Bansilal Ramnath Agarwal Charitable Trust's

VISHWAKARMA INSTITUTE OF INFORMATION TECHNOLOGY

Department of Engineering & Applied Sciences

F.Y.B.Tech (2020-21)

Course material (A brief reference version for students)

Course: Calculus

Unit 4: Curve Tracing

Curve Tracing

Curve Tracing means to find approximate shape of the curves using different features namely symmetry, intercepts, tangents, asymptotes, region of existence etc.
The knowledge of curve tracing is to avoid the labour of plotting a large number of points. It is helpful in finding the length of curve, area,volume and surface area. The limits of integration can be easily found on tracing the curve roughly.

Important Definitions:

1) **Singular point**: This is an unusual point on a curve.
2) **Multiple point**: A point through which a curve passes more than one time.
3) **Double Point**: If a curve passes two times through a point then it is called a double point.
   (a) **Node** : A double point at which two real tangents (not coincident) can be drawn.
   (b) **Cusp** : A double point is called cusp if the two tangents at it are coincident.
4) **Point of Inflection** : A point where the curve crosses the tangent is called point of inflection.
5) **Asymptote:** The tangent to the curve at infinity.

There are three types of curves viz cartesian,parametric and polar curves. Let's see one by one.

Cartesian Curves

Cartesian curves are given by the equations $f(x,y) = 0$.

I) Symmetry:

About $\boldsymbol{x}$ – axis	Powers of y are even everywhere in the equation of curve. e.g. $\boldsymbol{y^2 = 4ax}$
About $\boldsymbol{y}$ – axis	Powers of x are even everywhere in the equation of curve. e.g. $\boldsymbol{x^2 = 4ay}$
About both $\boldsymbol{x}$ – axis & $\boldsymbol{y}$ – axis	Powers of both x and y are even everywhere in the equation of curve. e.g. $\boldsymbol{x^2 + y^2 = 1}$
About Origin (opposite quadrants)	Equation of Curve remains unchanged after replacement of $\mathbf{x}$ by $-\mathbf{x}$ & $\boldsymbol{y}$ by $-\boldsymbol{y}$. e.g. $\boldsymbol{xy = 1}$
About line $\boldsymbol{y = x}$	Equation of Curve remains unchanged after replacement of $\boldsymbol{x}$ by $\boldsymbol{y}$ and $\boldsymbol{y}$ by $\boldsymbol{x}$. e.g. $\boldsymbol{xy = 1}$
About line $\boldsymbol{y = -x}$	Equation of Curve remains unchanged after replacement of $\boldsymbol{x}$ by $-\boldsymbol{y}$ & $\boldsymbol{y}$ by $-\boldsymbol{x}$. e.g. $\boldsymbol{xy = -1}$

II) Points of Intersection:

Origin $(\mathbf{0}, \mathbf{0})$	If constant term is absent in the equation of curve then curve passes through origin.
Intersection with $\boldsymbol{x}$ – axis	Put $\boldsymbol{y = 0}$ in the equation of curve

Intersection with $\boldsymbol{y}$ – axis	Put $\boldsymbol{x} = \boldsymbol{0}$ in the equation of curve
Intersection with line $\boldsymbol{y} = \boldsymbol{x}$	Put $\boldsymbol{y} = \boldsymbol{x}$ in the equation of curve
Intersection with line $\boldsymbol{y} = -\boldsymbol{x}$	Put $\boldsymbol{y} = -\boldsymbol{x}$ in the equation of curve

III) Tangents:

At Origin $(\mathbf{0}, \mathbf{0})$	Equate lowest degree term from the equation of curve to zero.	
At any arbitrary point of intersection $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})$	Find $\frac{dy}{dx}$ at point $P(x, y)$	
	$\left[\frac{dy}{dx}\right]_P = \mathbf{0}$	Tangent parallel to $\boldsymbol{x}$ – axis
	$\left[\frac{dy}{dx}\right]_P = \infty$	Tangent parallel to $\boldsymbol{y}$ – axis
	$\left[\frac{dy}{dx}\right]_P > 0$	Tangent makes acute angle with $\boldsymbol{x}$ – axis
	$\left[\frac{dy}{dx}\right]_P < 0$	Tangent makes Obtuse angle with $\boldsymbol{x}$ – axis

IV) Asymptotes:

Asymptote is the tangent to the curve at infinity.
Oblique Asymptote is the asymptote which is not parallel to co-ordinate axes.

Asymptotes parallel to $x -$ axis	Equate coefficient of highest degree term in x in equation of curve to zero.
Asymptotes parallel to $y -$ axis	Equate coefficient of highest degree term in y in equation of curve to zero.
Oblique Asymptote (When curve is not symmetric about x or y axis then we check for this asymptote)	Put $y = mx + c$ in the equation of curve & then find the values m & c by equating coefficients of two successive highest powers of x to zero.

Note : If $y \to \infty$ as $x \to a$ then $x = a$ is an asymptote.
If $x \to \infty$ as $y \to b$ then $y = b$ is an asymptote.

V) Region of Existence of the curve:

If curve is symmetric about x – axis then write down equation as of $y^2 = f(x)$	$y^2 < 0$ for $x > a$	Curve does not exists for $x > a$
If curve is symmetric about y – axis then write down equation as of $x^2 = f(y)$	$x^2 < 0$ for $y > b$	Curve does not exists for $y > b$

Example 1: Trace the curve $y^2(2a-x)=x^3$.

Solution:

1.**Symmetry**: About $x-$ axis (since all powers of y are even in the equation of curve)

2.**Points of intersection** :

(a) Origin: Constant term is absent in the equation.Curve is passing through origin.

(b) x-intercept : $y=0 \Rightarrow x=0$.

(c) y-intercept : $x=0 \Rightarrow y=0$.

3.**Tangent at origin** : $y^2(2a-x)=x^3 \;\Rightarrow 2ay^2-xy^2-x^3=0$

Lowest degree term in equation is $2ay^2$.

$\therefore 2ay^2=0 \;\Rightarrow y=0$ i.e. x axis is tangent to the curve at origin

4.**Asymptotes** :

(a) Asymptote parallel to x-axis :

Highest degree term in x is $-x^3$ and Coefficient of x^3 is -1. Therefore no asymptote parallel to x-axis

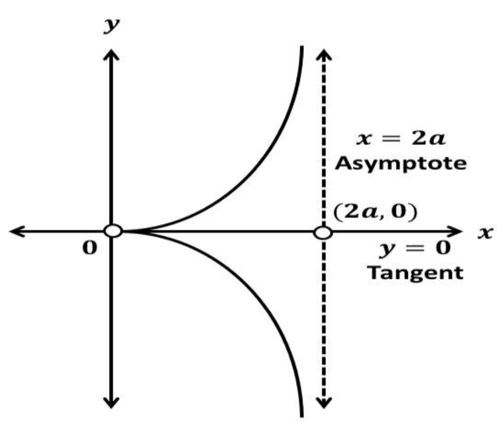
(b) Asymptote parallel to y-axis :

Highest degree term in y is $(2a-x)y^2$ and coefficient of y^2 is $2a-x$.

Therefore $2a-x=0 \Rightarrow x=2a$ is an asymptote parallel to y $-$ axis.

5.**Region of Existence**: $y^2=\frac{x^3}{2a-x}$

Values of x	**Sign of y^2**	**Conclusion**
$x<0$	$x^3<0$ and $2a-x>0$ $\therefore y^2<0$	Curve does not exist
$0<x<2a$	$x^3>0$ and $2a-x>0$ $\therefore y^2>0$	Curve exist
$x>2a$	$x^3>0$ and $2a-x<0$ $\therefore y^2<0$	Curve does not exist



Example 2: Trace the curve $x(x^2+y^2)=a(x^2-y^2)$.

Solution:

1.**Symmetry**: About $x-$ axis (since all powers of y are even in the equation of curve)

2.**Points of intersection** :

(a) Origin: Constant term is absent in the equation.Curve is passing through origin.

(b) x-intercept : $y=0 \Rightarrow x=0, a \Rightarrow x$ intercept is $(a,0)$.

(c) y-intercept : $x=0 \Rightarrow y=0$.

3.**Tangents** :

(a)**At origin** : Lowest degree term in the equation is $a(x^2-y^2)$.

$$\therefore a(x^2-y^2)=0 \Rightarrow x^2-y^2=0 \Rightarrow y^2=x^2 \Rightarrow y=\pm x$$

$\therefore y=\pm x$ are tangents to the curve at origin.

(b) **At** $(\boldsymbol{a},\mathbf{0})$: $x(x^2+y^2)=a(x^2-y^2) \Rightarrow y^2=\frac{ax^2-x^3}{x+a} \Rightarrow 2y\frac{dy}{dx}=d\left(\frac{ax^2-x^3}{x+a}\right)$

$$\Rightarrow \frac{dy}{dx}=\frac{d\left(\frac{ax^2-x^3}{x+a}\right)}{2y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a,0)}=\infty$$

$\Rightarrow$ Tangent at point $(\mathrm{a},0)$ to the curve is parallel to $\mathrm{y}-$ axis.

4.**Asymptotes**:

(a) Asymptote parallel to x-axis :

Highest degree term in x is x^3 and Coefficient of x^3 is 1. Therefore no asymptote parallel to x-axis

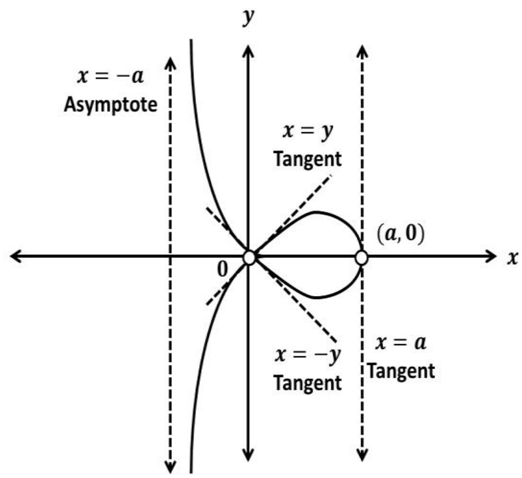
(b) Asymptote parallel to y-axis :

Highest degree term in y is $(x+a)y^2$ and coefficient of y^2 is $x+a$.

