec^2 u$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}= g(u)[g'(u)-1]$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\tan u\left[\frac{1}{2}\sec^2 u-1\right]$$

$$=\frac{1}{2}\tan u\left[\frac{1}{2}+\frac{1}{2}\tan^2 u-1\right]=\frac{1}{2}\tan u\left[\frac{1}{2}(1+\tan^2 u)-1\right]$$

$$=\frac{1}{2}\tan u\left[\frac{1}{2}\tan^2 u-\frac{1}{2}\right]$$

$$=\frac{1}{4}\tan^3 u-\frac{1}{4}\tan u$$

4. If $u = \operatorname{cosec}^{-1}\left[\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right]$ then prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\tan u}{144}(13+\tan^2 u)$.

Solution: Replacing x by tx and y by ty in u,

$$\operatorname{cosec}^{-1}\left[\sqrt{\frac{(tx)^{1/2}+(ty)^{1/2}}{(tx)^{1/3}+(ty)^{1/3}}}\right]=\operatorname{cosec}^{-1}\left[\sqrt{\frac{t^{1/2}}{t^{1/3}}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}\right]$$

$$=\operatorname{cosec}^{-1}\left[\sqrt{t^{\frac{1}{6}}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}\right]=\operatorname{cosec}^{-1}\left[t^{\frac{1}{12}}\sqrt{\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}\right]$$

$$\neq t^{\frac{1}{12}}\operatorname{cosec}^{-1}\left[\sqrt{\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}\right]$$

$\therefore u=\operatorname{cosec}^{-1}\left[\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right]$ is not a homogeneous function of x and y.

$$u = \operatorname{cosec}^{-1}\left[\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right] \Rightarrow cosec\, u = \sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

Let $f(u) = \operatorname{cosec} u$

Therfore, $f(u)$ is a homogeneous function of degree $n = \dfrac{1}{12}$.

By Deduction of Euler's Theorem,

Also $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where $g(u) = \dfrac{n\, f(u)}{f'(u)} = \dfrac{1}{12}\dfrac{\operatorname{cosec} u}{(-\operatorname{cosec} u \cot u)} = -\dfrac{1}{12}\tan u$

$\Rightarrow g'\ (u) = -\dfrac{1}{12}\sec^2 u$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

$$\Longrightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{1}{12}\tan u\left[-\frac{1}{12}\sec^2 u-1\right]$$

$$=\frac{1}{12}\tan u\left[\frac{1}{12}+\frac{1}{12}\tan^2 u+1\right]=\frac{1}{12}\tan u\left[\frac{1}{12}(1+\tan^2 u)+1\right]$$

$$=\frac{1}{12}\tan u\left[\frac{1}{12}\tan^2 u+\frac{13}{12}\right]$$

$$=\frac{\tan u}{144}[\tan^2 u+13]$$

5. If $u = \sec^{-1}\left(\dfrac{x^3 - y^3}{x + y}\right)$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\cot u$.

Also find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

Solution: Replacing x by tx and y by ty in u,

$$\sec^{-1}\left(\frac{(tx)^3-(ty)^3}{tx + ty}\right)$$

$$= \sec^{-1}\left(\frac{t^3}{t}\frac{x^3 - y^3}{x + y}\right) = \sec^{-1}\left(t^2\frac{x^3 - y^3}{x + y}\right) \neq t^2\sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right)$$

$\therefore u = \sec^{-1}\left(\dfrac{x^3 - y^3}{x + y}\right)$ is not a homogeneous function of x and y.

$$u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right) \Longrightarrow sec\, u = \frac{x^3 - y^3}{x + y}$$

Let $f(u) = \sec u$

Therfore, $f(u)$ is a homogeneous function of degree $n = 2$.

