Bansilal Ramnath Agarwal Charitable Trust,
Vishwakarma Institute of Information Technology, Pune – 48
Department: Engineering and Applied Sciences

F.Y.B.Tech

Course material (A brief reference version for students)

UNIT III

Reduction Formulae, Beta & Gamma Functions

1. Introduction:

   Many functions occur whose integrals are not immediately reducible to one or other standard forms and whose integrals are not directly obtainable. In some cases however, such integrals may be linearly connected by some algebraic formulae with the integral of another expression, which itself may be either immediately integrable or easier to integrate than the original function.

   For example $\int (a^2 + x^2)^{5/2}$ dx may be connected with $\int (a^2 + x^2)^{3/2} dx$ and this latter may be expressed in terms of $\int (a^2 + x^2)^{1/2}$

   Which is a standard form?

Similarly $\int \sin^n xdx$ may be ultimately connected with $\int \sin^2 x$ dx or $\int \sin x$ dx depending upon whether n is even or odd integer. Many such examples can b cited.

An algebraic relation connecting two integrals is termed as Reduction formula.

2. Reduction Formulae For Sinusoidal Function

   1. To find a reduction formula for $\int \sin^n x$ dx, where n is a positive integer ≥ 2 and to evaluate completely $\int_0^{\pi/2} \sin nx.dx$

Let $I_n = \int \sin^n x \text{ dx} = \int \sin^{n-1} x.\sin \text{x dx}$

$$I_n = \sin^{n-1} x.(-\cos x) - \int (n-1)\sin^{n-2} x.\cos x(-\cos x)dx$$

$$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x.(1-\sin^2 x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} xdx - (n-1)\int \sin^n xdx$$

$$I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n + (n-1)I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

$$nI_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

$$I_n = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}I_{n-2}$$

Thus the required reduction formula is

$$\int \sin^n x \text{ dx} = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x$$

From (1),

$$\int_0^{\pi/2} \sin^n x \text{ dx} = \left[-\frac{1}{n}\sin^{n-1}x\cos x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2} xdx$$

$$= \frac{n-1}{n}I_{n-2}$$

Now, $$\boxed{I_n = \frac{n-1}{n}I_{n-2}}$$

Changing n to n-2 in equation (2) successively we have,

$$I_{n-2} = \frac{n-3}{n-2}I_{n-4} : I_{n-4} = \frac{n-5}{n-4}I_{n-6}$$

$\therefore$ $I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}I_{n-6}$ and so on.

Now, consider two cases.

Case I: Let n be a positive even integer

If n is an even integer, putting n=4, in equation (2) we get,

$$I_4 = \frac{3}{4} I_2; \text{ Similarly } I_2 = \frac{1}{2}.I_0$$

$$I_0 = \int_0^{\pi/2} \sin^0 x.dx = \frac{\pi}{2}$$

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.............\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

Case II: Let n be an odd positive integer

Put n=5 in equation (2)

$$I_5 = \frac{4}{5} I_3; I_3 = \frac{2}{3}.I1$$

$$I_1 = \int \sin x dx = [-\cos x] \int_0^{\pi/2} = 1$$

$$I_n = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4}..............\frac{4}{5}.\frac{2}{3}.1$$

$$\boxed{\begin{aligned} \text{Hence } \int_0^{\pi/2} \sin^n x dx &= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...........\frac{3}{4}\frac{1}{2}\frac{\pi}{2}; \text{ if n is even} \\ &= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}..............\frac{4}{5}.\frac{2}{3}.1, \text{ if n is odd} \end{aligned}}$$

Note: Using the property $\int_0^a f(x) = \int_0^a f(a-x)dx$

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \sin n\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \cos^n x.dx$$

From equation (3)

$$\boxed{\begin{aligned} \int_0^{\pi/2} \sin^n x dx &= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...........\frac{3}{4}.\frac{1}{2}\frac{\pi}{2}.1; \text{ if n is even} \\ &= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}..............\frac{4}{5}.\frac{2}{3}.1, \quad \text{if n is odd} \end{aligned}}$$

For example,

$$\int_0^{\pi/2} \sin^9 xdx = \frac{8}{9}.\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1 = \frac{128}{315}$$

$$\int_0^{\pi/2} \cos^6 xdx = \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{5\pi}{32}$$

Now to evaluate $\int_0^a \sqrt{a^2 - x^2}\,dx,$

$$I = \int_0^a \sqrt{a^2 - x^2}\,dx \qquad \text{put } x = a\sin\theta \ \therefore dx = a\cos\theta d\theta$$

$$= \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin 2\theta}.a\cos\theta d\theta$$

$$= a^2 \int_0^{\pi/2} \cos^2\theta d\theta = a2.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi a^2}{4}$$

x	0	a
θ	0	$\pi/2$

3. REDUCTION FORMULAE FOR $\int \sin^n$ x dx or $\int \cos^n$ x dx BETWEEN THE LIMITS 0 TO π OR 0 to 2π :

1. Let $\quad In = \int_0^x \sin^n xdx$

By using the property of definite integral, $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$$I_n = \int_0^{\pi/2} \sin^n xdx + \int \sin^n(\pi - x)dx$$

$$= \int_0^{\pi/2} \sin^n xdx + \int_0^{\pi/2} \sin^n xdx (\because \sin(\pi - x) = \sin x)$$

$$= 2\int_0^{\pi/2} \sin^n xdx$$

Hence, $\int_0^x \sin^n xdx = 2\int_0^{\pi/2} \sin^n xdx$, for all n integral values of n.

2. Let $I_n = \int_0^{\pi} \cos nx dx$

$$= \int_0^{\pi/2} \cos nx dx + \int_0^{\pi/2} \cos^n (\pi - x) dx$$

But $\cos(\pi - x) = -\cos x$

$$= \int_0^{\pi/2} \cos^n x dx - \int_0^{\pi/2} \cos^n x dx,$$

$$I_n = 0$$

Also form equation (A)

$$I_n = \int_0^{\pi/2} \cos^n x dx + \int_0^{\pi/2} \cos^n x dx$$

, if n is even

$$I_n = 2\int_0^{\pi/2} \cos^n x dx$$

Hence, $\int_0^{\pi/2} \cos nx dx = 2\int_0^{\pi/2} \cos nx dx$ if n is even

$= 0$ if n is odd

3. Let $I_n = \int_0^{2\pi} \sin^n x dx$

$$= \int_0^{\pi} \sin^n x dx + \int_0^{\pi} \sin^n (2\pi - x) dx$$

But $\sin(2\pi - x) = -\sin x$

$$= \int_0^{\pi} \sin nx dx - \int_0^{\pi} \sin^n x dx, \text{ if n is odd}$$

$$= 0$$

Also from equation (B),

$$I_n = \int_0^{\pi} \sin^n x dx + \int_0^{\pi} \sin^n x dx, \text{ if n is even}$$

$$= 2\int_0^{\pi/2} \sin^n x dx = 4\int_0^{\pi/2} \sin^n x dx$$

By using result (1)

Hence, $\int_0^{2\pi} \sin^n x dx = 4\int_0^{\pi/2} \sin^n x dx$, if n is even

$= 0$, if n is odd

4. Similarly, by using method used in result (3), we get

$$\int_0^{2\pi} \cos^n x dx = 4\int_0^{\pi/2} \cos^n x dx \text{, if n is even}$$
$$= 0 \text{, if n is odd}$$

5. To find a reduction formula for $\int \sin^m x \cos^n x dx$, where m and n are positive integers ≥ 2 and to completely evaluate $\int \sin^m x \cos^n x.dx$

Let $$I_{m,n} = \int \sin^m x \cos^n x dx$$

$$= \int \sin^m x \cos^{n-1} x.\cos x.dx$$

$$= \int \cos^{n-1} x.(\sin^m x.\cos x).dx$$

Note that $\int \sin^m x \cos x dx = \frac{\sin^{m+1} x}{m+1}$ $\quad \because \int [f(x)]^m . f(x) dx = \frac{[f(x)]^{m+1}}{m+1}$

Now applying integration by parts,

$$I_{m,n} = \cos^{n-1} x.\frac{\sin^{m+1} x}{m+1} - \int (n-1)\cos^{n-2} x(-\sin x)\frac{\sin^{m+1} x}{m+1} dx.$$

$$I_{m,n} = \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x.\cos^{n-2} xdx$$

$$= \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x.\sin^2 x \cos^{n-2} x.dx$$

$$= \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x.(1-\cos^2 x)\cos^{n-2} x.dx$$

$$= \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} xdx - \frac{n-1}{m+1}\int \sin^m x \cos^n xdx$$

$$I_{m,n} = \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n}\left(\frac{m+1+n-1}{m+1}\right) = \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} + \frac{n-1}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n} = \frac{\cos^{n-1} x.\sin^{m+1} x}{m+1} + \frac{n-1}{m+1} + \frac{n-1}{m+n} I_{m,n-2}$$

$$\int \sin^m x \cos x dx = \frac{\cos^{n-1} x.\sin^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin mx.\cos^{n-2} x.dx$$

Which is the require reduction formula.

From equation (1)

$$\int_0^{\pi/2} \sin^m x \cos^n xdx = \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x.\cos^{n-2} xdx$$

$$= 0 + \frac{n-1}{m+n} I_{m,n-2}$$

$$\boxed{I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}}$$

Replace n by n-2 in equation (2)

$$I_{m,n-2} = \frac{n-3}{m+n-2} I_{m,n-4};$$

$$I_{m,n-4} = \frac{n-5}{m+n-4} I_{m,n-6}$$

$\therefore \quad I_{m,n} = \frac{n-1}{m+n}.\frac{n-3}{m+n-2}.\frac{n-5}{m+n-4} I_{m,n-6}$ and so on.