Therefore $x+a=0 \Rightarrow x=-a$ is an asymptote parallel to $\mathrm{y}-$ axis.

5.**Region of Existence**: $y^2=\frac{ax^2-x^3}{x+a}=\frac{x^2(a-x)}{x+a}$

Values of x	**Sign of y^2**	**Conclusion**
$x<-a$	$x^2>0, a-x>0$ and $x+a<0$ $\therefore y^2<0$	Curve does not exist
$-a<x<0$	$x^2>0, a-x>0$ and $x+a>0$ $\therefore y^2>0$	Curve exist
$0<x<a$	$x^2>0, a-x>0$ and $x+a>0$ $\therefore y^2>0$	Curve exist
$x>a$	$x^2>0, a-x<0$ and $x+a>0$ $\therefore y^2<0$	Curve does not exist



Example 3: Trace the curve $xy^2 = a^2(a - x)$.

Solution:

1.**Symmetry**: About $x -$ axis (since all powers of y are even in the equation of curve)

2.**Points of intersection** :

(a) Origin: Constant term is present in the equation.Curve not passes through origin.

(b) x-intercept : $y = 0 \Rightarrow x = a$.

(c) y-intercept : $x = 0 \Rightarrow y = \infty \;\Rightarrow$ No y-intercept.

3.**Tangent at $(a, 0)$** : $xy^2 = a^2(a - x) \Rightarrow y^2 = \frac{a^3 - a^2x}{x} \Rightarrow 2y\frac{dy}{dx} = d\left(\frac{a^3 - a^2x}{x}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{d\left(\frac{a^3 - a^2x}{x}\right)}{2y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a,0)} = \infty$$

$\Rightarrow$ Tangent at point $(a, 0)$ to the curve is parallel to y $-$ axis.

4.**Asymptotes** :

(a) **Asymptote parallel to x-axis** : $xy^2 = a^2(a - x) \Rightarrow x(y^2 + a^2) = a^3$

Highest degree term in x is $x(y^2 + a^2)$ and Coefficient of x is $y^2 + a^2$.

Therefore $y^2 + a^2 = 0$ is not possible in real number system. Hence no asymptote parallel to x-axis.

Alternatively, $x = \frac{a^3}{y^2 + a^2} \Rightarrow$ there is no y for which $x \to \infty$.

Hence no asymptote parallel to x-axis.

(b) **Asymptote parallel to y-axis** :

Highest degree term in y is xy^2 and coefficient of y^2 is x.

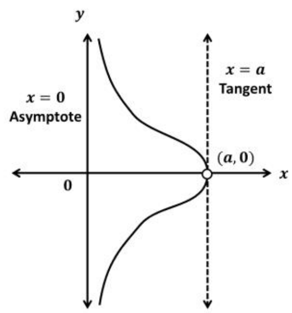
Therefore $x = 0$ i.e. y $-$ axis is an asymptote.

Alternatively, $y^2 = \frac{a^3 - a^2x}{x} \Rightarrow y \to \infty$ when $x \to 0$.

Therefore $x = 0$ i.e. y $-$ axis is an asymptote.

5.**Region of Existence**: $y^2 = \frac{a^2(a-x)}{x}$

Values of x	**Sign of y^2**	**Conclusion**
$x < 0$	$-x > 0$ and $a - x > 0$ $\therefore y^2 < 0$	Curve does not exists
$0 < x < a$	$x > 0$ and $a - x > 0$ $\therefore y^2 > 0$	Curve exists
$x > a$	$x > 0$ and $a - x < 0$ $\therefore y^2 < 0$	Curve does not exists



Example 4: Trace the curve $x^2y^2 = a^2(y^2 - x^2)$.

Solution:

1.Symmetry: (i) **About both the axes** (since all powers of x & y are even in the equation of curve)

(ii) **About opposite quadrants** (equation of curve remains unchanged after replacing x by –x and y by –y)

2.Points of intersection :

(a) Origin: Constant term is absent in the equation.Curve passes through origin.

(b) x-intercept : $y = 0 \Rightarrow x = 0$.

(c) y-intercept : $x = 0 \Rightarrow y = \infty$. Thus no y-intercept.

3.Tangent at origin :

Lowest degree term in the equation is $a^2(y^2 - x^2)$.

$\therefore a^2(y^2 - x^2) = 0 \Rightarrow y^2 - x^2 = 0 \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$

$\therefore y = \pm x$ are tangents to the curve at origin.

4.Asymptotes:

(a) Asymptote parallel to x-axis : $x^2y^2 = a^2(y^2 - x^2) \Rightarrow x^2(y^2 + a^2) = a^2y^2$

Highest degree term in x is $x^2(y^2 + a^2)$ and coefficient of x^2 is $y^2 + a^2$.

Therefore $y^2 + a^2 = 0$ is not possible in real number system. Hence no asymptote parallel to x-axis.

Alternatively, $x^2 = \frac{a^2y^2}{y^2+a^2} \Rightarrow$ there is no y for which $x \to \infty$.

Hence no asymptote parallel to x-axis.

(b) Asymptote parallel to y-axis : $x^2y^2 = a^2(y^2 - x^2) \Rightarrow y^2(x^2 - a^2) = -a^2x^2$

Highest degree term in y is $y^2(x^2 - a^2)$ and coefficient of y^2 is $x^2 - a^2$.

Therefore $x^2 - a^2 = 0 \Rightarrow x = \pm a$ are asymptotes parallel to y − axis.

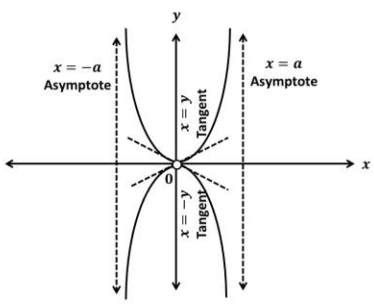
Alternatively, $y^2 = \frac{-a^2x^2}{x^2-a^2} \Rightarrow y \to \infty$ when $x \to \pm a$.

Therefore $x = \pm a$ are asymptotes parallel to y − axis.

5.Region of Existence: $y^2 = \frac{-a^2x^2}{x^2-a^2}$

Note that $a^2x^2 > 0 \Rightarrow -a^2x^2 < 0 \Rightarrow$ Numerator of y^2 is negative for all x. Therefore sign of y^2 depends on sign of $x^2 - a^2$.

Values of x	**Sign of y^2**	**Conclusion**
$x < -a$	$x + a < 0$ and $x - a < -2a < 0$ $\therefore x^2 - a^2 = (x+a)(x-a) > 0 \Rightarrow y^2 < 0$	Curve does not exists
$-a < x < 0$	$x + a > 0$ and $x - a < -a < 0$ $\therefore x^2 - a^2 = (x+a)(x-a) < 0 \Rightarrow y^2 > 0$	Curve exists
$0 < x < a$	$x + a > a$ and $x - a < 0$ $\therefore x^2 - a^2 = (x+a)(x-a) < 0 \Rightarrow y^2 > 0$	Curve exists
$x > a$	$x + a > a > 0$ and $x - a > 0$ $\therefore x^2 - a^2 = (x+a)(x-a) > 0 \Rightarrow y^2 < 0$	Curve does not exists



Exercise

Trace the following cartesian curves.

1) $y\left(1+x^2\right)=x$
2) $y^2=(x-1)(x-2)(x-3)$
3) $y^2\left(x^2-1\right)=x$
4) $x^2y^2=x^2+1$
5) $y=x\left(x^2-1\right)$
6) $y^2\left(a^2+x^2\right)=a^2x^2$
7) $y^2\left(a^2-x^2\right)=a^3x$
8) $a^2x^2=y^3(2a-y)$
9) $ay^2=x^2(a-x)$
10) $y^2=x^5(2a-x)$
11) $y\left(x^2+4a^2\right)=8a^3$
12) $y^2(4-x)=x(x-2)^2$
13) $ay^2=x(a^2-x^2)$
14) $xy^2=a^2(a-x)$
15) $xy^2=a(x^2-a^2)$
16) $a^2y^2=x^2(a^2-x^2)$
17) $x^3+y^3=3axy$

Parametric Curves

Equations of Parametric curve are given by $x = f(t),\ y = g(t)$.

Limitation of the curve: If possible find greatest and least values of x and y for a proper value of t and therefore draws the boundary lines parallel to x and y axes between which the curve lies.

I) Symmetry:

About	Condition
$\boldsymbol{x}$ – axis	If $x = f(t)$ is an even function of t and $y = g(t)$ is an odd function of t then the curve is symmetric about $x -$ axis. e.g. $x = at^2\ , y = 2at$
$\boldsymbol{y}$ – axis	If $x = f(t)$ is an odd function of t and $y = g(t)$ is an even function of t then the curve is symmetric about $y -$ axis. e.g. $x = 2at,\ \ y = at^2$
Opposite quadrants (Origin)	If both $x = f(t)$ and $y = g(t)$ are odd functions of t then the curve is symmetric about opposite quadrants. e.g. $x = t,\ \ y = \frac{1}{t}$
$\boldsymbol{y}$ – axis	After changing t by $\pi - t$ if $x = f(t)$ changes it's sign but $y = g(t)$ remains unchanged then curve is about $y -$ axis.