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\, f(u)}{f'(u)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sec u}{\sec u \tan u} = \frac{2}{\tan u} = 2\cot u$$

Also $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where $g(u) = \dfrac{n\, f(u)}{f'(u)} = \dfrac{2 \sec u}{\sec u \tan u} = \dfrac{2}{\tan u} = 2 \cot u$

$\Longrightarrow g'\ (u) = -2 \operatorname{cosec}^2 u$

$\therefore\ x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 2 \cot u \left[-2 \operatorname{cosec}^2 u - 1\right]$

$$= 2 \cot u \left[-2\,(1 + \cot^2 u) - 1\right]$$

$$= 2 \cot u \left[-2 - 2\cot^2 u - 1\right]$$

$$= 2 \cot u \left[-3 - 2\cot^2 u\right]$$

$$= -6 \cot u - 4 \cot^3 u$$

Exercise 3

1. Verify Euler′s Theorem for $u = 3x^2yz + 5xy^2z + 4z^4$.
2. If $u = \sin^{-1}\left[\frac{x}{y}\right] + \tan^{-1}\left[\frac{y}{x}\right]$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.
3. If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.
4. If $u = \sin^{-1}\left[\frac{x}{y}\right] + \tan^{-1}\left[\frac{y}{x}\right]$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.
5. If $u = e^{x^2+y^2}$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$.
6. If $u = f\left(\frac{y}{x}\right) + \sqrt{x^2+y^2}$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2+y^2}$.
7. If $u = \sin^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\tan u$

8. If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

9. If $u = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{\cos^3 u}$$

10. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$$

11. If $u = \sin^{-1}\left[\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}\right]$ then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u).$$

12. If $u=\sin^{-1}(x^3+y^3)^{2/5}$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{6}{5}\tan u\left(\frac{6}{5}\sec^2 u-1\right)$$

13. If $u=\tan^{-1}\left(\frac{y^2}{x}\right)$ then show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\sin^2 u\sin 2u$

13. If $u=\frac{x^3+y^3}{y\sqrt{x}}$ find the value of $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ at (1,2).

14. If $T=\sin\left(\frac{xy}{x^2+y^2}\right)+\sqrt{x^2+y^2}+\frac{x^2y}{x+y}$ then find the value of $x\frac{\partial T}{\partial x}+y\frac{\partial T}{\partial y}$

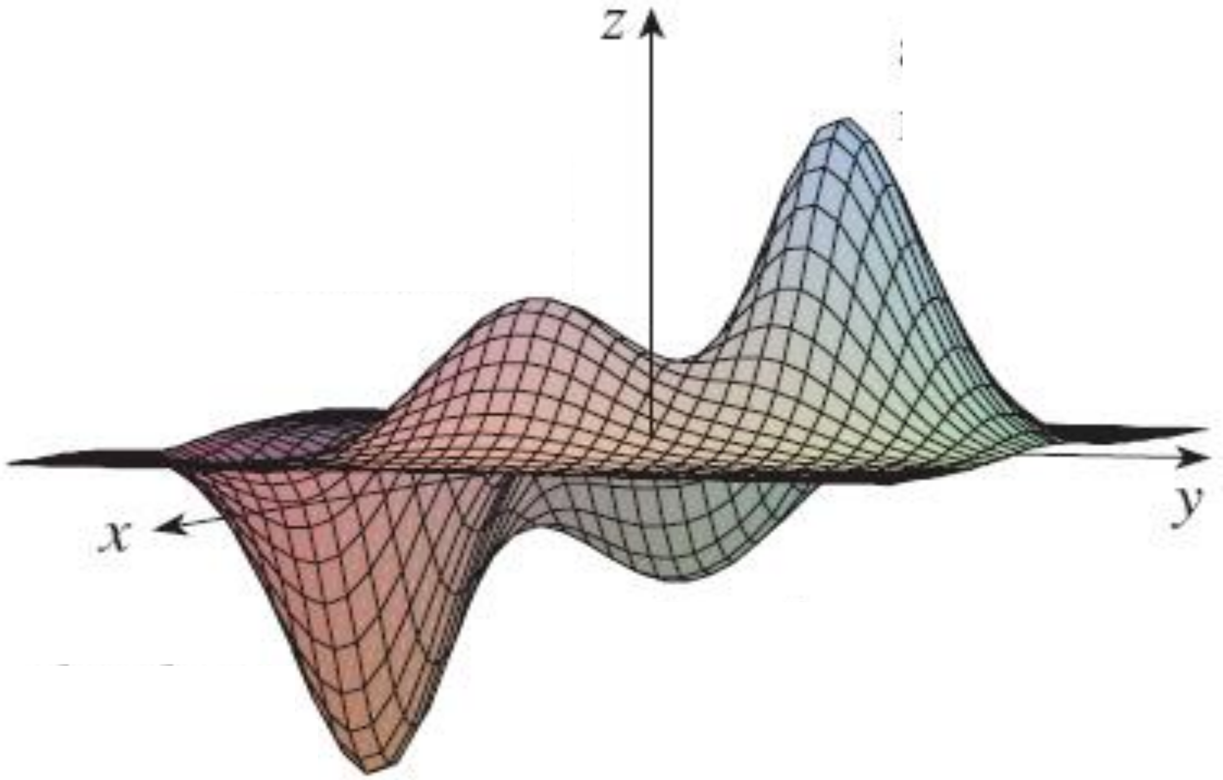
15. Verify Euler′s Theorem for $u=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$.