We now have the following cases:

Case I: Let n be an even positive integer.

$$I_{m,4} = \frac{3}{m+4} I_{m,2} = \frac{3}{m+4}.\frac{1}{m+2}.I_{m,0}$$

$$I_{m,n} = \frac{n-1}{m+n}.\frac{n-3}{m+n-2}.\frac{n-5}{m+n-4}............\frac{3}{m+4}.\frac{1}{m+2}.I_{m,0}$$

$$I_{m,0} = \int_0^{\pi/2} \sin_m x.\cos^0 x.dx = \int_0^{\pi/2} \sin^m xdx$$

$\mathrm{I}_{m,0} = \frac{m-1}{m}\frac{m-3}{m-2}\frac{m-5}{m-4}.........\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$, if m= even

$= \frac{m-1}{m}\frac{m-3}{m-2}\frac{m-5}{m-4}.............\frac{4}{5}.\frac{2}{3}.1$, if m= odd

$\therefore$ If both m and n are even,

$\therefore \quad I_{m,n} = \frac{n-1}{m+n}\frac{n-3}{m+n-2}.........\frac{1}{m+2}\frac{m-1}{m}\frac{m-3}{m-2}.............\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$

Which we write as,

$$\boxed{\mathrm{Im}, n = \frac{\{(n-1)(n-3).........3.1\}.\{(m-1)(m-3)............3.1\}}{\{(m+n)(m+n-2).............4.2\}}.\frac{\pi}{2} \quad \text{m, n both even}}$$

If m is odd and n be even,

$$I_{m,n}=\frac{n-1}{m+n}\frac{n-3}{m+n-2}.........\frac{1}{m+2}.\frac{m-1}{m}\frac{m-3}{m-2}.........\frac{4}{5}.\frac{2}{3}.1$$

$$\boxed{I_{m,n}=\frac{\{(n-1)(n-3).........3.1\}.\{(m-1)(m-3)............4.2\}}{\{(m+n)(m+n-2).............5.3.1\}}}$$

M odd, n even

Case II: Let n be an odd integer.

From equation (2)

$$I_{m,5}=\frac{4}{m+5}I_{m,3}$$

$$=\frac{4}{m+5}.\frac{2}{m+3}.I_{m,1}$$

$$I_{m,n}=\frac{n-1}{m+n}\frac{n-3}{m+n-2}\frac{n-5}{m+n-4}......\frac{4}{m+5}.\frac{2}{m+3}.I_{m,1}$$

$$I_{m,1}=\int_0^{\pi/2}\sin^m x.\cos xdx=\left[\frac{\sin^m+1x}{m+1}\right]_0^{\pi/2}=\frac{1}{m+1}$$

$\therefore$ If n is odd and m may be even or odd

$$\int_0^{\pi/2}\sin^m x\cos^n xdx=\left[\frac{n-1}{m+n}\frac{n-3}{m+n-2}.........\frac{2}{m+3}\right]\frac{1}{m+1}$$

This is also written as,

$$\boxed{\int_0^{\pi/2}\sin^m x\cos^n xdx=\left[\frac{\{(n-1)(n-3)......4.2\}\{(m-1)(m-3)......3.1\}}{(m+n)(m+n-2)......(m+3)(m+1)(m-1)(m-3)...3.1}\right]}$$

m= even; n= odd

$$\boxed{\int_0^{\pi/2}\sin^m x\cos^n xdx=\left[\frac{\{(n-1)(n-3)......4.2\}\{(m-1)(m-3)......4.2\}}{(m+n)(m+n-2)......(m+3)(m+1)(m-1)(m-3)...4.2}\right]}$$

If m= odd; n= odd

Note: From the above cases, it appears that the following working rule may be adopted for evaluation of integrals of the form

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{\{(m-1)(m-3)........2or1\}\{(n-1)(n-3)........2or1\}}{(m+n)(m+n-2)(m+n-4)...2or1} xP$$

Where $P = \frac{\pi}{2}$ if m and n are both even

$= 1$ For all other values of m and n

For example

$$\int_0^{\pi/2} \sin^6 x \cos^4 x dx = \frac{(5.3.1)(3.1)}{10.8.6.4.2} x \frac{\pi}{2} = \frac{3\pi}{512}$$

$$\int_0^{\pi/2} \sin^5 x \cos^6 x dx = \frac{(4.2)(5.3.1)}{11.9.7.5.3.1} x1 = \frac{8}{693}$$

$$\int_0^{\pi/2} \sin^4 x.\cos^5 x dx = \frac{(3.1)(4.2)}{9.7.5.3.1} x1 = \frac{8}{315}$$

$$\int_0^{\pi/2} \sin^3 x \cos^5 x dx = \frac{(2)(4.2)}{8.6.4.2} = \frac{1}{24}$$

Additional Results

I. $\int_0^{\pi/2} \sin^p x \cos x dx = \frac{1}{p+1} = \int_0^{\pi/2} \cos^p x.\sin x.dx$

II. $\int_0^{\pi/2} \sin mx \cos^n x dx = 2\int_0^{\pi/2} \sin mx.\cos^n x dx$ if n= even, m= even or odd

III. $\int_0^{\pi/2} \sin^m x \cos^n x dx = 4\int_0^{\pi/2} \sin^m x.\cos^n x dx$ if m, n= even

$= 0$ otherwise

5. LIST OF FORMULAE

Now, we tabulate the all reduction formulae for ready reference:

1. $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$

$= \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}...\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$, if n is even $\frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}.......\frac{4}{5}\frac{2}{3}1$, if n is odd

2. $\int_0^{\pi} \sin^n x dx = 2\int_0^{\pi/2} \sin^n x dx$, for all n integral values of n

3. $\int_0^{\pi} \cos^n x dx = 2\int_0^{\pi/2} \cos^n x dx$ if n is even

$= 0$ if n is odd

4. $\int_0^{2\pi} \sin^n x dx = 4\int_0^{\pi/2} \sin^n x dx$ if n is even

$= 0$, if n is odd

5. $\int_0^{2\pi} \cos^n x dx = 4\int_0^{\pi/2} \cos^n x dx$, if n is even

$= 0$, if n is odd

6. $\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{[(m-1)(m-3)......2 or 1][(n-1)(n-3)......2 or 1]}{(m+n)(m+n-2)......2 or 1} xP$

Where P $= \pi/2$ if m, n both even

= 1, otherwise

7. $\int_0^{\pi} \sin^m x \cos^n x dx = 2\int_0^{\pi/2} \sin^m x \cos^n x dx$, if n= even, m= even or odd

$= 0$ if n= odd, m= even or odd

8. $\int_0^{2\pi} \sin^m x \cos^n x dx = 4\int_0^{\pi/2} \sin^m x \cos^n x dx$ if m, n= even

$= 0$, otherwise

9. $\int_0^{\pi/2} \sin^p x \cos x dx = \frac{1}{p+1} = \int_0^{\pi/2} \cos^p x \sin x dx$

ILLUSTRATION

Type I: Using Trigonometric Formulae:

Ex.1: $I = \int_0^{\pi} \sin^2\theta(1+\cos\theta)^4 d\theta$

Sol: $I = \int_0^{\pi}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 (2\cos^2\theta/2)^4 d\theta = 2^6\int_0^{\pi}\sin^2\frac{\theta}{2}\cos^{10}\frac{\theta}{2} d\theta$

Put $\frac{\theta}{2}$=t $\theta = 2t \quad \therefore d\theta = 2dt$

θ	0	π
T	0	$\pi/2$

Changing limits

$$\therefore I = 64\int_0^{\pi/2} \sin^2 t\cos^{10} t2dt = 128\int_0^{\pi/2} \sin^2 t\cos^{10} tdt$$

$$= 128\frac{(1)(9.7.5.3.1)}{12.10.8.6.4.2} x\frac{\pi}{2} = \frac{21\pi}{16}$$

Ex.2: $I = \int_0^{\pi/4} \sin^7 2\theta d\theta$

Put $2\theta = t \qquad \therefore \theta = \frac{t}{2}$

$$\therefore \qquad I = \int_0^{\pi/2} \sin 7t.\frac{dt}{2} = \frac{1}{2}.\frac{6}{7}.\frac{4}{5}.\frac{2}{3} = \frac{8}{35}$$

Ex.3: $I = \int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2} d\theta$

Sol: $I = \int_0^{\pi} \frac{(2\sin\theta/2\cos\theta/2)^4}{(2\cos^2\theta/2)^2} d\theta$

$$= \frac{2^4}{2^2}\int_0^{\pi} \frac{\sin^2\theta/2\cos^4\theta/2}{\cos^4\theta/2} d\theta = 4\int_0^{\pi} \sin^2\theta/2 d\theta$$

Put $\theta/2 = t \qquad \therefore \theta = 2t$
$\qquad d\theta = 2dt$

θ	0	π
t	0	$\pi/2$

$$\therefore \qquad I = 4\int_0^{\pi/2} \sin 2t.2dt = 8.\frac{1}{2}.\frac{\pi}{2} = 2\pi$$

Ex.4: Evaluate $\int_0^{\pi/4} \cos^3 2\phi\sin^2 4\phi d\phi$

Sol.: Put $\quad 2\phi = \theta \qquad d\phi = \frac{1}{2}d\theta$

ϕ	0	$\pi/4$
θ	0	$\pi/2$

$$I = \int_0^{\pi/2} \cos^3\theta\sin^2 2\theta\left(\frac{1}{2}d\theta\right) = \frac{1}{2}\int_0^{\pi/2} \cos^3\theta(2\sin\theta\cos\theta)^2.d\theta$$

$$= 2\int_0^{\pi/2} \sin^2\theta\cos^5\theta d\theta = 2\frac{(1)(4.2)}{7.5.3.1} = \frac{16}{105}$$

Ex.5: Evaluate $\int_{-\pi/2}^{\pi/2} \cos^3\theta(1+\sin\theta)^2 d\theta$

Sol.: $$I = \int_{-\pi/2}^{\pi/2} \cos^3\theta(1+2\sin\theta+\sin^2\theta)d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos^3\theta d\theta + 2\int_{-\pi/2}^{\pi/2} \cos^3\theta\sin\theta d\theta + \int_{-\pi/2}^{\pi/2} \cos^3\theta\sin^2\theta.d\theta$$

$$= 2\int_0^{\pi/2} \cos^3\theta d\theta + 0 + 2\int_0^{\pi/2} \cos^3\theta\sin^2\theta d\theta$$

$$I = 2\left(\frac{2}{3}\right) + 2\frac{(2)(1)}{5.3.1} = \frac{8}{5}$$

Note: Here we have used $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if f(x) is even

$= 0$ if f(x) is odd

Ex.6: Evaluate $\int_0^{2\pi} \sin^2\theta(1+\cos\theta)^4 d\theta$

Sol.: $$I = \int_0^{2\pi} \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 \left(2\cos^2\frac{\theta}{2}\right)^4 d\theta$$

$$= 64\int_0^{2\pi} \sin^2\frac{\theta}{2}\cos^{10}\frac{\theta}{2} d\theta$$ put $\theta = 2t$ $d\theta = 2dt$

$$= 64\int_0^{\pi} \sin^2 t\cos^{10} t.2dt$$

$$= 128.2\int_0^{\pi/2} \sin^2\cos^{10} tdt = 256\frac{(1)(9.7.5.3.1)}{12.10.8.6.4.2}\frac{\pi}{2} = \frac{21\pi}{8}$$

θ	0	2π
t	0	π

Type II: Examples involving trigonometric and algebraic functions:

Ex.7: $I = \int_0^{\pi} x\sin^5 x\cos^4 xdx$

Sol. Using the property

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we get

$$I = \int_0^{\pi} (\pi - x)\sin^5(\pi - x)\cos^4(\pi - x)dx$$

$$=\int_0^{\pi}(\pi-x)\sin^5 x\cos^4 xdx$$

$$=\pi\int_0^{\pi}\sin^5 x\cos^4 xdx-\int_0^{\pi}x\sin^5 x\cos^4 xdx$$

$\therefore$ $$I=\pi.\int_0^{\pi}\sin^5 x\cos^4 xdx-I$$

$\therefore$ $$2I=\pi.2\int_0^{\pi/2}\sin^5 x\cos^4 xdx$$

$\therefore$ $$I=\pi\frac{(4.2)(3.1)}{9.7.5.3.1}=\frac{8\pi}{315}$$

Ex.8: Prove that $\int_0^{\pi}x\cos^6 xdx=\frac{5\pi^2}{32}$

Sol.: $$I=\int_0^{\pi}x\cos^5 xdx$$

$$=\int_0^{\pi}(\pi-x)\cos^6(\pi-x)dx$$

$$=\int_0^{\pi}(\pi-x)\cos^6 xdx$$

$$=\pi\int_0^{\pi}\cos^6 x-\int_0^{\pi}x\cos^6 xdx$$

$\therefore$ $$2I=\pi.2\int_0^{\pi/2}\cos 6xdx \quad \text{[n is even]}$$

$$I=\pi.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$I=\frac{5\pi^2}{32}$$

Ex.9: Evaluate $\int_0^{\pi}x\sin^7 x\cos^4 xdx$

Sol.: $$I=\int_0^{\pi}sxin^7 x\cos^4 xdx$$

$$=\int_0^{\pi}(\pi-x)\sin^7(\pi-x)\cos^4(\pi-x).dx \qquad \because\int_0^{a}f(x)dx=\int_0^{a}f(a-x)dx$$

$$I=\int_0^{\pi}(\pi-x)\sin^7 x\cos^4 x dx$$

$\sin(\pi-x)=\sin x$ $\cos(\pi-x)=-\cos x$

Adding (1) and (2),

$$2I=\int_0^{\pi}\pi\sin^7 x\cos^4 x dx.=\pi.2\int_0^{\pi/2}\sin^7 x\cos^4 x dx \qquad \text{(See additional result)}$$

$$I=\pi.\frac{6.4.2.3.1}{11.9.7.5.3.1}=\frac{16\pi}{1155}$$

Type III: Examples involving substitutions:

Note:

$$For a^2-x^2 put x=a\sin\theta$$

$$For a^2+x^2 put x=a\tan\theta$$

$$For x^2-a^2 put x=a\sec\theta$$

Ex.10: $\int_0^1 x^6\sqrt{1-x^2}\,dx$

Sol.: Put $x=\sin\theta$ $\therefore dx=\cos\theta d\theta$

x	0	1
θ	0	$\pi/2$

$$I=\int_0^{\pi/2}\sin^6\theta.\cos\theta.\cos\theta d\theta=\int_0^{\pi/2}\sin^6\theta\cos^2\theta d\theta=\frac{(5.3)(1)}{8.6.4.2}.\pi/2=\frac{5\pi}{256}$$

Ex.11: $I=\int_0^{1/2}x^3\sqrt{1-4x^2}\,dx$

Sol. Put $4x^2=\sin^2\theta$ OR $2x=\sin\theta$

$$\therefore \qquad x=\frac{1}{2}\sin\theta$$

$$dx=\frac{1}{2}\cos\theta d\theta$$

x	0	1/2
θ	0	$\pi/2$

$$\therefore \qquad I=\int_0^{\pi/2}\left(\frac{1}{2}\sin\theta\right)^3.\sqrt{1-\sin^2\theta}.\frac{1}{2}\cos\theta d\theta$$

$$=\frac{1}{16}\int_0^{\pi/2}\sin^3\theta.\cos^2\theta d\theta=\frac{1}{16}\frac{(2)(1)}{5.3.1}=\frac{1}{120}$$

Ex.12: Evaluate $I=\int_0^3\frac{x^{3/2}}{(3-x)^{1/2}}dx$

Sol.: Put $x = 3\sin^2\theta$, $dx = 6\sin\theta.\cos\theta d\theta$

Changing the limits

x	0	3
θ	0	$\pi/2$

$$I = \int_0^{\pi/2} \frac{(3\sin^2\theta)^{3/2}}{(3-3\sin^2\theta)^{1/2}}.6\sin\theta\cos\theta d\theta = 6\int_0^{\pi/2} \frac{3\sqrt{3}.\sin^3\theta}{\sqrt{3}(1-\sin^2\theta)^{1/2}}.\sin\theta\cos\theta d\theta$$

$$= 18\int_0^{\pi/2} \frac{\sin^4\theta.\cos\theta d\theta}{\cos\theta} = 18\int_0^{\pi/2} \sin^4\theta d\theta = 18.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{27\pi}{8}$$

Ex.13: Evaluate $\int_0^{2a} x\sqrt{2ax-x^2}\,dx.$

Sol.: $$I = \int_0^{2a} x\sqrt{2ax-x^2}\,dx = \int_0^{2a} x.\sqrt{x}\sqrt{2a-x}\,dx$$

$$= \int_0^{2a} x^{3/2}(2a-x)^{1/2}\,dx.$$ Put $x = 2a\sin^2\theta$

$dx = 4a\sin\theta\cos\theta d\theta$

x	0	2a
θ	0	$\pi/2$

$$I = \int_0^{\pi/2} (2a)^{3/2}\sin^3\theta(2a-2a\sin^2\theta)^{1/2}4a\sin\theta\cos\theta d\theta.$$

$$= 16a^3\int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta = 16a^3\frac{(3.1)(1)}{6.4.2}\frac{\pi}{2} = \frac{\pi a^3}{2}$$

Ex.14: Evaluate $\int_0^{\pi} \frac{x^2}{(1+x^6)^{7/2}}dx$

Sol.: Put $x^3 = \tan\theta, 3x^2dx = \sec^2\theta d\theta, x2dx = \frac{1}{3}\sec^2\theta d\theta$

x	0	∞
θ	0	$\pi/2$

$$I = \int_0^{\pi/2} \frac{\sec^2\theta d\theta}{(1+\tan^2\theta)^{7/2}3} = \frac{1}{3}\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{\sec^7\theta}$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^5 d\theta = \frac{1}{3}.\frac{4}{5}.\frac{2}{3}.1 = \frac{8}{45}$$

Ex.15: Evaluate $\int \frac{x^6 - x^3}{(1+x^3)^5} dx$

Sol: $$I = \int_0^{\infty} \frac{x^6 - x^3}{(1+x^3)^5} . x^2 dx \qquad x^3 = \tan^2 \theta$$

$$I = \int_0^{\pi/2} \frac{\tan^4 \theta - \tan^2 \theta}{(1+\tan^2 \theta)^5} \frac{2}{3} \tan \theta \sec^2 \theta d\theta$$

$$3x^2 dx = 2 \tan \theta \sec^2 \theta d\theta$$

x	0	∞
θ	0	$\pi/2$

$$= \frac{2}{3} \int_0^{\pi/2} \left(\tan^5 \theta - \tan^3 \theta \right) \cos^8 \theta d\theta$$

$$= \frac{2}{3} \left[\int_0^{\pi/2} \sin^5 \theta \cos^3 \theta d\theta - \int \sin^3 \theta \cos^5 \theta d\theta \right]$$

$$= 0$$

$$\because \int_0^a f(x) dx = \int_0^a f(a-x) dx.$$

$$\because \int_0^{\pi/2} \sin^5 \theta \cos^3 \theta d\theta = \int_0^{\pi/2} \sin^5 (\pi/2 - \theta) \cos^3 (\pi/2 - \theta) d\theta = \int_0^{\pi/2} \cos^5 \theta \sin^3 \theta . d\theta$$

Ex.16: Evaluate $\int_0^1 x^{4m+1} \sqrt{\frac{1-x^2}{1-x^2}} dx$

Sol.: $$I = \int_0^1 x4m \frac{(1-x^2)}{\sqrt{1-x^4}} . xdx \qquad \text{Put } x^2 = \sin \theta$$

$$2xdx = \cos \theta d\theta$$

x	0	1
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \sin^{2m} \theta \frac{(1-\sin \theta)}{\cos \theta} \frac{\cos \theta}{2} d\theta = \frac{1}{2} \left[\int_0^{\pi/2} \sin^{2m} \theta d\theta - \int_0^{\pi/2} \sin^{2m+1} \theta . d\theta \right]$$

$$= \frac{1}{2} \left[\frac{2m-1}{2m} . \frac{2m-3}{2m-2} \ldots\ldots \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2} - \frac{2m}{2m+1} . \frac{2m-2}{2m-1} \ldots\ldots\ldots \frac{4}{5} . \frac{2}{3} . 1 \right]$$

Type IV: Examples involving inverse functions:

Ex.17: Evaluate $\int_0^1 x^4 \cos^{-1} x dx$

Sol.: $$I = \int_0^1 x^4 \cos^{-1} x dx$$

Integrating by parts we get

$$= \left[\cos^{-1} x \frac{x^5}{5}\right]_0^1 - \int_0^1 \frac{x^5}{5} \frac{-1}{\sqrt{1-x^2}} dx$$

$$= 0 + \frac{1}{5}\int_0^1 \frac{x^5}{\sqrt{1-x^2}} dx [\because \cos^{-1} 1 = 0]$$

Put $x = \sin\theta, dx = \cos\theta d\theta$

x	0	1
θ	0	$\pi/2$

$\therefore$ $$I = \frac{1}{5}\int_0^{\pi/2} \frac{\sin^5\theta}{\cos\theta} . \cos\theta d\theta$$

$$= \frac{1}{5}.\frac{4}{5}.\frac{2}{3}.1 = \frac{8}{75}$$

Ex.18: Evaluate $\int_0^1 x^5 . \sin^{-1} x dx$

Sol.: $$I = \int_0^1 \sin^{-1} x . x^5 dx$$ (integrate by parts)

$$= \left[\left(\sin^{-1} x\right)\frac{x^6}{6}\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x2}} \frac{x^6}{6} dx$$

$$= \frac{1}{6}.\frac{\pi}{2} - \frac{1}{6}\int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx$$

Put $x = \sin\theta$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2} \frac{\sin^6\theta}{\cos\theta} \cos\theta d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\left(\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right) = \frac{11\pi}{192}$$

$dx = \cos\theta d\theta$

x	0	1
θ	0	$\pi/2$

Type V: Miscellaneous Examples:

Ex.19: Prove that $\int_0^1 (1-x^{1/n})^m dx = \frac{m!n!}{(m+n)!}$

Sol. $I = \int_0^1 (1-x^{1/n})^m dx$ Put $x^{1/n} = \sin^2\theta$

$\therefore x = \sin^{2n}\theta$

$dx = 2n\sin^{2n-1}\theta\cos\theta d\theta$

$$I = \int_0^{\pi/2} (1-\sin 2\theta)^m .2n\sin^{2n-1}\theta\cos\theta d\theta$$

x	0	1
θ	0	$\pi/2$

$$= 2n\int_0^{\pi/2} \sin^{2n-1}\theta\cos^{2m+1}\theta d\theta$$

$$= \frac{2n[(2n-2)(2n-4)...2][(2m)(2m-2)....2]}{(2m+2n)(2m+2n-2)..........2}$$

$$= \frac{2^n.n!2m.m!}{2^{m+n}(m+n)!} = \frac{m!+n!}{(m+n)!}$$

Ex.20: Prove that $\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}} dx = \frac{(2n)!}{2^{2n}(n!)^2}.\frac{\pi}{2}$

Sol.: $I = \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}} dx$ Put $x = \sin\theta$

$\therefore dx = \cos\theta d\theta$

$$I = \int_0^{\pi/2} \frac{\sin^{2n}\theta}{\cos\theta}.\cos\theta d\theta$$

x	0	1
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \sin^{2n}\theta d\theta = \frac{2n-1}{2n}\frac{2n-3}{2n-2}\frac{2n-5}{2n-4}.........\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

In order to get (2n)! In the numerator multiply and divide by the terms required.

$$I = \frac{2n}{2n}\frac{2n-1}{2n}\frac{2n-2}{2n-2}\frac{2n-3}{2n-2}.......\frac{3}{4}.\frac{2}{2}.\frac{1}{2}.\pi/2$$

$$= \frac{(2n)!}{(2n)^2(2n-2)^2......2^2}.\pi/2$$

$$= \frac{(2n)!}{2^{2n}[n(n-1)....1]}\pi/2 = \frac{(2n)!}{22n(n!)^2}.\pi/2$$

Ex.21: Prove that $\int_0^{\infty}\frac{dx}{(x^2+1)n}=\frac{(2n-2)!}{2^{2n-2}[(n-1)!]^2}.\frac{\pi}{2}$

Sol.: $I=\int_0^{\infty}\frac{dx}{(x^2+1)n}$ Put $x=\tan\theta$. $dx=\sec^2\theta d\theta$

x	0	∞
θ	0	$\pi/2$

$$I=\int_0^{\pi/2}\frac{\sec^2\theta d\theta}{(\tan^2\theta+1)n}=\int_0^{\pi/2}\cos^{2n-2}\theta.d\theta$$

$$=\frac{2n-3}{2n-2}.\frac{2n-5}{2n-4}.........\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

Multiply numerator and denominator by $(2n-2)(2n-4)...........4.2$

$$I=\frac{(2n-2)(2n-3)(2n-4)(2n-5)............4.3.2.1}{[(2n-2)(2n-4).........4.2]^2}$$

$$=\frac{(2n-2)!}{[2(n-1)2(n-2).........2(2).2(1)^2}.\frac{\pi}{2}$$

$$=\frac{(2n-2)!}{[2^{n-1}(n-1)!]^2}x\frac{\pi}{2}$$

$$=\frac{(2n-2)!}{2^{2n-2}[(n-2)!]^2}.\frac{\pi}{2}$$

Ex.22: If $U_n=\int_0^{\pi/2}\theta\cos^n\theta d\theta$. Prove that $U_n=-\frac{1}{n^2}+\frac{n-1}{n}U_{n-2}$

Sol.: $u_n=\int_0^{\pi/2}\theta\cos^n\theta d\theta$

$$=\int_0^{\pi/2}\theta\cos^{n-1}\theta d\theta\text{, Integrating by parts}$$

$$u_n=[\theta\cos^{n-1}\theta.\sin\theta]_0^{\pi/2}\sin\theta[\cos^{n-1}\theta+\theta(n-1)\cos^{n-2}\theta(-\sin\theta)d\theta$$

$$u_n=0-\int_0^{\pi/2}\cos^{n-1}\theta.\sin\theta d\theta+(n-1)\int_0^{\pi/2}\theta.\cos^{n-2}\theta.\sin^2\theta d\theta$$

$$=-\int_0^{\pi/2}\cos^{n-1}\theta\sin\theta d\theta+(n-1)\int_0^{\pi/2}\theta.\cos^{n-2}\theta.(1-\cos^2\theta)d\theta$$

Put $\cos\theta=t$ in 1st integral, $\therefore -\sin\theta d\theta=dt$

θ	0	$\pi/2$
t	1	0

$$\therefore \quad u_n = \int_1^0 t^{n-1}dt + (n-1)\int_0^{\pi/2} \theta\cos^{n+2}\theta d\theta - (n-1)\int_0^{\pi/2}\theta.\cos^n\theta d\theta$$

$$= \left[\frac{t^n}{n}\right]_1^0 + (n-1)u_{n-2} - (n-1)u_n$$

$$\therefore u_n(1+n-1) = -\frac{1}{n} + (n-1)u_{n-2}$$

$$\therefore u_n = \frac{-1}{n^2} + \frac{n-1}{n}u_{n-2} \quad \text{(Proved)}$$

Ex.23: Evaluate $\int_0^1 \frac{x^2(4-x^4)}{\sqrt{1-x^2}}$

Sol. $I = \int_0^1 \frac{4x^2 - x^6}{\sqrt{1-x^2}}dx$ Put $x = \sin\theta$ $dx = \cos\theta d\theta$

x	0	1
θ	0	$\pi/2$

$$\therefore \quad I = \int_0^{\pi/2} \frac{4\sin^2\theta - \sin^6\theta}{\cos\theta}.\cos\theta d\theta$$

$$\therefore \quad I = 4\int_0^{\pi/2} \sin^2\theta d\theta - \int_0^{\pi/2}\sin^6\theta d\theta$$

$$= 4.\frac{1}{2}.\frac{\pi}{2} - \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \pi - \frac{5\pi}{32} = \frac{27\pi}{32}$$

Ex.24: Evaluate $\int_0^1 x^5\sqrt{\frac{1+x^2}{1-x^2}}.dx$

Sol.: $I = \int_0^1 x5\sqrt{\frac{1+x^2}{1-x^2}}dx$

Multiply and divide by $\sqrt{1+x^2}$

$\therefore \quad I = \int_0^1 x^5\frac{(1+x^2)}{\sqrt{1-x^4}}dx$ Put $x^2 = \sin\theta$ $\therefore 2xdx = \cos\theta d\theta$

x	0	1
θ	0	$\pi/2$

$$I=\int_0^1 \frac{x^4(1+x^2)xdx}{\sqrt{1-x^4}}$$

$$=\int_0^{\pi/2} \frac{\sin^2\theta(1+\sin\theta)}{\sqrt{1-\sin^2\theta}}\frac{\cos\theta d\theta}{2}$$

6. REDUCTION FORMULA FOR $\int \tan^{n} x \ dx$

Let $$I_n=\int \tan^n xdx=\int \tan^{n-2} x.\tan^2 x.dx$$

$$=\int \tan^{n-2} x.(\sec^2 x-1)dx.$$

$$=\int \tan^{n-2} x\sec^2 xdx-\int \tan^{n-2} xdx$$

$$I_n=\frac{\tan^{n-1} x}{n-1}-I_{n-2}$$

$$\int \tan^n xdx=\frac{\tan^{n-1} x}{n-1}-\int \tan^{n-2} xdx$$

Which is the required reduction formula?

ILLUSTRATIONS

Ex.32: If $U_n=\int_0^{\pi/4}\tan^n\theta d\theta$, then show that, $n(U_{n+1}+U_{n-1})=1$ and hence, find $\int_0^{\pi/4}\tan^6\theta d\theta$ And also evaluate $\int x^5(2a^2-x^2)^{-3}dx.$

$$U_n=\int_0^{\pi/4}\tan^n\theta d\theta$$

Sol.: $$U_{n+1}=\int_0^{\pi/4}\tan^{n+1}\theta d\theta$$

$$U_{n+1}=\int_0^{\pi/4}\tan^{n-1}\theta\tan^2\theta d\theta=\int_0^{\pi/4}\tan^{n-1}\theta(\sec^2\theta-1)d\theta$$

$$=\int_0^{\pi/2}\tan^{n-1}\theta\sec^2\theta d\theta-\int_0^{\pi/2}\tan^{n-1}\theta d\theta$$

$$=\frac{\tan^n\theta}{n}-U_{n-1}=\frac{1}{n}-U_{n-1}$$

$$U_{n+1}+U_{n-1}=\frac{1}{n}\Rightarrow n\left(U_{n+1}+U_{n-1}\right)=1$$

We have, $$U_{n+1}=\frac{1}{n}-U_{n-1}$$

Put n=5, $$U_6=\frac{1}{5}-U_4=\frac{1}{5}-\left(\frac{1}{3}-U_2\right) \qquad \left(\because U_4=\frac{1}{3}-U_2\right)$$

$$=-\frac{2}{15}+U_2=-\frac{2}{15}+1-U_0$$

$$=\frac{13}{12}-\int_0^{\pi/4}d\theta=\frac{13}{15}-\frac{\pi}{4}$$

Let $$I=\int_0^a x^5(2a^2-x^2)^{-3}dx$$

Put $x=\sqrt{2}\;\;a\sin\theta$ $\qquad dx=\sqrt{2}a\cos\theta d\theta$

x	0	a
θ	0	$\pi/4$

$$=\int_0^{\pi/4}\left(\sqrt{2}\right)^5 a^5\sin^5\theta 2^{-3}a^{-6}(\cos^2\theta)^{-3}\sqrt{2}.a\cos\theta d\theta$$

$$=\int_0^{\pi/4}\sin^5\theta\cos^{-5}\theta d\theta=\int_0^{\pi/4}\tan^5\theta d\theta=U_5$$

Put n=4, $$U_5=\frac{1}{4}-U_3=\frac{1}{4}-\left(\frac{1}{2}-U_1\right)$$

$$=-\frac{1}{4}+\int_0^{\pi/4}\tan\theta.d\theta=-\frac{1}{4}+\left[\log\sec\theta\right]_0^{\pi/4}$$

$$=-\frac{1}{4}+\log\sqrt{2}=\frac{1}{2}\left[-\frac{1}{2}+\log^2\right]$$

$\therefore$ $$\int_0^a x^5(2a^2-x^2)^{-3}dx=\frac{1}{2}\left[-\frac{1}{2}+\log 2\right]$$

Ex.33: If $I_n=\int_0^{\pi/4}\tan^n x dx$. Show that $I_n+I_{n-2}=\frac{1}{n-1,}$, hence evaluate I_5.