II) Points of Intersection:

Point	Condition
Origin (0, 0)	If both x & y are zero for some value of t then curve passes through origin.
$\boldsymbol{x}$ intercept	Put $\boldsymbol{y} = \boldsymbol{0}$ and find values of $\boldsymbol{t}$ .Then put these values of $\boldsymbol{t}$ in $\boldsymbol{x}$.
$\boldsymbol{y}$ intercept	Put $\boldsymbol{x} = \boldsymbol{0}$ and find values of $\boldsymbol{t}$ .Then put these values of $\boldsymbol{t}$ in $\boldsymbol{y}$.

III) Tangents: Find $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$.

$\left(\frac{dy}{dx}\right)_{for\ some\ t} = 0$	Tangent at a point corresponding to this t is parallel to x-axis
$\left(\frac{dy}{dx}\right)_{for\ some\ t} = \pm\infty$	Tangent at a point corresponding to this t is parallel to y-axis

IV) Table: Prepare a table for values of $t, x, y, \frac{dy}{dx}$. Take values of t from origin & points of intersections.

V) Find asymptotes if any.

VI) Find region af absence of the curve.Draw a curve from x,y coordinates & tangent position.

Solved Examples:

Example 1: Trace the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.(**Astriod**)

Solution: Parametric equations of the curve are $x = a\cos^3 t,\;\; y = a\sin^3 t$.

1.Symmetry:

(a) **About x-axis** : $x = a\cos^3 t$ is an even function of t and $y = a\sin^3 t$ is an odd function of t. Therefore curve is symmetric about x-axis.

(b) **About y-axis** : Changing t by $\pi - t$ in x and y we get,

$$x = a\cos^3(\pi - t) = -a\cos^3 t$$

$$y = a\sin^3(\pi - t) = a\sin^3 t$$

x changes it's sign but y remains unchanged.Hence curve is symmetric about y-axis also.

2.Points of intersection:

(a) Origin: $\sin t$ & $\cos t$ never becomes zero at a time for any value of t. Therefore x and y are not zero simultaneously. Curve not passes through origin.

(b) x-intercept:

$$y = 0 \;\Rightarrow\; a\sin^3 t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0, \pi$$

$$t = 0 \;\Rightarrow\; x = a\cos^3 0 = a$$

$$t = \pi \Rightarrow\; x = a\cos^3 \pi = -a$$

Points of intersection with x-axis are $(a, 0)$ and $(-a, 0)$.

(c) y-intercept:

$$x = 0 \;\Rightarrow\; a\cos^3 t = 0 \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$t = \frac{\pi}{2} \Rightarrow\; y = a\sin^3\left(\frac{\pi}{2}\right) = a$$

$$t = \frac{3\pi}{2} \Rightarrow\; y = a\sin^3\left(\frac{3\pi}{2}\right) = -a$$

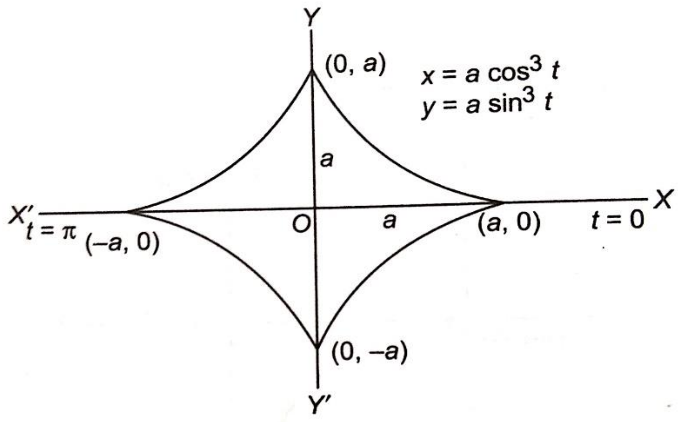
Points of intersection with y-axis are $(0, a)$ and $(0, -a)$.

3.Tangents: $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{3a\sin^2 t.\cos t}{-3a\cos^2 t.s\,int} = -\tan t$$

t	**x**	**y**	$\frac{dy}{dx}$	**Conclusion**
0	a	0	0	Tangent at point (a ,0) is parallel to x-axis
$\frac{\pi}{2}$	0	a	$-\infty$	Tangent at point (0, a) is parallel to y-axis
π	$-a$	0	0	Tangent at point ($-a$,0) is parallel to x-axis
$\frac{3\pi}{2}$	0	$-a$	∞	Tangent at point (0, $-a$) is parallel to y-axis

4.Existence of Curve:

Since $|\sin t| \le 1$ & $|\cos t| \le 1$, $|x| \le a$ & $|y| \le a$ .Therefore curve lies between $x = -a$ to a and $y = -a$ to a



Example 2: Trace the curve: $x = a(t - \sin t), y = a(1 + \cos t)$.

Solution:

1.Symmetry: (a) About y-axis : $x = a(t - \sin t)$ is an odd function of t and $y = a(1 + \cos t)$ is an even function of t. Therefore curve is symmetric about y-axis.

2.Points of intersection:

(a) Origin: $\sin t$ & $\cos t$ never becomes zero simultaneously for any value of t. Therefore x and y are not zero simultaneously. Therefore curve not passes through origin.

(b) x-intercept:

$$y = 0 \Rightarrow a(1 + \cos t) = 0 \Rightarrow 1 + \cos t = 0 \Rightarrow \cos t = -1 \Rightarrow t = \pi$$

$$t = \pi \Rightarrow x = a(\pi - \sin \pi) = a\pi$$

Point of intersection with x-axis is $(a\pi, 0)$.

(c) y-intercept:

$$x = 0 \Rightarrow a(t - \sin t) = 0 \Rightarrow t - \sin t = 0 \Rightarrow t = 0$$

$$t = 0 \Rightarrow y = a(1 + \cos 0) = 2a$$

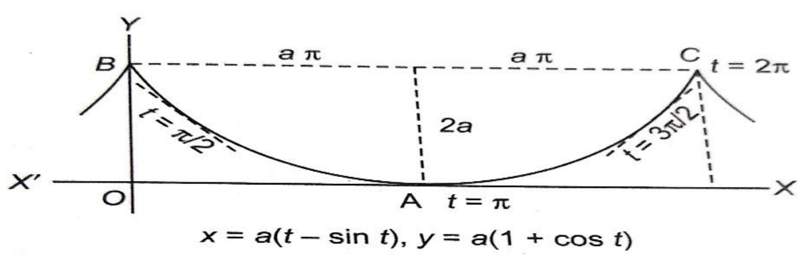
Point of intersection with y-axis is $(0, 2a)$.

3.Tangents: $\dfrac{dy}{dx}=\dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)}=\dfrac{-a\sin t}{a(1-\cos t)}=-\dfrac{2\sin\left(\dfrac{t}{2}\right)\cos\left(\dfrac{t}{2}\right)}{2\sin^2\left(\dfrac{t}{2}\right)}=-\cot\left(\dfrac{t}{2}\right).$

t	x	y	$\dfrac{dy}{dx}$	Conclusion
0	0	$2a$	$-\infty$	Tangent at point $(\mathbf{0},\mathbf{2a})$ is parallel to y-axis
$\dfrac{\pi}{2}$	$a\left(\dfrac{\pi}{2}-1\right)$	0	$\mathbf{-1}$	Tangent at point $\left(a\left(\dfrac{\pi}{2}-1\right),0\right)$ makes obtuse angle with positive x-axis
π	$a\pi$	0	0	Tangent at point ($a\pi$,0) is parallel to x-axis
$\dfrac{3\pi}{2}$	$a\left(\dfrac{3\pi}{2}+1\right)$	a	$\mathbf{1}$	Tangent at point $\left(a\left(\dfrac{3\pi}{2}+1\right),a\right)$ makes an acute angle with positive x-axis
2π	$2a\pi$	$2a$	$-\infty$	Tangent at point $(2a\pi,2a)$ is parallel to y-axis

4.Existence of Curve:

Maximum value of $\cos t$ is 1 and minimum value of $\cos t$ is -1 .
Maximum value of y is $2a$ and minimum value of y is 0.
Therefore curve lies in between the lines $y=0$ and $y=2a$.



Example 3: Trace the curve: $x = a(t + \sin t), y = a(1 + \cos t)$.

Solution:

1.Symmetry: (a) About y-axis : $x = a(t + \sin t)$ is an odd function of t and $y = a(1 + \cos t)$ is an even function of t. Therefore curve is symmetric about y-axis.

2.Points of intersection:

(a) Origin: $\sin t$ & $\cos t$ never becomes zero simultaneously for any value of t. Therefore x and y are not zero simultaneously. Therefore curve not passes through origin.

(b) x-intercept:

$$y = 0 \Rightarrow a(1 + \cos t) = 0 \Rightarrow 1 + \cos t = 0 \Rightarrow \cos t = -1 \Rightarrow t = \pi$$

$$t = \pi \Rightarrow x = a(\pi + \sin \pi) = a\pi$$

Point of intersection with x-axis is $(a\pi, 0)$

(c) y-intercept:

$$x = 0 \Rightarrow a(t + \sin t) = 0 \Rightarrow t + \sin t = 0 \Rightarrow t = 0$$

$$t = 0 \Rightarrow y = a(1 + \cos 0) = 2a$$

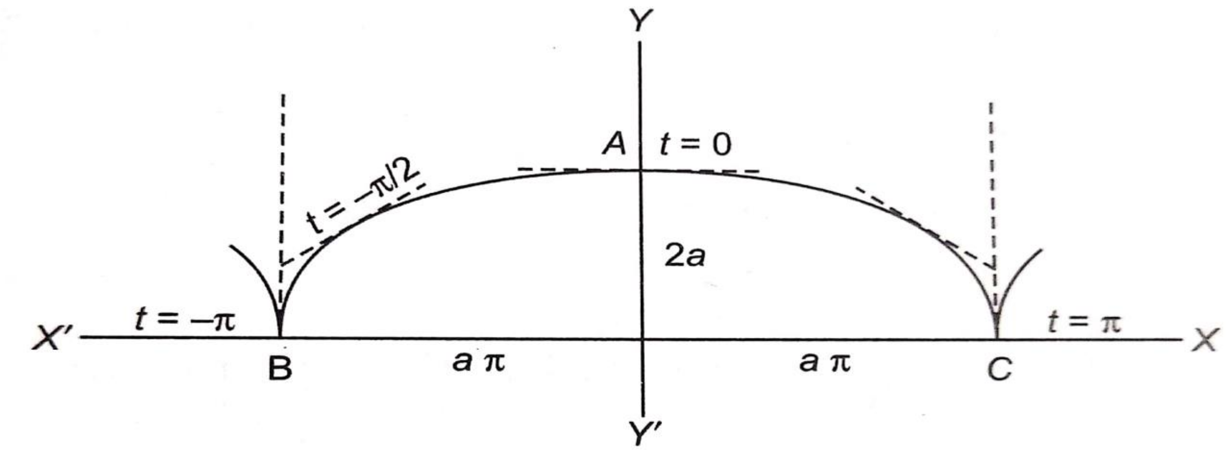
Point of intersection with y-axis is $(0, 2a)$.