Applications of Partial Derivatives

1. Maxima and Minima of function of two variables
2. Lagranges Method of undetermined multiplier
3. Errors
4. Approximations
5. Jacobian

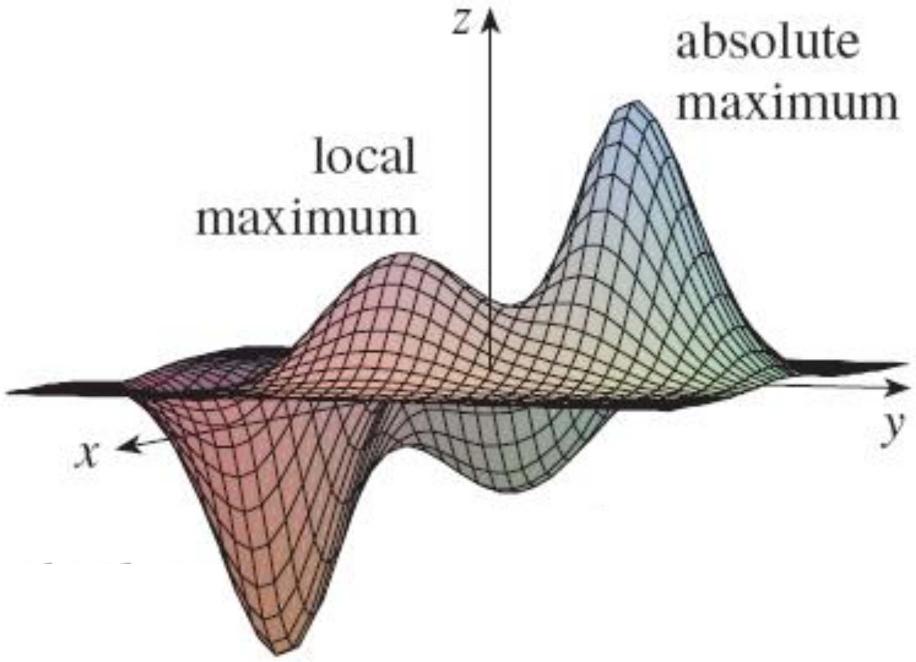
1.Maxima and Minima of function of two variables

Look at the hills and valleys in the graph of *f* shown here.



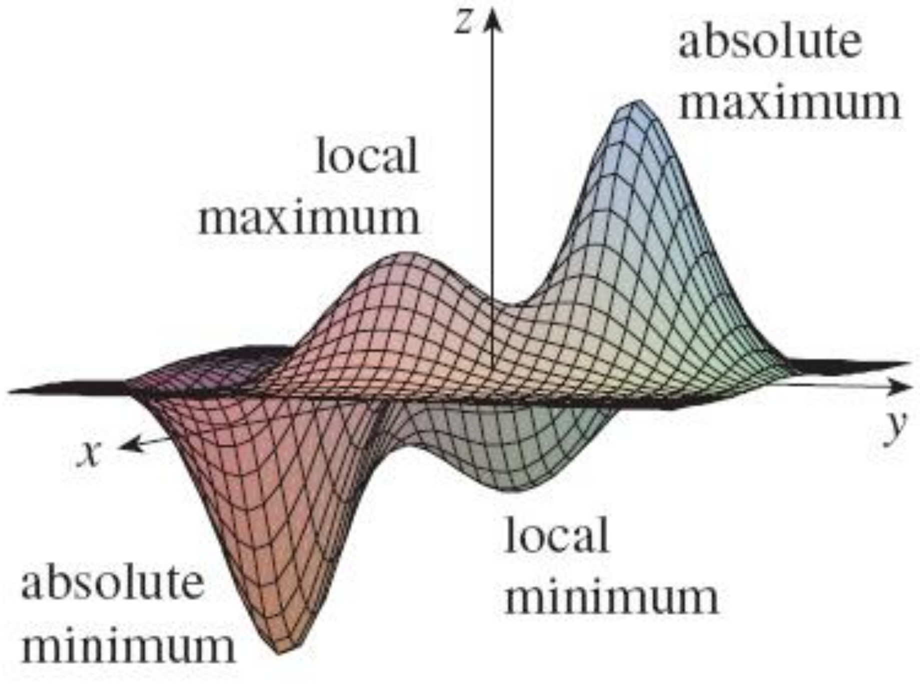
There are two points (a, b) where f has a local maximum that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$.