Sol.: $$I_n=\int_0^{\pi/4}\tan^n\theta.d\theta$$

$$=\int_0^{\pi/4}\tan^{n-2}\theta.\tan^2\theta d\theta=\int_0^{\pi/4}\tan^{n-2}\theta(\sec^2\theta-1)d\theta$$

$$I_n=\int_0^{\pi/4}\tan^{n-2}\theta\sec^2\theta d\theta-\int_0^{\pi/4}\tan^{n-2}\theta d\theta$$

Put $\tan\theta = t$ $\sec^2\theta = d\theta = dt,$

θ	0	$\pi/4$
t	0	1

$$\therefore I_n = \int_0^1 t^{n-2}dt - I_{n-2}$$

$$I_n = \left[\frac{t^{n-1}}{n-1}\right]_0^1 - I_{n-2}$$

$$\boxed{I_n = \frac{1}{n-1} - I_{n-2}}$$

Putting n+1 instead of n, the same result can be written in the form:

$$I_{n+1} = \frac{1}{n} - I_{n-1}$$

$$\therefore \quad I_{n+1} + I_{n-1} = \frac{1}{n}$$

$$\therefore \quad \boxed{n[I_{n+1} + I_{n-1}] = 1}$$

Now, $\quad I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \left[\frac{1}{2} - I_1\right] = -\frac{1}{4} + I_1$

And $\quad I_1 = \int_0^{\pi/4} \tan x dx = [\log\sec x]_0^{\pi/4}$

$$= \log\sec\frac{\pi}{4} - \log\sec\theta = \log\sqrt{2} - \log 1 = \log\sqrt{2}$$

$$\therefore \quad I_5 = -\frac{1}{4} + \log\sqrt{2}$$

Ex.34: Evaluate $\int_0^{\pi/4} \tan^8\theta d\theta$

Sol.: If $\quad I_n = \int_0^{\pi/4} \tan^n\theta d\theta,$ then

$$I_n = \frac{1}{n-1} - I_{n-2} \text{(Refer Ex. 34)}$$

Put n=8, $\quad I_8 = \frac{1}{7} - I_6 = \frac{1}{7} - \left[\frac{1}{5} - I4\right] = \frac{1}{7} - \frac{1}{5} + \left[\frac{1}{3} - I_2\right]$

$$=\frac{1}{7}-\frac{1}{5}+\frac{1}{3}+\left[I-I_n\right]=\frac{1}{7}-\frac{1}{5}+\frac{1}{3}-1+\int_0^{\pi/4}\tan^0\theta d\theta$$

$$=-\frac{76}{105}+\left[\theta\right]_0^{\pi/4}=-\frac{76}{105}+\frac{\pi}{4}$$

EXERCISE

1. Prove that $\int_0^1 \frac{x^7}{\sqrt{1-x^4}}dx=\frac{1}{3}$.

2. Evaluate $\int_0^{\pi}(1-\cos\theta)^3 d\theta$

3. Evaluate $\int_0^{\pi/4}\sin^7 2\theta d\theta$

   Hint: Put $2\theta=t$. Ans.: 8/35

4. Evaluate $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}dx$

   Ans.: $\frac{3\pi a^4}{16}$

5. Evaluate $\int_0^{\pi/6}\sin^6 3\theta d\theta$

   Ans.: $\frac{5\pi}{96}$

6. Evaluate $\int_0^1 \frac{x^9}{\sqrt{1-x^4}}dx.$

   Ans.: $\frac{3\pi}{2}$

7. Evaluate $\int_0^{\pi}(1+\cos\theta)^3 d\theta$

   Ans.: $\frac{5\pi}{2}$

8. Evaluate $\int_0^{\infty}\frac{x2}{(a^2+x^6)5/2}dx$

   Ans.: $\frac{2}{9a^4}$

9. Evaluate $\int_0^{2a}x^{7/2}(2a-x)^{-1/2}dx$

   Hint: Put $x=2a\sin^2\theta$

Ans.: $\dfrac{35\pi a^4}{8}$

10. Evaluate $\int_0^{2a} x^3(2ax-x^2)^{3/2}\,dx.$

Hint: Put x=2 a $\sin^2\theta$

Ans.: $\dfrac{9\pi}{16}a^7$

11. Evaluate $\int_0^{\pi} x\sin^5 x\cos^8 x dx.$

Ans.: $\dfrac{8\pi}{1287}$

12. Evaluate $\int_0^{\pi/2} \cos^3 2x\sin^4 4x dx.$

Ans.: 0

$\because \int_0^a f(x)dx = 0$ if $f(a-x) = -f(x)$

13. Prove that $\int_0^1 \dfrac{x^8}{\sqrt{1-x^2}}dx = \dfrac{35\pi}{256}$

14. Prove that $\int_0^\infty \dfrac{t^4}{(1+t^2)^3}dt = \dfrac{3\pi}{16}$

15. Prove that $\int_0^1 \dfrac{x^7}{\sqrt{1-x^4}}dx = \dfrac{1}{3}$

16. Prove that $\int_0^3 \dfrac{x^{3/2}}{(3-x)^{1/2}}dx = \dfrac{27\pi}{8}$

17. Prove that $\int_0^1 \dfrac{x^{2n}}{\sqrt{1-x^2}}dx = \dfrac{(2n)!}{22n(n!)^2}\dfrac{\pi}{2}$

18. Prove that $\int_0^\infty \dfrac{x^2}{(1+x^2)^{7/2}}dx = \dfrac{2}{15}$

19. Prove that $\int_0^\infty \left(\dfrac{t}{a+t2}\right)^6 dt = \dfrac{3\pi}{512}$

20. Prove that $\int_0^\infty \dfrac{x^7-x^8}{(1+x)^{17}}dx = 0$

21. Prove that $\int_0^\infty \dfrac{x^7(1-x^{12})}{(1+x)^{28}}dx = 0$

22. Prove that $\int_0^{2a} x\sqrt{2ax-x^2}\,dx = \frac{\pi a^2}{2}$

23. Prove that $\int_0^{2a} x^n \sqrt{2ax-x^2}\,dx = \frac{a^{n+2}\pi(2n+1)!}{2^n n!(n+2)!}$

24. Prove that $\int_{-1}^{1} (1+x)^m (1-x)^n\,dx = 2^{m+n+1}\frac{m!n!}{(m+n+1)!}$ where m and n are positive integers.
Hint: Put x= cos 2 θ

25. Considering $\int_0^1 (1-x^2)^n\,dx$

$$1-\frac{n}{1.3}+\frac{n(n-1)}{1.2.5}-\frac{n(n-1)(n-2)}{1.2.3.7}+......=\frac{2}{3}.\frac{4}{5}.\frac{6}{7}............\frac{2n}{2n+1}$$

26. Show that $\int_{-\pi/2}^{\pi/2} \sin^4 x\cos^2 x dx = \frac{\pi}{16}$

27. Show that $\int_{-\pi/2}^{\pi/2} \sin^5 x dx = 0$

28. Show that $\int_{-\pi/2}^{\pi} \sin^4 x\cos^2 x dx = \frac{\pi}{8}$

29. Show that $\int_0^{2\pi} \sin^4 x\cos^2 x dx = \frac{\pi}{8}$

30. Show that $\int_0^{\pi/4} \sin^7 2\theta \mathrm{d}\theta = \frac{8}{35}$

31. Show that $\int_0^{\pi} \sin 2\theta(1+\cos\theta)^4\,d\theta = \frac{21\pi}{16}$

32. Show that $\int_0^{\infty} \frac{x^2}{(1+x^6)^{7/2}}\,dx = \frac{8}{45}$

33. Prove that $\int_4^6 \sin^4 \pi x\cos^2 2\pi x dx = \frac{7}{16}$

34. Prove that $\int_0^{\pi} x\sin^5 x\cos^4 x dx = \frac{8\pi}{315}$

35. Prove that $\int_0^{\pi} x\cos^6 x dx = \frac{5\pi 2}{12}$

36. Prove that $\int_0^1 x^6\sqrt{1-x2}dx = \frac{5\pi}{256}$

37.

GAMMA AND BETA FUNCTION

1. GAMMA FUNCTIONS

Consider the definite integral $\int_0^\infty e^{-x} x^{n-1}$ dx, it is denoted by the symbol Γn (we read it as Γn and is called Gamma function of n. Thus,

$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx (n > 0)}$$

Gamma function is also called Euler's integral of the second kind.

2. PROPERTIES OF GAMMA FUNCTIONS

$$\Gamma n = 2\int_0^\infty e^{-x^2} . x^{2n-1} dx$$

Proof: We have, $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$ Put x=t^2 dx= 2t dt.

$$= \int_0^\infty e^{-t^2} t^{2n-2} 2t dt = 2\int_0^\infty e^{-t^2} t^{2n-1} dt$$

$$\boxed{\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx}$$

(It may be borne in mind that variable of integration is immaterial in a definite integral.) Relations (1) and (2) are both considered as definitions of Gamma functions.

2. $\Gamma 1 = 1$

Proof: $n = \int_0^\infty e^{-x} x^{n-1} dx$

$$1 = \int_0^\infty e^{-x} x^0 dx = [-e^{-x}]_0^\infty = (-e^{-\infty} + e^0) = 0 + 1 = 1$$

3. Reduction formula for gamma functions:

$$\Gamma(n+1) = n\Gamma n$$

Proof: $n = \int_0^\infty e^{-x} x^{n-1} dx$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx.$$

Now integrating by parts,

$$= \left\{x^n(-e^{-x})\right\}_0^\infty - \int_0^\infty n x^{n-1}(-e^{-x})dx.$$

Now, $\lim_{x \to \infty} \frac{x^n}{e^x} = 0$

Also if n>0, $\frac{x^n}{e^x} = 0$ for x=0 $\therefore \left[\frac{x^n}{e^x}\right]_0^\theta = 0$

$$\therefore \quad \Gamma(n+1) = 0 + n\int_0^\infty e^{-x} x^{n-1} dx = n\,\Gamma n$$

$$\boxed{\therefore \quad \Gamma(n+1) = n\,\Gamma n}$$

If n is a positive integer,

$$\Gamma(n+1) = n(n-1)\,\Gamma(n-1) \qquad \therefore \Gamma n = (n-1)\,\Gamma(n-1)$$

$$= n(n-1)(n-2)\,\Gamma(n-2) = n(n-1)(n-2)(n-3)(n-4)$$

$$n(n-1)(n-2)(n-3)..............................3.2.1 \qquad \therefore \Gamma 1 = 1$$

$\Gamma(n+1) = n!$ If n is a positive integer.