3.Tangents: $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-a\sin t}{a(1 + \cos t)} = -\frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\cos^2\left(\frac{t}{2}\right)} = -\tan\left(\frac{t}{2}\right).$$

t	x	y	$\frac{dy}{dx}$	Conclusion
0	0	$2a$	0	Tangent at point $(0, 2a)$ is parallel to x-axis
$\frac{\pi}{2}$	$a\left(\frac{\pi}{2} + 1\right)$	a	-1	Tangent at point $\left(a\left(\frac{\pi}{2} + 1\right), a\right)$ makes obtuse angle with positive x-axis
π	$a\pi$	0	$-\infty$	Tangent at point ($a\pi$,0) is parallel to y-axis
$\frac{3\pi}{2}$	$a\left(\frac{3\pi}{2} - 1\right)$	a	1	Tangent at point $\left(a\left(\frac{3\pi}{2} - 1\right), a\right)$ makes an acute angle with positive x-axis
2π	$2a\pi$	$2a$	0	Tangent at point $(2a\pi, 2a)$ is parallel to x-axis

4.Existence of Curve:

Maximum value of $\cos t$ is 1 and minimum value of $\cos t$ is -1 .
Maximum value of y is $2a$ and minimum value of y is 0.
Therefore curve lies in between the lines $y = 0$ and $y = 2a$.



$x = a(t + \sin t),\ y = a(1 + \cos t)$

Exercise

Q. Trace the following curves

1) $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ 2) $x = t^2,\ y = t - \frac{t^3}{3}$ 3) $x = at,\ y = \frac{a}{t}$

4) $x = a(t + \sin t),\ y = a(1 + \cos t)$ 5) $x = a(t + \sin t), y = a(1 - \cos t)$

6) $x = a(t - \sin t), y = a(1 - \cos t)$ 7) $x = a\left[\cos t + \frac{1}{2}\log\left(\tan^2 \frac{t}{2}\right)\right], y = a\sin t$

Polar Curves

To trace the polar curve given by the equation $r = f(\theta)$.

1.**Pole** – Origin O(0,0) is called as pole.
2.**Initial Line** – Positive x –axis is called as initial line (i.e line corresponding to $\theta = 0$).
3. Line perpendicular to initial line ($\theta = 0$) is y – axis ($\theta = \frac{\pi}{2}$)
4. r is called radius vector.

I) Symmetry:

About	Condition
Pole (Origin)	All the powers of **r** are even in the equation of curve. e.g. $\mathbf{r^2 = a\cos 2\theta}$, $\mathbf{r^4 = a^2 \sin 3\theta}$
Initial Line ($\theta = 0$)	Equation of curve remains unchanged after replacement of $\boldsymbol{\theta}$ by $-\boldsymbol{\theta}$ in the equation of curve. e.g. $r = 2\cos\theta$, $r = a\sin^2\theta$
y – axis ($\theta = \frac{\pi}{2}$)	Equation of curve remains unchanged after replacement of **r** by **– r** and $\boldsymbol{\theta}$ by $-\boldsymbol{\theta}$ in the equation of curve. e.g. $\mathbf{r = 2\sin\theta}$, $\mathbf{r^2 = a\cos 2\theta}$ **OR** Equation of curve remains unchanged after replacement of $\boldsymbol{\theta}$ by $\boldsymbol{\pi - \theta}$ in the equation of curve. e.g. $\mathbf{r = 1 + \sin\theta}$

II) Pole:

If $r = 0$ for some value of θ, then curve passes through the pole.
e.g Curve $r = a\cos\theta$ passes through the pole as $\theta = \frac{\pi}{2} \Rightarrow r = 0$.

III) Tangent at Pole:

Put $\mathbf{r = 0}$ in the equation of curve and find $\boldsymbol{\theta}$.Values of $\boldsymbol{\theta}$ gives the tangent at the pole.
e.g. For the curve $\mathbf{r = a\sin 3\theta}$,

$\mathbf{r = 0 \Rightarrow \sin 3\theta = 0 \Longrightarrow 3\theta = 0, \pi, 2\pi, 3\pi, \ldots \Longrightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi}$, ...are tangents at pole.

IV)Angle ϕ between radius vector & tangent :

Find angle ϕ between radius vector & tangent using, $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$.

V)Formation of table:

Prepare a table of various values of θ, r and $\phi = \tan^{-1}\left(r\frac{d\theta}{dr}\right)$

Find the values of θ for which $\phi = 0$ or $\phi = \frac{\pi}{2}$
$\phi = 0$ for $\theta = \theta_1$ ⇒Tangent coincides with radius vector at $\theta = \theta_1$.
$\phi = \frac{\pi}{2}$ for $\theta = \theta_2$ ⇒ Tangent is perpendicular to radius vector at $\theta = \theta_2$.

VI) Asymptotes:

Find asymptotes if any.

VII) Region of absence of the curve

1) If $r^2 < 0$ for $\alpha < \theta < \beta$ then no branch of curve exists between lines $\theta = \alpha$ & $\theta = \beta$.
2) If maximum numerical value of r is a, then entire curve will lie within a circle of radius a.

Example 1: Trace the curve $r = a(1+\cos\theta)$.

Solution:

Symmetry: Equation of curve remains unchanged after replacement of θ by – θ in the equation of curve.**Therefore curve is symmetric about initial line $\theta = 0$.**

Pole: $\theta = \pi \Rightarrow r = 0 \Rightarrow$ Curve passes through the pole.

Tangent at pole:

$$r = 0 \Rightarrow a(1+\cos\theta) = 0 \Rightarrow 1+\cos\theta = 0 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$$

$\theta = \pi$ is the tangent to the curve at pole.

Angle ϕ between radius vector & tangent:

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(dr/d\theta\right)} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = -\cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

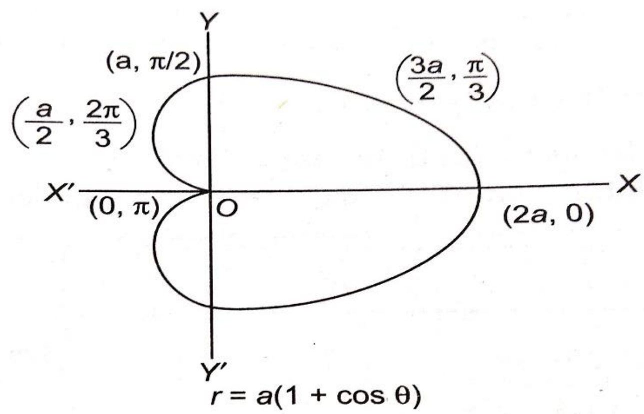
$$\therefore\ \phi = \frac{\pi}{2}+\frac{\theta}{2}.$$

Table:

$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\boldsymbol{r}$	$2a$	a	0	a	$2a$
ϕ	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$ **or** $-\frac{\pi}{2}$

From table we get the information that, as θ increases from 0 to π, r decreases from $2a$ to 0.
Also as θ increases from π to 2π, r increases from 0 to $2a$.
Maximum value of r is $2a$ and minimum value of r is 0.
Therefore entire curve will lie within a circle of radius $2a$.

At $\theta = 0$, $\phi = \frac{\pi}{2}$. Therefore tangent at $(2a, 0)$ is perpendicular to initial line $\theta = 0$.



Example 2: Trace the curve $r = a(1+\sin\theta)$.

Solution:

Symmetry: Equation of curve remains unchanged after replacement of **θ** by **π− θ** in the equation of curve.**Therefore curve is symmetric about the line θ =** $\frac{\pi}{2}$.

Pole: $\theta = \frac{3\pi}{2} \Rightarrow r = 0 \Rightarrow$ Curve passes through the pole.

Tangent at pole:

$$r = 0 \Rightarrow a(1+\sin\theta) = 0 \Rightarrow 1+\sin\theta = 0 \Rightarrow \sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}$$

$\theta = \frac{3\pi}{2}$ is the tangent to the curve at pole.