The larger of these two values is the **absolute maximum.**



Similarly there are two points (*a*, *b*) where *f* has a local minimum that is, where *f*(*a*, *b*) is smaller than nearby values of *f*(*x*, *y*).

The smaller of these two values is the **absolute minimum.**



Let f be a function defined on a region R containing the point (a, b).

Then, f has a local maximum at (a, b) if

$f(x, y) \leq f(a, b)$ for all points (x, y) that are sufficiently close to (a, b).

The number $f(a, b)$ is called a **local maximum value.**

Similarly, f has a local minimum at (a, b) if

$f(x, y) \geq f(a, b)$ for all points (x, y) that are sufficiently close to (a, b).

The number $f(a, b)$ is called a **local minimum value**.

Let f be a function defined on a region R containing the point (a, b).

If $f(x, y) \leq f(a, b)$ for all points (x, y) in the domain R of the function f

then number $f(a, b)$ is called a **absolute maximum value.**

If $f(x, y) \geq f(a, b)$ for all points (x, y) in the domain R of the function f

then number $f(a, b)$ is called a **absolute minimum value.**

Working rule for finding extreme values

Let $f(x, y)$ be a given function of x and y.

1. Find partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}$.

2. Let $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

3. Solve these equations for x and y. Let (a, b) be the values of (x, y).

4. Evaluate $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at point (a, b).

5. Find $rt - s^2$ at point (a, b)

Working rule for finding extreme values

Then

$a)\ rt - s^2 > 0$ and $r < 0$ implies that $f(x, y)$ has maximum value at the point (a, b).

$b)\ rt - s^2 > 0$ and $r > 0$ implies that $f(x, y)$ has minimum value at the point (a, b).

$c)\ rt - s^2 < 0$ implies that $f(x, y)$ has neither a maximum nor a minimum at the point (a, b). Such a point is called as a **saddle point**.

$d)\ rt - s^2 = 0$ then test gives no information. $f(x, y)$ could have a maximum or minimum at (a, b), or (a, b) could be a saddle point of f. Further investigation is needed.

Note: The point (a, b) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ is called as stationary point

1. Find extreme values of the function $f(x,y) = x^2 + y^2$.

Solution : $f(x,y) = x^2 + y^2$

$\dfrac{\partial f}{\partial x} = 2x$ and $\dfrac{\partial f}{\partial y} = 2y$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$$

Stationary point is $(0,0)$.