Hence	$\Gamma(n+1) = n\,\Gamma n$, in general
	$= n!$ if n is positive integer.

4. $\boxed{\Gamma 0 = \infty}$ $\quad \therefore \Gamma n = \frac{\Gamma(n+1)}{n} \qquad \therefore \Gamma 0 = \frac{\Gamma 1}{0} = \frac{1}{0} = \infty$

5. $\Gamma \frac{1}{2} = \sqrt{\pi}$

6. $\left(\therefore \Gamma(n+1) = n!\right) \therefore \Gamma 6 = 5!$

$\Gamma 8 = 7!$, $\qquad \Gamma 2 = 1! = 1$

$$\Gamma\left(\frac{5}{3}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}.\frac{1}{2}.\sqrt{\pi}$$

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}$$

7. For negative fraction n, we use

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \quad \Gamma\left(\frac{5}{3}\right) = \left(-\frac{3}{5}\right)\Gamma\left(-\frac{2}{3}\right) = \left(-\frac{3}{5}\right)\left(-\frac{3}{2}\right)\Gamma\left(\frac{1}{3}\right) = \frac{9}{10}\Gamma\left(\frac{1}{3}\right)$$

3. TRANSFORMATION OF GAMMA FUNCTIONS

1. We know that $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ Put x= ky $\therefore dx = kdy$

x	0	∞
y	0	∞

$$= \int_0^\infty e^{-ky} k^{n-1}.y^{n-1}.k.dy = k^n \int_0^\infty e^{-ky}.y^{n-1} dy$$

$$\boxed{\therefore \int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma(n)}{k^n}}$$

Note: Students are advised to remember this formula.

2. We know that $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ Put $xn = y \therefore nx^{n-1}dx = dy.$

Also $x = y^{1/n}$

x	0	∞
y	0	∞

$$= \int_0^\infty e^{-y^{1/n}} \frac{dy}{n}$$

$$\therefore \quad \boxed{\int_0^\infty e^{-y^{1/n}} dy = n\Gamma(n) = \Gamma(n+1)}$$

Put $n=\frac{1}{2}\;\therefore \int_0^\infty e^{-y^2}dy=\frac{1}{2}.\left|\overline{\frac{1}{2}}\right.$ $\qquad\left(but\int_0^\infty e^{-y^2}dy=\frac{1}{2}\sqrt{\pi}\right)$

$$\frac{1}{2}\sqrt{\pi}=\frac{1}{2}\left|\overline{\frac{1}{2}}\right. \qquad \therefore \left|\overline{\frac{1}{2}}\right.=\sqrt{\pi}$$

3. We know that $\left|\overline{n}\right.=\int_0^\infty e^{-x}x^{n-1}dx.$ Put $e^{-x}=y$ $\quad\therefore -e^{-x}dx=dy.e^{x}=\frac{1}{y}\therefore x=\log\frac{1}{y}$

$$=\int_1^0\left(\log\frac{1}{y}\right)^{n-1}(-dy)$$

x	0	∞
y	1	0

$$\boxed{\left|\overline{n}\right.=\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy}$$

Additional Results

$$\boxed{\left|\overline{P}\right.\;\left|\overline{1-P}\right.=\frac{\pi}{\sin p\pi}\quad\text{if}\quad 0<P<1}$$

e.g. $\left|\overline{\frac{1}{4}}\right.\left|\overline{\frac{3}{4}}\right.=\left|\overline{\frac{1}{4}}\right.\left|\overline{1-\frac{1}{4}}\right.$ $\qquad$ Let $P=\frac{1}{4}<1=\frac{\pi}{\sin\left(\frac{1}{4}\pi\right)}=\frac{\pi}{\frac{1}{\sqrt{2}}}=\sqrt{2}.\pi$

4 BETA FUNCTION

Definition: Consider the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx, m>0, n>0.$

It is denoted by the symbol B (m, n) (we read it as Beta (m, n) and is called Beta function.

$$\boxed{B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx.m>0n>0}$$

The Beta function is also called Euler's integral of the first kind.

e.g. (1) $\qquad B,\left(3,\frac{3}{2}\right)=\int_0^1 x^2(1-x)^{1/2}dx.$

(2) $\qquad \int_0^1 t^4(1-t)^{3/2}dt=B\left(5,\frac{5}{2}\right)$

5. PROPERTIES OF BETA FUNCITONS

1. $b(m,n) = b(n,m)$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx. = \int_0^1 (1-x)^{m-1}(1-(1-x))^{n-1}dx$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$= \int_0^1 (1-x)^{m-1}.x^{n-1}dx = \int_0^1 x^{n-1}(1-x)^{m-1}dx = B(n,m)$$

$$\boxed{\therefore B(m,n) = B(n,m)}$$

2. $\int_0^1 x^m(1-x)^n dx = B(m+1, n+1)$

3. $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ Put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta.d\theta$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta d\theta$$

x	0	1
θ	0	$\pi/2$

$$\boxed{B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta.d\theta}$$

We consider the as definition of Beta function.

Further, Let $2m-1 = p, 2n-1 = q$ $\therefore m = \frac{p+1}{2}, n = \frac{q+1}{2}$

$$B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta.\cos^q\theta d\theta.$$

Standard Formula:

$$\boxed{\int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+2}{2}\right)}$$

Note: Students are advised to remember this formula.

4. $$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx.$ Put $\boxed{x = \frac{t}{1+t}}$ (remember this substitution)

$\therefore$ $x(1+t) = t$ i.e. $x + xt = t$ $\therefore x = t - xt$ Or $\boxed{t = \frac{x}{1-x}}$

When x=0 $t = \frac{0}{1-0} = 0$ and when x=1, $t = \frac{1}{1-1} = \frac{1}{0} = \infty$

x	0	1
t	0	∞

$$dx = \frac{(1+t)(1) - t(1)}{(1+t)^2} dt = \frac{1}{(1+t)^2} dt$$

$$B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1}} \left(1 - \frac{t}{1+t}\right)^{n-1} . \frac{dt}{(1+t)^2}$$

$$= \int_0^\infty \frac{t^{m-1} dt}{(t+1)^{m-1}(1+t)^{n-1}(1+t)^2} = \int_0^\infty \frac{t^{m-1} dt}{(1+t)^{m+n}}$$

$$\boxed{B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx}$$

We consider this result also as another definition of Beta function.

5. Relation between Beta and Gamma function:
We have

$$\boxed{B(m,n) = \frac{\left|\overline{m}\right.\left|\overline{n}\right.}{\left|\overline{m+n}\right.}}$$

6. $$\int_0^{p/2} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{1}{2} \frac{\left|\overline{\frac{p+1}{2}}\right., \left|\overline{\frac{q+1}{2}}\right.}{\left|\overline{\frac{p+q+2}{2}}\right.}$$

Put $p = q = 0$

$$\int_0^{p/2} d\theta = \frac{1}{2}\frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = \frac{1}{2}\left(\Gamma(1/2)\right)^2 \qquad \therefore \Gamma(1/2) = \sqrt{\pi}$$

6 DUPLICATION FORMULA OF GAMMA FUNCITONS

$$\boxed{\Gamma(m)\quad \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}.\Gamma(2m)}$$

Proof: Consider $\dfrac{1}{2}.\dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\pi/2} \sin^p\theta\cos^q\theta.d\theta$

Put p= 2m-1, q= 2m-1, i.e. $\dfrac{p+1}{2} = m, \dfrac{q+1}{2} = m$

$$\frac{1}{2}\frac{\Gamma(m).\Gamma(m)}{\Gamma(2m)} = \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2m-1}\theta.d\theta$$

$$\frac{\Gamma(m).\Gamma(m)}{\Gamma(2m)} = \frac{2}{2^{2m-1}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2m-1}d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2^{m-1}}d\theta \qquad \text{Put } 2\theta = t;\ d\theta = \frac{1}{2}dt$$

θ	0	$\pi/2$
t	0	π

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} (\sin t)^{2m-1}dt = \frac{1}{2^{2m-1}}2\int_0^{\pi/2} \sin^{2m-1}t.dt \qquad [\because f(\pi - t) = f(t)]$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}t.\cos^0 t.dt \qquad \text{Note this step}$$

$$= \frac{2}{2^{2m-1}}\frac{1}{2}\frac{\Gamma\left(\frac{2m-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{2m-1+0+2}{2}\right)}$$

$$\frac{\Gamma(m).\Gamma(m)}{\Gamma(2m)}=\frac{1}{2^{2m-1}}\frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\boxed{\therefore \Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)}$$

Additional Results

Show that, $\therefore \Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\ }}{2^{2m-1}}\Gamma(2m)$, given that $\int_0^\infty \frac{x^{p-1}}{1+x}dx=\frac{\pi}{\sin p\pi}$ for 0<p<1

Proof: Consider $I=\int_0^\infty \frac{x^{p-1}}{1+x}dx$ Put $x=\tan^2\theta$, $dx=2\tan\theta\sec^2\theta d\theta$

$$=\int_0^{\pi/2}\frac{\tan^{2p-2}\theta.2\tan\theta\sec^2\theta d\theta}{1+\tan^2\theta}$$

x	0	∞
θ	0	$\pi/2$

$$=2\int_0^{\pi/2}\tan^{2p-1}\theta.d\theta=2\int_0^{\pi/2}\sin 2p-1\theta.\cos^{1-2p}\theta.d\theta$$

$$=2.\frac{1}{2}B\left(\frac{2p-1+1}{2},\frac{1-2p+1}{2}\right)=B(p,1-p)=\frac{\Gamma(p).\Gamma(1-p)}{\Gamma(p+1-p)}$$

$$\boxed{\frac{\pi}{\sin p\pi}=\Gamma(p)\Gamma(1-p)}\qquad\left(\because \int_0^\infty\frac{x^{p-1}}{1+x}dx=\frac{\pi}{\sin p\pi}\right)\text{(given)}$$

(Note: this formula is to be used only when 0<p<1)

ILLUSTRATIONS ON GAMMA FUNCTIONS

Ex.1: Evaluate $\int_0^\infty \sqrt[4]{x}e^{\sqrt{x}}dx.$

Sol.: $I=\int x^{\frac{1}{4}}e^{\sqrt{x}}dx$ Put $\sqrt{x}=t$ or x= t^2; dx=2t dt

x	0	∞
T	0	∞

$$=\int_0^\infty t^{1/2}.e^{-t}2tdt=2\int_0^\infty e^{-t}t^{3/2}dt=2\Gamma\left(\frac{5}{2}\right)=2.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}=\frac{3\sqrt{\pi}}{2}$$