Angle ϕ between radius vector & tangent:

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(dr/d\theta\right)} = \frac{a(1+\sin\theta)}{a\cos\theta} = \frac{\sin^2\left(\frac{\theta}{2}\right)+\cos^2\left(\frac{\theta}{2}\right)+2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{\left[\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right]^2}{\left[\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right]\left[\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right]} = \frac{\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)} = \frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

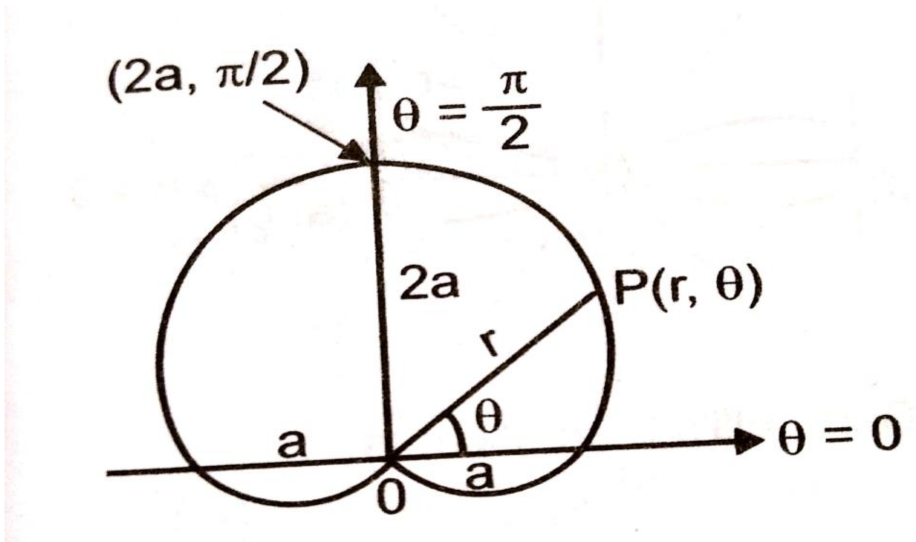
$$\therefore\ \phi = \frac{\pi}{4}+\frac{\theta}{2}.$$

Table:

$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\boldsymbol{r}$	a	$2a$	a	0	a
ϕ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$ **or** $-\frac{3\pi}{4}$

From table,we get the information that,maximum value of r is $2a$ and minimum value of r is 0. Therefore entire curve will lie within a circle of radius $2a$.

At $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$. Therefore tangent at $\left(2a, \frac{\pi}{2}\right)$ is perpendicular to line $\theta = \frac{\pi}{2}$ i.e. y-axis.



Example 3: Trace the curve $r^2 = a^2 \cos 2\theta$.

Solution:

Symmetry:

(1) Equation of curve remains unchanged after replacement of **θ** by **– θ** in the equation of curve. **Therefore curve is symmetric about initial line θ = 0.**

(2) All the powers of **r** are even in the equation of curve. **Therefore curve is symmetric about pole.**

(3) Equation of curve remains unchanged after replacement of **r** by **– r** and **θ** by **– θ** in the equation of curve. **Therefore curve is symmetric about the line θ =** $\frac{\pi}{2}$.

Pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4} \Rightarrow r = 0 \Rightarrow$ Curve passes through the pole.

Tangent at pole:

$$r = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots\ldots \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \ldots\ldots$$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \ldots\ldots$ are the tangents to the curve at pole.

Angle ϕ between radius vector & tangent:

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(dr/d\theta\right)} = \frac{a\sqrt{\cos 2\theta}}{a\left(-\frac{2\sin 2\theta}{2\sqrt{\cos 2\theta}}\right)} = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$\therefore \ \phi = \frac{\pi}{2} + 2\theta.$

Table:

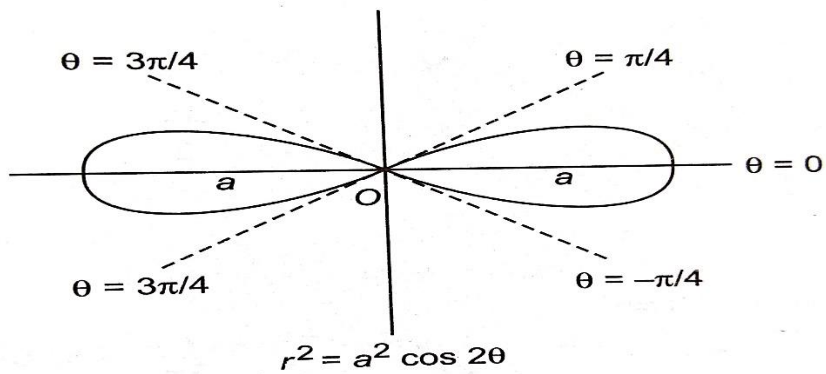
$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$ **or** $-\frac{\pi}{4}$	2π
$\boldsymbol{r}$	a	0	Imaginary	0	$-a$	0	Imaginary	0	a
ϕ	$\frac{\pi}{2}$	π	–	2π	$-\frac{\pi}{2}$	π	–	**0**	$\frac{\pi}{2}$

From table we get the information that,maximum value of r is a and minimum value of r is $-a$. Therefore entire curve will lie within a circle of radius a.

Note that, when $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ then $r^2 < 0$.Therefore no branch of curve exists for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

Similarly when $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$ then $r^2 < 0$.Therefore no branch of curve exists for $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$.

At $\theta = 0,\ \phi = \frac{\pi}{2}$. Therefore tangent at $(a,0)$is perpendicular to the initial line $\theta = 0$ i.e. x-axis.



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Example 4: Trace the curve $r^2 = a^2 \sin 2\theta$.

Solution:

Symmetry: All the powers of **r** are even in the equation of curve. **Therefore curve is symmetric about the pole.**

Pole: $\theta = 0, \frac{\pi}{2}, \pi \Rightarrow\ r = 0\ \Rightarrow$ Curve passes through the pole.

Tangent at pole:

$$r = 0\ \Rightarrow\ \sin 2\theta = 0 \Rightarrow 2\theta = 0, \pi, 2\pi\ ..... \Rightarrow \theta = 0, \frac{\pi}{2}, \pi, ...$$

$\theta = 0, \frac{\pi}{2}, \pi, ...$ are the tangents to the curve at pole.

Angle ϕ between radius vector & tangent:

$$\tan\phi = r\frac{d\theta}{dr} = r\frac{1}{\left(\frac{dr}{d\theta}\right)} = \frac{a\sqrt{\sin 2\theta}}{a\left(\frac{2\cos 2\theta}{2\sqrt{\sin 2\theta}}\right)} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

$\therefore\ \phi = 2\theta.$

Table:

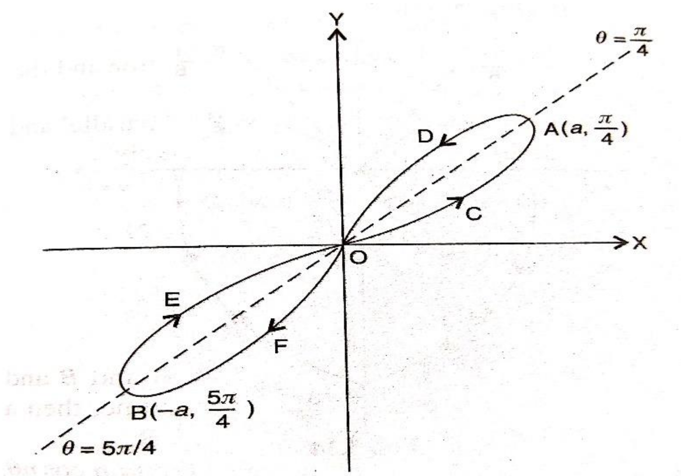
$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$ **or** $-\frac{\pi}{4}$	2π
$\boldsymbol{r}$	0	a	0	Imaginary	0	$-a$	0	Imaginary	0
ϕ	0	$\frac{\pi}{2}$	π	–	2π **or** 0	$\frac{5\pi}{2}$	π	–	2π **or** 0

From table we get the information that,maximum value of r is a.
Therefore entire curve will lie within a circle of radius a.

Note that, when $\frac{\pi}{2} < \theta < \pi$ then $r^2 < 0$.Therefore no branch of curve exists for $\frac{\pi}{2} < \theta < \pi$.

Similarly when $\frac{3\pi}{2} < \theta < 2\pi$ then $r^2 < 0$.Therefore no branch of curve exists for $\frac{3\pi}{2} < \theta < 2\pi$.

At $\theta = \frac{\pi}{4},\ \phi = \frac{\pi}{2}$. Therefore tangent at $A\left(a, \frac{\pi}{4}\right)$ is perpendicular to the corresponding radius vector.



Rose Curves

To trace the Curves of the type $r = a \cos n\theta$ or $r = a \sin n\theta$.

Limitation of the curve: $|\sin n\theta| \le 1$ & $|\cos n\theta| \le 1$

Therefore $0 \le r \le a$ .Hence entire curve lie within a circle of radius a. Draw a circle of radius a.

I) Symmetry:

Initial Line $(\boldsymbol{\theta = 0})$	Equation of Curve remains unchanged after replacement of $\boldsymbol{\theta}$ by $\boldsymbol{-\theta}$. e.g. $r = a \cos n\theta$
$\boldsymbol{y}$ **– axis** $(\boldsymbol{\theta = \frac{\pi}{2}})$	Equation of Curve remains unchanged after replacement of **r** by **– r** and $\boldsymbol{\theta}$ by $\boldsymbol{-\theta}$. e.g.$\mathbf{r = a \sin n}\theta$

II) Points of Intersection

Pole	If $r = 0$ some value of θ, then curve passes through the pole. e.g 1) For the curve $r = a \cos n\theta,\ r = 0\ when\ \theta = \frac{\pi}{2n}$ 2) For the curve $r = a \sin n\theta,\ r = 0\ when\ \theta = 0$ ∴ Both the curve passes through the pole.

III) Number of loops:

1) If n is even then there are 2n loops
2) If n is odd then there are n loops only.

IV) Drawing of loops:

Step1: Divide each quadrant into n equal parts by drawing $n - 1$ lines in each quadrant.

Draw the lines as $\theta = \frac{\pi}{2n}, \theta = \frac{2\pi}{2n}, \theta = \frac{3\pi}{2n}, \theta = \frac{4\pi}{2n}, \ldots.$

Step2: a) If n is even then draw loops in two sectors consecutively from $\theta = 0$ to $\theta = 2\pi$.

b) If n is odd then draw loops in two sectors alternatively keeping two sectors between the loops vacant.

Step3 : a) If $r = a \cos n\theta$ then draw the first loop on the initial line.

b) If $r = a \sin n\theta$ then draw the first loop on the line $\theta = \frac{\pi}{2n}$.

Example 1: Trace the curve $r = a\sin 2\theta$.