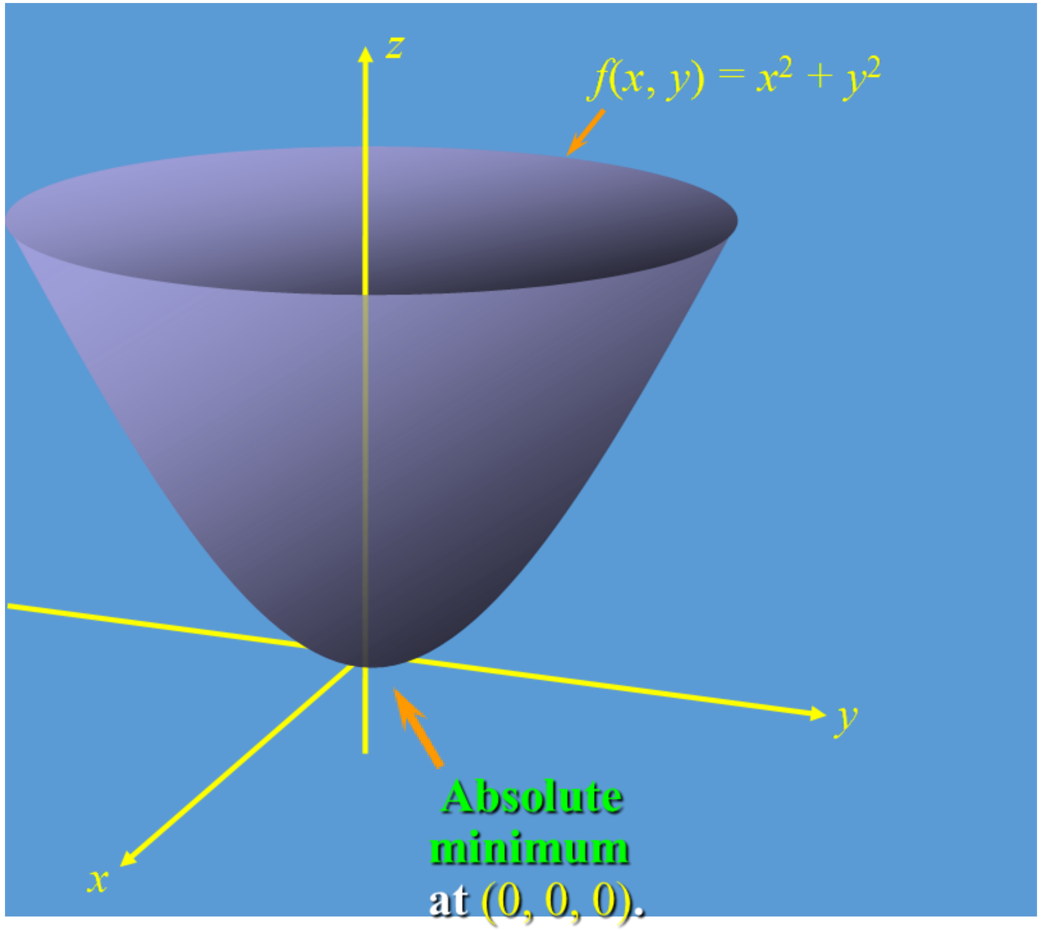
Now $r = \dfrac{\partial^2 f}{\partial x^2} = 2,\qquad s = \dfrac{\partial^2 f}{\partial x \partial y} = 0$ and $t = \dfrac{\partial^2 f}{\partial y^2} = 2$

Thus $rt - s^2 = (2)(2) - (0)^2 = 4$

At point (0,0), $rt - s^2 = 4 > 0$ and $r = 2 > 0$.

$\Longrightarrow f(x, y)$ has minimum value at the point (0,0).

$f_{min} = f(0,0) = 0$



2. Find extreme values of the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution : $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$\dfrac{\partial f}{\partial x} = y - 2x - 2$ and $\dfrac{\partial f}{\partial y} = x - 2y - 2$

$\dfrac{\partial f}{\partial x} = 0 \Longrightarrow y - 2x - 2 = 0 \ldots\ldots\ldots (1)$

$\dfrac{\partial f}{\partial y} = 0 \Longrightarrow x - 2y - 2 = 0 \ldots\ldots\ldots (2)$

Solving (1)and (2) simultaneously we get, $x = -2$ and $y = -2$

Therefore stationary point is $(-2,-2)$.

Now $r = \dfrac{\partial^2 f}{\partial x^2} = -2,\qquad s = \dfrac{\partial^2 f}{\partial x \partial y} = 1$ and $t = \dfrac{\partial^2 f}{\partial y^2} = -2$

Thus $rt - s^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3$

At point $(-2,-2)$, $rt - s^2 = 3 > 0$ and $r = -2 < 0$.

$\Longrightarrow f(x,y)$ has maximum value at the point $(-2,-2)$.

$$
\begin{aligned}
f_{max} &= f(-2,-2) \\
&= (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4 \\
&= 4 - 4 - 4 + 4 + 4 + 4 \\
&= 8
\end{aligned}
$$

3. Find extreme values of the function $f(x, y) = x^2 + y^2 - 4y + 9$

Solution : $f(x, y) = x^2 + y^2 - 4y + 9$

$$\frac{\partial f}{\partial x} = 2x \text{ and } \frac{\partial f}{\partial y} = 2y - 4$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \ldots\ldots\ldots (1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y - 4 = 0 \Rightarrow y = 2 \ldots\ldots\ldots (2)$$

From (1)and (2), stationary point is $(0, 2)$.