Ex.2: Evaluate $\int_0^\infty e^{-x^2}dx$

Sol. Put x^2=t, $\therefore 2xdx = dt \quad \therefore dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

x	0	∞
t	0	∞

$$\therefore I = \int_0^\infty e^{-t}\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty t^{-1/2}e^{-t}dt = \frac{1}{2}\left|\overline{\frac{1}{2}}\right. = \frac{\sqrt{\pi}}{2}$$

Ex.3: Evaluate $\int_0^\infty \sqrt{x}e^{-x^2}dx$

Sol.: Put x2= t, $\therefore x = t^{1/2}$ 2x dx= dt $\quad \therefore dx = \frac{1}{2\sqrt{t}}dt$

$\therefore$ $$I = \int_0^\infty (t^{1/2})^{1/2}e^{-t}\frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty t^{-1/4}e^{-t}dt = \frac{1}{2}\left|\overline{3/4}\right.$$

x	0	∞
t	0	∞

Ex.4: Evaluate $\int_0^\infty \frac{x^a}{a^x}dx (a > 1)$

Sol.: Put ax= et or x log a= t or $x = \frac{t}{\log a} \quad \therefore dx = \frac{dt}{\log a}$

x	0	∞
T	0	∞

$$I = \int_0^\infty \frac{x^a}{a^x}dx$$

$$I = \int_0^\infty \frac{ta}{(\log a)^a}.\frac{dt}{\log a}.\frac{1}{e^t}$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}.t^a dt = \frac{1}{(\log a)^{a+1}}\left|\overline{a+1}\right. = \frac{\left|\overline{a+1}\right.}{(\log a)^{a+1}}$$

Ex.5: Evaluate $\int_0^1 (x\log x)^3 dx.$

Sol.: $I = \int_0^1 x^3(\log x)^3 dx$ Put log x= -t, x=e^{-t}, dx= -e^{-t} dt

x	0	1
T	∞	0

[To find limits Put x=0 $\therefore \log 0 = -t, -\infty = -t \therefore t = \infty, x = 1 \log 1 = -t \therefore t = 0$]

$\therefore$ $$I = \int_{\infty}^{0} e - 3t(-t)^3.(-e^{-t})dt$$

$$= -\int_{0}^{\infty} e^{-4t} t^3 dt \quad \text{Put 4t=y, } t = \frac{y}{4} \qquad \therefore dt = \frac{dy}{4}$$

t	0	∞
y	0	∞

$\therefore$ $$I = -\int_{0}^{\infty} e^{-y} (y/4)^3 \frac{dt}{4} = -\frac{1}{4^4}.\int_{0}^{\infty} y^3 e^{-y} dy$$

$$= -\frac{1}{256}\left|\overline{4}\right. = -\frac{6}{256} = \frac{-3}{128}$$

Ex.6: Evaluate $\int_{0}^{1} x^m (\log x)^n dx.$

Sol.: $$I = \int_{0}^{1} x^m (\log x)^n dx$$

Put $\log x = -t \Rightarrow x = e^{-t} \quad dx = -e^{-t} dt$

x	0	1
T	∞	0

$$I = \int_{\infty}^{0} e^{-mt} (-t)^n (-e^{-t} dt) = \int_{0}^{\infty} (-1)^n e^{-(m+1)t} t^n dt$$

$$= \frac{(-1)^n \left|\overline{n+1}\right.}{(m+1)^{n+1}} = \frac{(-1)^n \left|\overline{n+1}\right.}{(m+1)^{n+1}}$$

Ex.7: Show that $\int_{0}^{1} \left(\log \frac{1}{y}\right)^{n-1} dy = \left|\overline{n}\right.$

Sol.: $$I = \int_{0}^{1} \left(\log \frac{1}{y}\right)^{n-1} dy \qquad \log \frac{1}{y} = t \quad \text{Or} \quad \frac{1}{y} = e^t \quad y = e^{-t} dy = -e^{-t} dt$$

$$= \int_{\infty}^{0} t^{n-1}(-e^{-t})dt = \int_{0}^{\infty} e^{-t} t^{n-1} dt = \left|\overline{n}\right.$$

y	0	1
t	∞	0

Ex.8: Evaluate $\int_{0}^{\infty} e^{-x^4} dx.$

Sol.: Put $x^4 = t, x = t^{1/4}, \therefore dx = \frac{1}{4}t^{-3/4}dt$

x	0	∞
T	0	∞

$$\therefore \qquad I = \int_0 e^{-t}\frac{1}{4}t^{-3/4}dt = \frac{1}{4}\int_0^\infty e^{-t}t^{-3/4}dt = \frac{1}{4}\overline{|1/4}$$

Ex.9: Evaluate $\int_0^\infty 7^{-4x^2}dx.$

Sol.: Put $7^{-4x^2} = e^{-t};\ +4x^2\log 7 = +t$

x	0	1
t	∞	0

$$x^2 = \frac{t}{4\log 7}\, x = \frac{\sqrt{t}}{2\sqrt{\log 7}}\, dx = \frac{1}{4\sqrt{t}\sqrt{\log 7}}dt$$

$$\therefore I = \int_0^\infty e^{-t}\frac{dt}{4\sqrt{t}\sqrt{\log 7}}$$

$$= \frac{1}{4\sqrt{\log 7}}.\int_0^\infty t^{-1/2}e^{-t}dt = \frac{1}{4\sqrt{\log 7}}.\overline{|1/2}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\log 7}}$$

Example10: $\int_0^1 x^3[\log(1/x)]^4dx.$

Sol.: Put $\log\frac{1}{x} = t, \therefore x = e^{-t}, dx = -e^{-t}dt$

x	0	1
t	∞	0

$$I = \int_\infty^0 e - 3t.t4.(-e-t)dt = \int_0^\infty t4e - 4tdt = \frac{\overline{|5}}{4^5}$$

Example11: $\int_0^\infty \sqrt{x}e^{-x^3}dx.$

Sol.: Put $x^3 = t, x = t^{1/3}, \therefore dx = \frac{1}{3}t^{-2/3}dt$

x	0	∞
T	0	∞

$$I = \int_0^\infty t^{1/5}.e - t.\frac{1}{3}t^{-2/3}dt$$

$$= \frac{1}{3}\int_0^\infty t^{-1/2}.e^{-t}dt = \frac{1}{3}\left|\overline{1/2}\right. = \frac{\sqrt{\pi}}{3}$$

Example12: If $In = \dfrac{\frac{\sqrt{\pi}}{2}\left|\overline{\frac{n+1}{2}}\right.}{\left|\overline{\frac{n}{2}+1}\right.}$, show that $I_{n+2} = \dfrac{n+1}{n+2} I_n$ and hence find I_5

Sol.: $I_n = \dfrac{\frac{\sqrt{\pi}}{2}\left|\overline{\frac{n+1}{2}}\right.}{\left|\overline{\frac{n}{2}+1}\right.}$, Replace n by n+2 then.

$$I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\left|\overline{\frac{n+3}{2}}\right.}{\left|\overline{\frac{n+2}{2}+1}\right.} = \frac{\frac{\sqrt{\pi}}{2}\left|\overline{\frac{n+2}{2}+1}\right.}{\frac{n+2}{2}\left|\overline{\frac{n+2}{2}}\right.}$$

$$= \frac{\sqrt{\pi}}{n+2}.\frac{\frac{n+1}{2}.\left|\overline{\frac{n+1}{2}}\right.}{\left|\overline{\frac{n}{2}+1}\right.} = \frac{n+1}{n+2}.\frac{\frac{\sqrt{\pi}}{2}.\left|\overline{\frac{n+1}{2}}\right.}{\left|\overline{\frac{n}{2}+1}\right.}$$

$$I_{n+2} = \frac{n+1}{n+2}.I_n$$ Now put n=3, then

$$I_5 = \frac{4}{5}.I3 = \frac{4}{5}.\frac{2}{3}.I_1 = \frac{8}{15}.\frac{\frac{\sqrt{\pi}}{2}.\left|\overline{1}\right.}{\left|\overline{\frac{1}{2}+1}\right.}$$

$\therefore \quad I5 = \dfrac{8}{15}$

Example13: show that $\dfrac{2n\left|\overline{\left(n+\frac{1}{2}\right)}\right.}{\sqrt{\pi}} = 1.3.5..........(2n-1)$

Sol.: $\left|\overline{\left(n+\frac{1}{2}\right)}\right. = \left(n-\frac{1}{2}\right)\left|\overline{\left(n-\frac{1}{2}\right)}\right.$ $\qquad \left(\because \left|\overline{(n+1)}\right. = n\left|\overline{n}\right.\right)$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right)\ldots\ldots\ldots\ldots\ldots\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2} \qquad \because \left(\Gamma\frac{1}{2}=\sqrt{\pi}\right)$$

$$= \frac{2n-1}{2}.\frac{2n-3}{2}.\frac{2n-5}{2}\ldots\ldots\ldots\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}$$

$$= \frac{(2n-1)(2n-3)....3.1}{2^n}\sqrt{\pi}$$

$$\therefore \qquad \frac{2n\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}} = 1.3.5\ldots\ldots\ldots\ldots\ldots(2n-1)$$

Example14: Evaluate $\int_0^\infty x^m e^{-ax^n}dx(a>0)$

So.: $I = \int_0^\infty x^m e^{-ax^n}dx$ Put $ax^n = t$, or $x = \frac{t^{1/n}}{a^{1/n}}$, $dx = \frac{1}{n}\frac{t^{\frac{1}{n}-1}}{a^{1/n}}dt$

$$= \int_0^\infty \frac{t^{m/n}}{a^{m/n}}.e-t.\frac{t^{\frac{1}{n}-1}dt}{n.a^{1/n}}$$

x	0	∞
t	0	∞

$$= \frac{1}{n.(a)^{\frac{m+1}{n}}}\int_0^\infty e^{-t}.t^{\left(\frac{m+1}{n}-1\right)}dt = \frac{1}{n}.\frac{1}{(a)^{\frac{m+1}{n}}}\Gamma\left(\frac{m+1}{n}\right)$$