Solution:

Loops: Since $n=2$, curve consists of $2n=4$ equal and equidistant loops.

Symmetry: Equation of curve remains unchanged after replacement of **r** by **– r** and **θ** by **– θ** in the equation of curve.**Therefore curve is symmetric about line** $\theta=\frac{\pi}{2}$ **i.e. y-axis.**

Pole: $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \Rightarrow r=0 \Rightarrow$ Curve passes through the pole.

Tangent at pole:

$$r=0 \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \Rightarrow \theta = 0, \frac{\pi}{4}, \frac{2\pi}{4}=\frac{\pi}{2}, \frac{3\pi}{4}$$

$\theta = 0, \frac{\pi}{4}, \frac{2\pi}{4}=\frac{\pi}{2}, \frac{3\pi}{4}$ are the tangents to the curve at pole.

Angle ϕ between radius vector & tangent:

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{a\sin 2\theta}{2a\cos 2\theta} = \frac{1}{2}\tan 2\theta$$

(1) $\tan\phi = 0$ when $\tan 2\theta = 0 \Rightarrow 2\theta = 0, \pi, 2\pi, 3\pi, ... \Rightarrow \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, ...$

$\therefore \phi = 0$ when $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, ... \Rightarrow$ Tangents are coincident with radius vectors when

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, ...$$

(2) $\tan\phi$ is infinite when

$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \pm\infty \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, ... \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, ...$

$\therefore \phi = \pm\frac{\pi}{2}$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \Rightarrow$ Tangents are perpendicular to radius vectors when

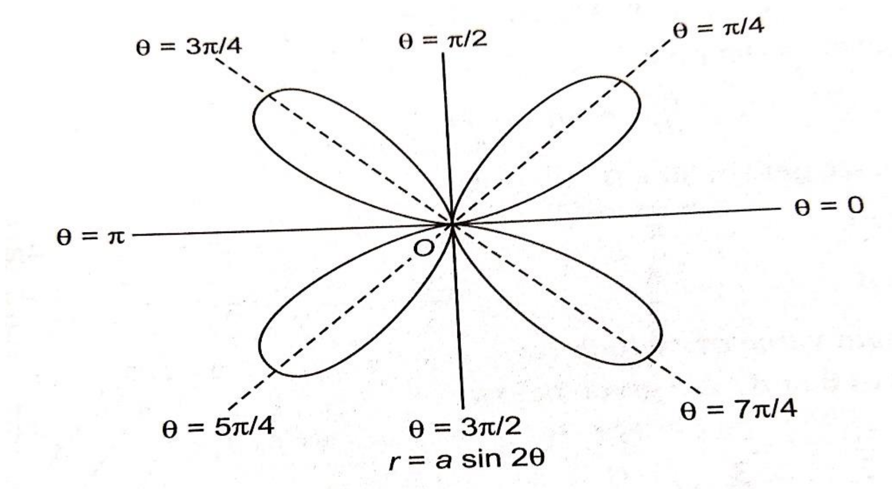
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

Table:

$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\boldsymbol{r}$	0	a	0	$-a$	0	a	0	$-a$	0

From table we get the information that,maximum value of r is a and minimum value of r is $-a$.
Therefore entire curve will lie within a circle of radius a.

Draw first loop on line $\theta = \frac{\pi}{2n} = \frac{\pi}{4}$. Then draw remaining three loops by making distance of $\frac{\pi}{2}$ radians between two consecutive loops.



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Example 2: Trace the curve $r = a\cos 2\theta$.

Solution:

Loops: Since $n=2$, curve consists of $2n=4$ equal and equidistant loops.

Symmetry: Equation of curve remains unchanged after replacement of **θ** by **– θ** in the equation of curve. **Therefore curve is symmetric about initial line θ = 0.**

Pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \Rightarrow r = 0 \Rightarrow$ Curve passes through the pole.

Tangent at pole:

$$r = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots\ldots \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \ldots\ldots$$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \ldots\ldots$ are the tangents to the curve at pole.

Angle ϕ between radius vector & tangent:

$$\tan\phi = r\frac{d\theta}{dr} = r\frac{1}{\left(dr/d\theta\right)} = \frac{a\cos 2\theta}{-2a\sin 2\theta} = -\frac{\cot 2\theta}{2} = \frac{1}{2}\tan\left(\frac{\pi}{2} + 2\theta\right)$$

(1) $\tan\phi = 0$ when

$$\tan\left(\frac{\pi}{2} + 2\theta\right) = 0 \Rightarrow \frac{\pi}{2} + 2\theta = 0, \pi, 2\pi, 3\pi, \ldots \Rightarrow 2\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots \Rightarrow \theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$$

$\therefore\ \phi = 0$ when $\theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots \Rightarrow$ Tangents are coincident with radius vectors when

$$\theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$$

(2) $\tan\phi$ is infinite when

$$\tan\left(\frac{\pi}{2}+2\theta\right)=\frac{\sin\left(\frac{\pi}{2}+2\theta\right)}{\cos\left(\frac{\pi}{2}+2\theta\right)}=\pm\infty \Rightarrow \cos\left(\frac{\pi}{2}+2\theta\right)=0 \Rightarrow \frac{\pi}{2}+2\theta=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},.....$$

$$\Rightarrow 2\theta=0,\frac{\pi}{2},2\pi,3\pi,...\Rightarrow \theta=0,\frac{\pi}{4},\pi,\frac{3\pi}{2},...$$

$\therefore\ \phi=\pm\frac{\pi}{2}$ when $\theta=0,\frac{\pi}{4},\pi,\frac{3\pi}{2},...\Rightarrow$ Tangents are perpendicular to radius vectors when

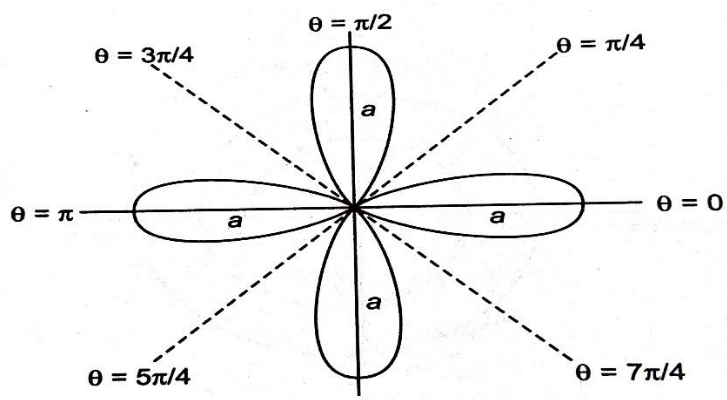
$$\theta=0,\frac{\pi}{4},\pi,\frac{3\pi}{2},...$$

Table:

$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$ **or** $-\frac{\pi}{4}$	2π
$\boldsymbol{r}$	a	0	$-a$	0	a	0	$-a$	0	a

From table we get the information that,maximum value of r is a and minimum value of r is $-a$. Therefore entire curve will lie within a circle of radius a.

Draw first loop on line $\theta=0$. Then draw remaining three loops by making distance of $\frac{\pi}{2}$ radians between two consecutive loops.



Example 3: Trace the curve $r = a\sin 3\theta$.

Solution:

Loops: Since $n=3$, curve consists of $n=3$ equal and equidistant loops.

Symmetry: Equation of curve remains unchanged after replacement of **r** by **– r** and **θ** by **– θ** in the equation of curve. **Therefore curve is symmetric about line** $\theta = \frac{\pi}{2}$ **i.e. y-axis.**

Pole: $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3} \Rightarrow r = 0 \Rightarrow$ Curve passes through the pole.

Tangent at pole:

$$r = 0 \Rightarrow \sin 3\theta = 0 \Rightarrow 3\theta = 0, \pi, 2\pi, \ldots \Rightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots\ldots$$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots\ldots$ are the tangents to the curve at pole.

Angle ϕ between radius vector & tangent:

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(dr/d\theta\right)} = \frac{a\sin 3\theta}{3a\cos 3\theta} = \frac{1}{3}\tan 3\theta$$

(1) $\tan\phi = 0$ when $\tan 3\theta = 0 \Rightarrow 3\theta = 0, \pi, 2\pi, \ldots\ldots \Rightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3} \ldots\ldots$

$\therefore \phi = 0$ when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots\ldots \Rightarrow$ Tangents are coincident with radius vectors at points $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots\ldots\ldots.$

(2) $\tan\phi = \pm\infty$ when $\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots\ldots \Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \ldots\ldots$

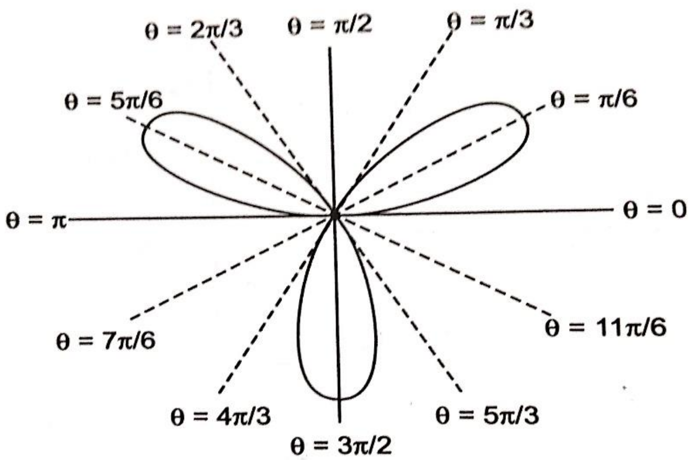
$\therefore \phi = \pm\frac{\pi}{2}$ when $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \ldots\ldots\ldots \Rightarrow$ Tangents are perpendicular to radius vectors at points $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \ldots\ldots\ldots$

Table:

$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}=\frac{\pi}{3}$	$\frac{3\pi}{6}=\frac{\pi}{2}$	$\frac{4\pi}{6}=\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}=\frac{4\pi}{3}$	$\frac{9\pi}{6}=\frac{3\pi}{2}$	$\frac{10\pi}{6}=\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\boldsymbol{r}$	0	a	0	$-a$	0	a	0	$-a$	0	a	0	$-a$	0

From table we get the information that,maximum value of r is a and minimum value of r is $-a$. Therefore entire curve will lie within a circle of radius a.