Now $r = \frac{\partial^2 f}{\partial x^2} = 2,\quad s = \frac{\partial^2 f}{\partial x \partial y} = 0$ and $t = \frac{\partial^2 f}{\partial y^2} = 2$

Thus $rt - s^2 = (2)(2) - (0)^2 = 4$

At point $(0, 2)$, $rt - s^2 = 4 > 0$ and $r = 2 > 0$.

$\Longrightarrow f(x, y)$ has minimum value at the point $(0,2)$.

$$f_{min} = f(0, 2)$$

$$= 0 + (2)^2 - 4(2) + 9$$

$$= 5$$

4. Find extreme values of the function $f(x,y) = x^2 + 4y^3 - 12y^2 - 36y + 2$

Solution : $f(x,y) = x^2 + 4y^3 - 12y^2 - 36y + 2$

$\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 12y^2 - 24y - 36$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \ldots\ldots\ldots (1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 12y^2 - 24y - 36 = 0 \Rightarrow y^2 - 2y - 3 = 0 \ldots\ldots\ldots (2)$$

From(2) we get, $y = -1$ or $y = 3$

Therefore stationary points are $(0,-1)$ and $(0,3)$

Now $r = \frac{\partial^2 f}{\partial x^2} = 2,\qquad s = \frac{\partial^2 f}{\partial x \partial y} = 0$ and $t = \frac{\partial^2 f}{\partial y^2} = 24y - 24 = 24(y-1)$

Thus $rt - s^2 = 2\big(24(y-1)\big) - (0)^2 = 48(y-1)$

At point $(0,-1)$, $rt - s^2 = 48(-1-1) = -96 < 0.$

$\Longrightarrow f(x,y)$ has neither maxima nor minima at the point $(0,-1)$.

Point $(0,-1)$ is a saddle point.

At point $(0,3)$, $rt - s^2 = 48(3-1) = 96 > 0$ and $r = 2 > 0$.

$\Longrightarrow f(x,y)$ has minimum value at the point $(0,3)$.

$$
\begin{aligned}
f_{min} &= f(0,3) \\
&= 0 + 4(3)^3 - 12(3)^2 - 36(3) + 2 \\
&= 108 - 108 - 108 + 2 \\
&= -106
\end{aligned}
$$

5. Find extreme values of the function $f(x,y) = x^3 + y^3 - 3axy$ where $a > 0$

Solution : $f(x,y) = x^3 + y^3 - 3axy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay \quad \text{and} \quad \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3ay = 0 \Rightarrow x^2 = ay \ldots\ldots\ldots(1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 3ax = 0 \Rightarrow y^2 = ax \ldots\ldots\ldots(2)$$

From(1), $\quad x^4 = a^2y^2 \ldots\ldots\ldots(3)$

Putting value of y^2 from (2) in (3) we get,

$$x^4 = a^2ax = a^3x \Rightarrow \quad x^4 - a^3x = 0$$

$$\Rightarrow x(x^3 - a^3) = 0 \Rightarrow x(x-a)(x^2 + ax + a^2) = 0$$

$\Longrightarrow x = 0$ or $x - a = 0$ or $x^2 + ax + a^2 = 0$

Now $x^2 + ax + a^2 = 0$ have complex (imaginary) roots.

$\Longrightarrow x = 0$ or $x - a = 0$ $i.e.$ $x = a$

$x = 0$ & $y^2 = ax \Longrightarrow y = 0$

$x = a$ & $y^2 = ax \Longrightarrow y^2 = a^2 \Longrightarrow y = \pm a$

Therefore stationary points are $(0, 0), (a, a)$ and $(a, -a)$

Now $r = \dfrac{\partial^2 f}{\partial x^2} = 6x,$ $\quad s = \dfrac{\partial^2 f}{\partial x \partial y} = -3a$ and $t = \dfrac{\partial^2 f}{\partial y^2} = 6y$

Thus $rt - s^2 = (6x)(6y) - (-3a)^2 = 36xy - 9a^2$

Point	$\boldsymbol{rt - s^2 = 36xy - 9a^2}$	$\boldsymbol{r = 6x}$	**Conclusion**
$(0,0)$	$-9a^2 < 0$	$-$	Neither maxima nor minima at $(0,0)$. $(0,0)$ is a saddle point.
(a,a)	$27a^2 > 0$	$6a > 0$	$f(x,y)$ has minimum value at (a,a). $f_{min} = a^3 + a^3 - 3a^3 = -a^3$
$(a,-a)$	$-45a^2 < 0$	$-$	Neither maxima nor minima at $(a,-a)$ $(a,-a)$ is a saddle point.

6. Find extreme values of the function $f(x,y) = x^4 + y^4 - 4xy + 1$

Solution : $f(x,y) = x^4 + y^4 - 4xy + 1$

$$\frac{\partial f}{\partial x} = 4x^3 - 4y \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y^3 - 4x$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 - 4y = 0 \Rightarrow x^3 = y \ldots\ldots\ldots (1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 - 4x = 0 \Rightarrow y^3 = x \ldots\ldots\ldots (2)$$

Putting value of y from (1) in (2) we get,

$$(x^3)^3 = x \Rightarrow \quad x^9 = x \Rightarrow \quad x^9 - x = 0$$

$$\Rightarrow x(x^8 - 1) = 0 \Rightarrow x(x^4 - 1)(x^4 + 1) = 0 \Rightarrow x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$$

$\Longrightarrow x = 0$ or $x^2 - 1 = 0$ or $x^2 + 1 = 0$ or $x^4 + 1 = 0$

Now $x^2 + 1 = 0$ and $x^4 + 1 = 0$ have complex (imaginary) roots.

$\Longrightarrow x = 0$ or $x^2 - 1 = 0 \Longrightarrow x = 0, x = \pm 1$

$x = 0$ & $y^3 = x \Longrightarrow y^3 = 0 \Longrightarrow y = 0$

$x = 1$ & $y^3 = x \Longrightarrow y^3 = 1 \Longrightarrow y^3 - 1 = 0 \Longrightarrow (y - 1)\ (y^2 + y + 1) = 0$

$\Longrightarrow y = 1 \quad (\because y^2 + y + 1 = 0$ has no real roots)

$x = -1$ & $y^3 = x \Longrightarrow y^3 = -1 \Longrightarrow y^3 + 1 = 0 \Longrightarrow (y + 1)\ (y^2 - y + 1) = 0$

$\Longrightarrow y = -1 \quad (\because y^2 - y + 1 = 0$ has no real roots)

Therefore stationary points are $(0, 0), (1,1)$ and $(-1, -1)$

Now $r = \frac{\partial^2 f}{\partial x^2} = 12x^2,$ $s = \frac{\partial^2 f}{\partial x \partial y} = -4$ and $t = \frac{\partial^2 f}{\partial y^2} = 12y^2$

Thus $rt - s^2 = (12x^2)(12y^2) - (-4)^2 = 144x^2y^2 - 16$

Point	$rt - s^2 = 144x^2y^2 - 16$	$r = 12x^2$	Conclusion
$(0,0)$	$-16 < 0$	$-$	Neither maxima nor minima at $(0,0)$. $(0,0)$ is a saddle point.
$(1,1)$	$128 > 0$	$12 > 0$	$f(x,y)$ has minimum value at $(1,1)$. $f_{min} = 1^2 + 1^2 - 4(1)(1) + 1 = -1$
$(-1,-1)$	$128 > 0$	$12 > 0$	$f(x,y)$ has minimum value at $(-1,-1)$ $f_{min} = (-1)^2+(-1)^2-4(-1)(-1) + 1 = -1$

Point	$\boldsymbol{rt - s^2} = 144x^2y^2 - 16$	$\boldsymbol{r} = 12x^2$	**Conclusion**
$(0,0)$	$-16 < 0$	$-$	Neither maxima nor minima at $(0,0)$. $(0,0)$ is a saddle point.
$(1,1)$	$128 > 0$	$12 > 0$	$f(x,y)$ has minimum value at $(1,1)$. $f_{min} = 1^2 + 1^2 - 4(1)(1) + 1 = -1$
$(-1,-1)$	$128 > 0$	$12 > 0$	$f(x,y)$ has minimum value at $(-1,-1)$ $f_{min} = (-1)^2 + (-1)^2 - 4(-1)(-1) + 1 = -1$

Point	$rt - s^2 = 36xy - 9a^2$	$r = 6x$	Conclusion
$(0,0)$	$-9a^2 < 0$	$-$	Neither maxima nor minima
$(a,0)$	$-9a^2 < 0$	$-$	Neither maxima nor minima
$(0,a)$	$-9a^2 < 0$	$-$	Neither maxima nor minima
(a,a)	$27a^2 > 0$	$6a > 0$	$f(x,y)$ has minimum value at (a,a) and $f_{min} = -a^3$

At point $(0, -1)$, $rt - s^2 = 48(-1 - 1) = -96 < 0$.