Draw first loop on line $\theta = \frac{\pi}{2n} = \frac{\pi}{6}$. Then draw remaining two loop by making distance of $\frac{2\pi}{3}$ radians between two consecutive loops.



Example 4: Trace the curve $r = a\cos 3\theta$.

Solution:

Loops: Since $n = 3$, curve consists of $n = 3$ equal and equidistant loops.

Symmetry: Equation of curve remains unchanged after replacement of **θ** by **– θ** in the equation of curve. **Therefore curve is symmetric about initial line θ = 0.**

Pole: $\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \Rightarrow r = 0 \Rightarrow$ Curve passes through the pole.

Tangent at pole:

$$r = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} ..... \Rightarrow \theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} .....$$

$\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ ..... are the tangents to the curve at pole.

Angle ϕ between radius vector & tangent:

$$\tan\phi = r\frac{d\theta}{dr} = r\frac{1}{\left(dr/d\theta\right)} = \frac{a\cos 3\theta}{-3a\sin 3\theta} = -\frac{\cot 3\theta}{3} = \frac{1}{3}\tan\left(\frac{\pi}{2} + 3\theta\right)$$

(1) $\tan\phi = 0$ when

$$\tan\left(\frac{\pi}{2} + 3\theta\right) = 0 \Rightarrow \frac{\pi}{2} + 3\theta = 0, \pi, 2\pi, 3\pi, ... \Rightarrow 3\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} .... \Rightarrow \theta = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, ...$$

$\therefore\ \phi = 0$ when $\theta = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, ... \Rightarrow$ Tangents are coincident with radius vectors when

$$\theta = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, ...$$

(2) $\tan\phi$ is infinite when

$$\tan\left(\frac{\pi}{2}+3\theta\right)=\frac{\sin\left(\frac{\pi}{2}+3\theta\right)}{\cos\left(\frac{\pi}{2}+3\theta\right)}=\pm\infty \Rightarrow \cos\left(\frac{\pi}{2}+3\theta\right)=0 \Rightarrow \frac{\pi}{2}+3\theta=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},.....$$

$$\Rightarrow 3\theta=0,\frac{\pi}{2},2\pi,3\pi,...\Rightarrow \theta=0,\frac{\pi}{6},\frac{2\pi}{3},\pi,...$$

$\therefore\ \phi=\pm\frac{\pi}{2}$ when $\theta=0,\frac{\pi}{6},\frac{2\pi}{3},\pi,...\Rightarrow$ Tangents are perpendicular to radius vectors when

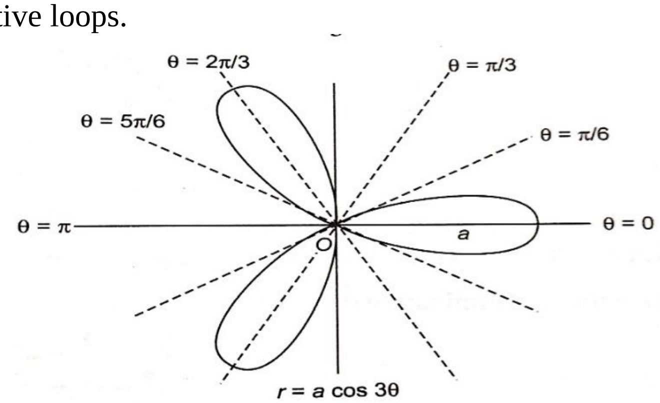
$$\theta=0,\frac{\pi}{6},\frac{2\pi}{3},\pi,...$$

Table:

$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}=\frac{\pi}{3}$	$\frac{3\pi}{6}=\frac{\pi}{2}$	$\frac{4\pi}{6}=\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}=\frac{4\pi}{3}$	$\frac{9\pi}{6}=\frac{3\pi}{2}$	$\frac{10\pi}{6}=\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\boldsymbol{r}$	a	0	$-a$	0	a	0	$-a$	0	a	0	$-a$	0	a

From table we get the information that,maximum value of r is a and minimum value of r is $-a$. Therefore entire curve will lie within a circle of radius a.

Draw first loop on line $\theta=0$. Then draw remaining two loop by making distance of $\frac{2\pi}{3}$ radians between two consecutive loops.



Exercise

Trace the following curves.

1) $r=a(1-\cos\theta)$ 2) $r=a(1-\sin\theta)$ 3) $r=a(1+2\cos\theta)$ 4) $r=a\left(\sqrt{3}+2\cos\theta\right)$

5) $r=a\sin\theta$ 6) $r=a\cos\theta$ 7) $r^2\sin 2\theta=a^2$ 8) $r^2\cos 2\theta=a^2$

9) $r=a\sin 4\theta$ 10) $r=a\cos 4\theta$ 11) $r=a\sin 5\theta$ 12) $r=a\cos 5\theta$

Rectification of curves

The process to find the lengths of arcs of plane curves whose equations are given is known as *Rectification*.
There are three types of curves. Depending on type, formulae for arc length are given as follows.

1) **Cartesian Equations** $y = f(x)$:
The length of the arc of the curve $y = f(x)$ included between $x = a$ and $x = b$ is given by

$$l = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

2) **Cartesian Equations** $x = g(y)$:
The length of the arc of the curve $x = g(y)$ included between $y = c$ and $y = d$ is given by

$$l = \int_{y=c}^{y=d} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

3) **Parametric Equations**:
The length of the arc of the curve $x = f(t)$, $y = g(t)$ included between $t = t_1$ and $t = t_2$ is given by

$$l = \int_{t=t_1}^{t=t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

4) **Polar Equations** $r = f(\theta)$:
The length of the arc of the curve $r = f(\theta)$ included between $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$l = \int_{\theta=\theta_1}^{\theta=\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

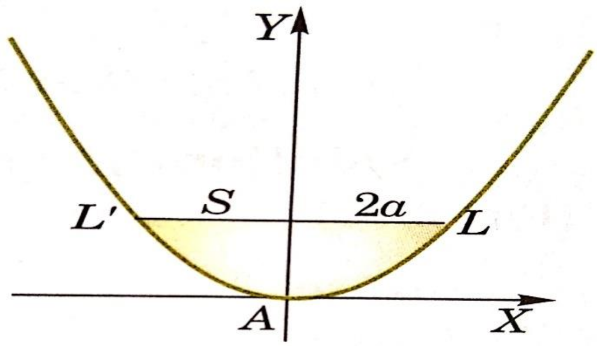
5) **Polar Equations** $\theta = g(r)$:
The length of the arc of the curve $\theta = g(r)$ included between $r = r_1$ and $r = r_2$ is given by

$$l = \int_{r=r_1}^{r=r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr$$

Solved Examples:

Example 1: Find the length of the arc of the parabola $x^2 = 4ay$ measured from vertex to one extremity of the latus rectum.

Solution: Let A be the vertex and L an extremity of the latus-rectum so that at A, x=0 and at L, x=2a .



Length of the arc AL $= \int_{x=0}^{x=2a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

Now differentiating the equation of curve w.r.t.x,

$$x^2 = 4ay \Rightarrow y = \frac{x^2}{4a} \Rightarrow \frac{dy}{dx} = \frac{x}{2a}$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{4a^2} = \frac{4a^2+x^2}{4a^2}$$

$$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{4a^2+x^2}}{2a}$$

Length of the arc AL

$$= \int_{x=0}^{x=2a} \frac{\sqrt{4a^2+x^2}}{2a} dx = \frac{1}{2a}\left[\frac{x}{2}\sqrt{4a^2+x^2} + \frac{4a^2}{2}\log\left(x+\sqrt{4a^2+x^2}\right)\right]_0^{2a}$$

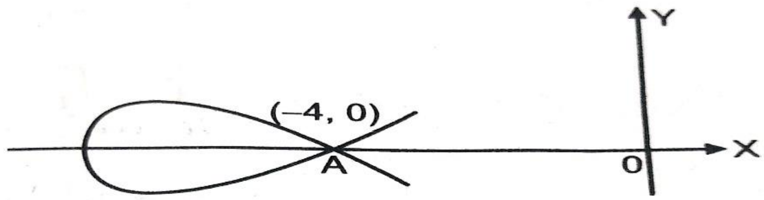
$$= \frac{1}{2a}\left\{\left[\frac{2a}{2}\sqrt{4a^2+4a^2} + 2a^2\log\left(2a+\sqrt{4a^2+4a^2}\right)\right] - \left[0 - 2a^2\log\left(\sqrt{4a^2}\right)\right]\right\}$$

$$= \frac{1}{2a}\left\{\left[2\sqrt{2}a^2 + 2a^2\log\left(\frac{2a+2\sqrt{2}a}{2a}\right)\right]\right\}$$

$$= a\left[\sqrt{2} + \log(1+\sqrt{2}\right]$$

Example 2: Find the whole length of the loop of the curve $9y^2=(x+7)(x+4)^2$.

Solution: Given curve is symmetric about x axis.Also it passes through the points $(-7,0)$ & $(-4,0)$. Thus loop of the curve is around x axis between $x=-7$ & $x=-4$.



Whole length of the loop $=2\times$ length of upper arc of loop

$$=2\int_{x=-7}^{x=-4}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Now differentiating the equation of curve w.r.t.x,

$$9y^2=(x+7)(x+4)^2 \Rightarrow 18y\frac{dy}{dx}=(x+4)^2+2(x+4)(x+7)=(x+4)(x+4+2x+14)=(x+4)(3x+18)$$

$$\Rightarrow \frac{dy}{dx}=\frac{(x+4)(3x+18)}{18y}=\frac{(x+4)(x+6)}{6y}$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2=1+\frac{(x+4)^2(x+6)^2}{36y^2}=1+\frac{(x+4)^2(x+6)^2}{4(x+7)(x+4)^2}=1+\frac{(x+6)^2}{4(x+7)}=\frac{4x+28+x^2+12x+36}{4(x+7)}$$

$$=\frac{x^2+16x+64}{4(x+7)}=\frac{(x+8)^2}{4(x+7)}.$$

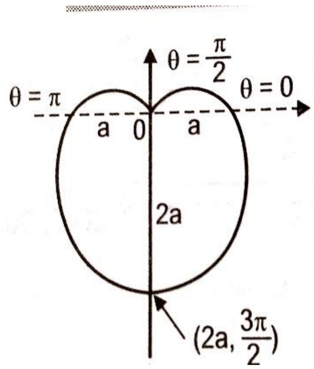
Whole length of the loop

$$=2\int_{x=-7}^{x=-4}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=2\int_{-7}^{-4}\sqrt{\frac{(x+8)^2}{4(x+7)}}dx=\int_{-7}^{-4}\frac{x+8}{\sqrt{x+7}}dx=\int_{-7}^{-4}\frac{(x+7)+1}{\sqrt{x+7}}dx$$

$$\int_{-7}^{-4}\left(\sqrt{x+7}+\frac{1}{\sqrt{x+7}}\right)dx=\left(\frac{(x+7)^{3/2}}{3/2}+2(x+7)^{1/2}\right)_{-7}^{-4}=\frac{2}{3}3\sqrt{3}+2\sqrt{3}=4\sqrt{3}$$

Example 3: Find the perimeterof the cardiode $r = a(1-\sin\theta)$.

Solution:



Perimeter = $2\times$length of left arc of cardiode $= 2\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

$$r = a(1-\sin\theta) \Rightarrow \frac{dr}{d\theta} = -a\cos\theta.$$

$$\therefore\ r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1-\sin\theta)^2 + a^2\cos^2\theta = a^2(1-2\sin\theta+\sin^2\theta+\cos^2\theta)$$

$$= a^2(1-2\sin\theta+1)$$

$$= a^2(2-2\sin\theta)$$

$$= 2a^2(1-\sin\theta)$$

$$= 2a^2\left[\sin^2\left(\frac{\theta}{2}\right)+\cos^2\left(\frac{\theta}{2}\right)-2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]$$

$$= 2a^2\left[\sin\left(\frac{\theta}{2}\right)-\cos\left(\frac{\theta}{2}\right)\right]^2$$

$$\therefore\ \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{2a^2\left[\sin\left(\frac{\theta}{2}\right)-\cos\left(\frac{\theta}{2}\right)\right]^2} = \sqrt{2}\ a\left[\sin\left(\frac{\theta}{2}\right)-\cos\left(\frac{\theta}{2}\right)\right]$$

$$\therefore\ \ Perimeter = 2\times\int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = 2\int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}\sqrt{2}\ a\left[\sin\left(\frac{\theta}{2}\right)-\cos\left(\frac{\theta}{2}\right)\right]d\theta$$

$$= 2\sqrt{2}\ a\int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}\left[\sin\left(\frac{\theta}{2}\right)-\cos\left(\frac{\theta}{2}\right)\right]d\theta$$

$$= 2\sqrt{2}\ a\left[-2\cos\left(\frac{\theta}{2}\right)-2\sin\left(\frac{\theta}{2}\right)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= -4\sqrt{2}\ a\left[\cos\left(\frac{3\pi}{4}\right)+\sin\left(\frac{3\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\right]$$

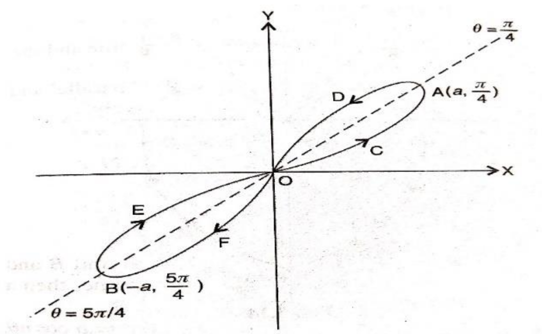
$$= -4\sqrt{2}\ a\left[\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right]$$

$$= -4\sqrt{2}\ a\left[-2\frac{1}{\sqrt{2}}\right]$$

$$= 8a$$

Example 4: Find the length of one loop of the Lemniscate $r^2 = a^2 \sin 2\theta$.

Solution:



Length of one loop, $l = \int_{\theta=0}^{\theta=\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

$$r^2 = a^2 \sin 2\theta \Rightarrow r = a\sqrt{\sin 2\theta} \Rightarrow \frac{dr}{d\theta} = \frac{2a\cos 2\theta}{2\sqrt{\sin 2\theta}} = \frac{a\cos 2\theta}{\sqrt{\sin 2\theta}}.$$

$$\therefore\ r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 \sin 2\theta + \frac{a^2\cos^2 2\theta}{\sin 2\theta} = \frac{a^2\sin^2 2\theta + a^2\cos^2 2\theta}{\sin 2\theta} = \frac{a^2}{\sin 2\theta}$$

$$\therefore\ \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \frac{a}{\sqrt{\sin 2\theta}}$$

$$\therefore\ \ l = 2\times \int_{\theta=0}^{\theta=\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 2\int_{\theta=0}^{\theta=\pi/4} \frac{a}{\sqrt{\sin 2\theta}} d\theta$$

$$Put\ \ 2\theta = t \Rightarrow d\theta = \frac{dt}{2}.$$

$$\theta = 0 \Rightarrow t = 0\ \ \&\ \theta = \frac{\pi}{4} \Rightarrow t = \frac{\pi}{2}.$$

$$\therefore\ \ l = 2\int_{t=0}^{t=\pi/2} \frac{a}{\sqrt{\sin t}}\frac{dt}{2} = a\int_0^{\pi/2} \sin^{-1/2} t \cos^0 t\, dt = a\frac{1}{2}B\left(\frac{-\frac{1}{2}+1}{2}, \frac{0+1}{2}\right) = \frac{a}{2}B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{a}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$= \frac{a}{2}\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)} = \frac{a}{2}\sqrt{\pi}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \frac{a}{2}\sqrt{\pi}\frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\left(\pi \big/ \sin\left(\frac{\pi}{4}\right)\right)} = \frac{a}{2\sqrt{\pi}}\frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\left(\frac{1}{1/\sqrt{2}}\right)} = \frac{a}{2\sqrt{2}\sqrt{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2$$

Example 5: Find the length of the arc of the curve $x = e^t \cos t$, $y = e^t \sin t$ from $t = 0$ to $t = \dfrac{\pi}{2}$.

Solution:

The length of the arc of the curve $x = e^t \cos t$, $y = e^t \sin t$ from $t = 0$ to $t = \dfrac{\pi}{2}$ is

given by $l = \displaystyle\int_{t=0}^{t=\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$x = e^t \cos t \Rightarrow \frac{dx}{dt} = e^t \cos t - e^t \sin t$$

$$y = e^t \sin t \Rightarrow \frac{dy}{dt} = e^t \sin t + e^t \cos t$$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2$$

$$= e^{2t}(\cos^2 t + \sin^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\sin t \cos t)$$

$$= 2e^{2t}$$

$$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2}e^t$$

$$l = \int_{t=0}^{t=\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t=0}^{t=\frac{\pi}{2}} \sqrt{2}e^t dt = \sqrt{2}\left[e^t\right]_0^{\frac{\pi}{2}} = \sqrt{2}\left(e^{\frac{\pi}{2}} - 1\right)$$

Exercise:

1) Find perimeter of the circle $x^2 + y^2 = a^2$. (**Ans**: $2\pi a$)

2) Show that whole length of the loop of the curve $3y^2 = x(x-1)^2$ is $\dfrac{4}{\sqrt{3}}$.

3) Show that length of the arc of the parabola $y^2 = 4x$ cut off by the line $3y = 8x$ is $\log 2 + \dfrac{15}{16}$.

4) Find the length of the arc of the cycloid $x = a\left(t - \sin t\right)$, $y = a\left(1 - \cos t\right)$ between two consecutive cusps. (**Ans**: 8a)

5) Find the length of the loop of the curve $x = t^2$, $y = t - \dfrac{t^3}{3}$ (**Ans**: $4\sqrt{3}$)

6) Find the length of the arc of the curve $x = a\left(\cos t + t\sin t\right)$, $y = a\left(\sin t - t\cos t\right)$ from $t = 0$ to $t = 2\pi$ (**Ans**: $2\pi^2 a$)

7) Find the total length of the of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (**Ans**: 6a)

8) Find the length of the of the arc of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ in the positive quadrant

Ans: $\dfrac{a^2 + ab + b^2}{a+b}$

9) Show that the length of the arc of the tractrix $x = a\left(\cos t + \log\tan\left(\frac{t}{2}\right)\right)$, $y = a\sin t$ from

$t = \dfrac{\pi}{2}$ to any point t is $a\log\sin t$.

10) Find the length of the arc of the cardioide $r = a(1+\cos\theta)$ which lies outside the circle $r + a\cos\theta = 0$ *i.e.* $r = -a\cos\theta$. (**Ans :** $4\sqrt{3}\, a$)

11) Find the length of the arc of the cardioide $r = a(1+\sin\theta)$ which lies inside the circle $r + a\sin\theta = 0$. (**Ans :** $4a(2-\sqrt{3})$)

12) Find the length of the upper arc of one loop of Lemniscate $r^2 = a^2\cos 2\theta$.

(**Ans :** $\dfrac{a}{4}\dfrac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$)

13) Find perimeter of the cardioide $r = a(1-\cos\theta)$. (**Ans :** $8\,a$)