$\Longrightarrow f(x, y)$ has neither maxima nor minima at the point $(0, -1)$.

Point $(0, -1)$ is a saddle point.

At point $(0, 3)$, $rt - s^2 = 48(3 - 1) = 96 > 0$ and $r = 2 > 0$.

$\Longrightarrow f(x, y)$ has minimum value at the point $(0,3)$.

$$f_{min} = f(0, 3)$$

$$= 0 + 4(3)^3 - 12(3)^2 - 36(3) + 2$$

$$= 108 - 108 - 108 + 2$$

$$= -106$$

5. Find extreme values of the function $f(x, y) = xy(a - x - y)$ where $a > 0$

Solution : $f(x, y) = xy(a - x - y) = axy - x^2y - xy^2$

$$\frac{\partial f}{\partial x} = ay - 2xy - y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = ax - x^2 - 2xy$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow ay - 2xy - y^2 = 0 \Rightarrow y(a - 2x - y) = 0 \ldots\ldots\ldots (1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow ax - x^2 - 2xy = 0 \Rightarrow x(a - x - 2y) = 0 \ldots\ldots\ldots (2)$$

From(1), $y = 0$ or $a - 2x - y = 0$

From(2), $x = 0$ or $a - x - 2y = 0$

$y = 0$ and $x = 0 \Longrightarrow (x, y) = (0, 0)$

$y = 0$ and $a - x - 2y = 0 \Longrightarrow (x, y) = (a, 0)$

$a - 2x - y = 0$ and $x = 0 \Longrightarrow (x, y) = (0, a)$

$a - 2x - y = 0$ and $a - x - 2y = 0 \Longrightarrow (x, y) = \left(\frac{a}{3}, \frac{a}{3}\right)$

Therefore stationary points are $(0, 0), (a, 0), (0, a)$and $\left(\frac{a}{3}, \frac{a}{3}\right)$.

Now $r = \frac{\partial^2 f}{\partial x^2} = -2y, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y$ and $t = \frac{\partial^2 f}{\partial y^2} = -2x$

Thus $rt - s^2 = (-2y)(-2x) - (a - 2x - 2y)^2 = 4xy - (a - 2x - 2y)^2$

Point	$\boldsymbol{rt - s^2} = 4xy - (a - 2x - 2y)^2$	$\boldsymbol{r} = -2y$	**Conclusion**
$(0,0)$	$-a^2 < 0$	$-$	Neither maxima nor minima at $(0,0)$. $(0,0)$ is a saddle point.
$(a,0)$	$-a^2 < 0$	$-$	Neither maxima nor minima at $(a,0)$. $(a,0)$ is a saddle point.
$(0,a)$	$-a^2 < 0$	$-$	Neither maxima nor minima at $(0,a)$. $(0,a)$ is a saddle point.
$\left(\frac{a}{3}, \frac{a}{3}\right)$	$4\frac{a}{3}\frac{a}{3} - \left(a - \frac{2a}{3} - \frac{2a}{3}\right)^2 = a^2 > 0$	$-\frac{2a}{3} = \begin{cases} < 0 \text{ if } a > 0 \\ > 0 \text{ if } a < 0 \end{cases}$	$f(x,y)$ has maximum value if $a > 0$ and minimum value if $a < 0$

Exercise 4

Find extreme values of the following functions

1. $f(x, y) = x^2 + y^2 - 6x + 12.$
2. $f(x, y) = x^3y^2(1 - x - y)$
3. $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$
4. $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$
5. $f(x, y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$
6. $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$
7. $f(x, y) = x^2 + y^2 + xy + x - 4y + 5.$
8. $f(x, y) = x^3 + 3x^2 + y^2 + 4xy.$