Example15: Evaluate $\int_0^1 x^{a-1}\left(\log\frac{1}{x}\right)^{n-1}dx(a>0).$

Sol.: $I = \int_0^1 x^{a-1}\left(\log\frac{1}{x}\right)^{n-1}dx.$ Put $\log\frac{1}{x} = t$ or $\frac{1}{x} = e^t$, $x = e^{-t}dx = -e^{-t}dt$

$$= \int_\infty^0 (e^{-t})^{a-1}t^{n-1}(-e^{-t})dt$$

x	0	1
t	∞	0

$$= \int_0^\infty e^{-at}e^t t^{n-1}e^{-t}dt = \int_0^\infty e^{-at}t^{n-1}dt \qquad \left(\because \int_0^\infty e^{-ky}y^{n-1}dy = \frac{\Gamma n}{k}\right)$$

$$I = \frac{1}{a^n}\overline{|n}$$

Example16: Prove that $\int_0^\infty e^{-h^2}x^2 dx = \frac{\sqrt{\pi}}{2h}$

Sol.: $I = e^{-h^2}x^2 dx$ $\quad h^2x^2 = t; x = \frac{t^{1/2}}{h}; \therefore dx = \frac{1}{2h}t^{-1/2}dt$

x	0	∞
T	0	∞

$$= \int_0^\infty e^{-t}.\frac{t^{-1/2}}{2h} = \frac{1}{2h}\int_0^\infty e^{-t}t^{-1/2}dt = \frac{1}{2h}\overline{\left|\frac{1}{2}\right.} = \frac{\sqrt{\pi}}{2h} \qquad \left(\because \overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}\right)$$

Example17: show that $\int_0^\infty x^{m-1}\cos axdx = \frac{\overline{|m}}{am}\cos\frac{m\pi}{2}$

Sol.: $\quad e^{-iax} = \cos ax - i\sin ax$

$\therefore \quad \cos ax =$ Real part of $e^{-i}ax$

$$I = \int_0^\infty x^{m-1}\cos axdx.$$

$= \text{Real part of } \int_0^\infty x^{m-1}.e^{-iax}dx$ (note this step carefully.)

$= \text{Real part of } \int_0^\infty \frac{t^{m-1}}{(ia)^{m-1}}e^{-t}.\frac{dt}{ia}.$ $\qquad iax = t$ or $x = \frac{t}{ia} dx = \frac{dt}{ia}$

x	0	∞
t	0	∞

$= \text{Real part of } \frac{1}{i^m a^m}\int_0^\infty e^{-t}t^{m-1}dt.$

$I = \text{Real part of } \frac{1}{i^m a^m}\overline{|m} = \text{Real part of } \frac{\overline{|m}}{a^m}.\left(\frac{1}{i^m}\right)$

But $\qquad i = \cos\pi/2 + i\sin\pi/2$

$\therefore \qquad i^m = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^m$

$\therefore \qquad i^m = \cos\frac{m\pi}{2} + i\sin m\frac{m\pi}{2}......$

$$\therefore \qquad \frac{1}{i^m} = \frac{1}{\cos\frac{m\pi}{2} + i\sin\frac{m\pi}{2}} = \cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}$$

$$\therefore \qquad I = \text{Real part of } \frac{\lfloor m}{a^m}\left(\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right)$$

$$= \text{Real part of } \left[\frac{\lfloor m}{a^m}\cos\frac{m\pi}{2} - i\frac{\lfloor m}{a^m}\sin\frac{m\pi}{2}\right]$$

$$I = \frac{\lfloor m}{a^m}\cos\frac{m\pi}{2}$$

ILLUSTRATIONS OF BETA FUNCTIONS

Type 1: Examples Involving substitutions:

Ex.18: Evaluate $\int_0^1 x^3 \left(1-\sqrt{x}\right)^5 dx.$

Sol.: This can be reduced to beta function by putting $\sqrt{x} = t, x = t^2, dx = 2tdt$

X	0	1
T	0	1

$$\therefore \qquad I = \int_0^1 (t^2)^3 (1-t)^5 .2tdt = 2\int_0^1 t^7 (1-t)^5 dt = 2\beta(8.6)$$

$$= 2\frac{\lfloor 8 \lfloor 6}{\lfloor 14} = \frac{2(7!)(5!)}{13!} = \frac{1}{5148}$$

Ex.19: Evaluate $\int_0^a (a^6 - x^6)^{1/6} dx.$

Sol: This can be expressed in beta function consider:

$$I = \int_0^a (a^6 - x^6) dx = \int_0^a a\left(1 - \left(\frac{x}{a}\right)^6\right)^{1/6} dx$$

$$\therefore \text{ Put } \left(\frac{x}{a}\right)^6 = t, \therefore \frac{x}{a} = t^{1/6}, x = at^{1/6}, dx = a\frac{1}{6}t^{-5/6}dt$$

x	0	a
t	0	1

$$I=\int_0^1 a(1-t)^{1/6}.\frac{1}{6}t^{-3/5}dt$$

$$=\frac{a^2}{6}\int_0^1 t^{-5/6}(1-t)^{1/6}dt=\frac{a^2}{6}\beta\left(\frac{1}{6},\frac{7}{6}\right)$$

$$=\frac{a^2}{6}\frac{\Gamma(1/6)\Gamma(7/6)}{\Gamma(8/6)}=\frac{a^2}{6}\frac{\Gamma(1/6).\frac{1}{6}\Gamma(1/6)}{\frac{1}{3}\Gamma(1/3)}$$

$$=\frac{a^2}{12}\frac{\left(\Gamma\frac{1}{6}\right)^2}{\Gamma(1/3)}$$

Ex.20: Evaluate $\int_0^1 x^2(1-x^2)^4dx.$

Sol.: $x^2=t, 2xdx=dt, dx=\dfrac{dt}{2\sqrt{t}}$

x	0	1
T	0	1

$$\therefore\ I=\int_0^1 t.(1-t)^4\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_0^1 t^{1/2}(1-t)^4dt$$

$$=\frac{1}{2}\beta(3/2,5)=\frac{1}{2}\frac{\Gamma(3/2)\Gamma(5)}{\Gamma(13/2)}=\frac{128}{3465}$$

Ex.21: Evaluate $\int_0^1 x4(1-x)^{3/2}dx.$

Sol.: $I=\int_0^1 x4(1-x)3/2dx=\beta(5,5/2)=\dfrac{\Gamma(5)\Gamma(5/2)}{\Gamma(15/2)}$

$$=\frac{4!\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}{\frac{13}{2}.\frac{11}{2}.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}}$$

$$=\frac{24x32}{13.11.9.7.5}=\frac{256}{15015}$$

Ex.22: Evaluate $x^7(16-x^4)^{10}dx$

Sol.: This can be reduced to beta function by substitution

$x^4 = 16t$

$\therefore\ 4x^3\,dx = 16\,dt$,

$x^3\,dx = 16/4\,dt = 4\,dt$

x	0	2
t	0	1

$$I = \int_0^2 x^4.(16-x^4)^{10}.x^3 dx = \int_0^1 16t(16-16t)^{10} 4dt$$

$$= 16^{11} x 4 \int_0^1 t(1-t)^{10} dt = 4x16^{11}.\beta(2,11)$$

$$= 4x16^{11} \frac{\overline{|1}\,\overline{|10}}{\overline{|12}} = 4x16^{11} \frac{1x9!}{11!}$$

$$= 4x16^{11} \frac{9!}{9!x10x11} = \frac{2x16^4}{55}$$

Ex.23: Evaluate $\int_0^1 xm(1-x^n)^p dx.$

sol.: Put x^n=t, x=$t^{1/n}$, $\therefore\ dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

x	0	1
T	0	1

$$\therefore\ I = \int_0^1 (t^{1/n})m(1-t)^p . \frac{1}{n} t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{n} \int_0^1 t^{\frac{m+1}{n}-1} (1-t)^p dt$$

$$= \frac{1}{n} \beta\left(\frac{m+1}{n}, p+1\right) = \frac{1}{n} \frac{\overline{\left|\frac{m+1}{n}\right.}\,\overline{|p+1}}{\overline{\left|\frac{m+1}{n} + p+1\right.}}$$

Ex.24: Evaluate $\int_0^1 x^5(1-x^3)^{10} dx$

Sol.: Put x^3= t, $3x^3$ dx= dt, x^2 dx= $\frac{d\pi}{3}$

x	0	1
T	0	1

$$\therefore \qquad I=\int_0^1 x^3(1-x^3)^{10}.x^2dx=\int_0^1 t(1-t)^{10}\frac{dt}{3}$$

$$=\frac{1}{3}\int_0^1 t(1-t)^{10}dt=\frac{1}{3}\beta(2,11)=\frac{\Gamma 2\,\Gamma 11}{3\,\Gamma 13}=\frac{1}{396}$$

Ex.25: Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^m}}$

Sol.: $I=\int_0^1 (1-xm)^{-1/2}dx$ Put $x^m=t, x=t^{1/m}, dx=\frac{1}{m}t^{\frac{1}{m}-1}dt$

$$I=\int_0^1 (1-t)^{-1/2}.\frac{1}{m}t^{\frac{1}{m}-1}dt$$

x	0	1
t	0	1

$$=\frac{1}{m}\int_0^1 t^{\frac{1}{m}-1}(1-t)^{-1/2}dt=\frac{1}{m}\beta\left(\frac{1}{m},\frac{1}{2}\right)$$

$$=\frac{1}{m}\frac{\Gamma(1/m)\,\Gamma(1/2)}{\Gamma\left(1/m+\frac{1}{2}\right)}=\frac{1}{m}\frac{\Gamma(1/m)\sqrt{\pi}}{\Gamma\left(\frac{1}{m}+\frac{1}{2}\right)}$$

Ex.26: Evaluate $\int_0^1 \sqrt{1-x^4}\,dx$

Sol.: $I=\int_0^1 (1-x^4)^{1/2}dx$ Put $x^4=t, x=t^{1/4}, \therefore dx=\frac{1}{4}t^{-3/4}dt$

x	0	1
T	0	1

$$I=\int_0^1 (1-t)^{1/2}.\frac{1}{4}t^{-3/4}dt=\frac{1}{4}\int_0^1 t^{-3/4}(1-t)^{1/2}dt$$

$$=\frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right) \qquad =\frac{1}{4}\frac{\Gamma(1/4)\,\Gamma(3/2)}{\Gamma(7/4)}$$

$$=\frac{1}{4}.\frac{\Gamma(1/4).2\,\Gamma(1/2)}{\frac{3}{4}\Gamma(3/4)} \qquad =\frac{1}{6}\frac{\Gamma(1/4)}{\Gamma(3/4)} \qquad \sqrt{\pi}$$

Ex.27: Evaluate $\int_0^{2a} x\sqrt{2ax-x^2}\,dx$

Sol.: $I=\int_0^{2a} x.\sqrt{x}(2a-x)^{1/2}dx$ Put $x=2at, dx=2adt$

x	0	2a
T	0	1

$$I=\int_0^1 (2at)^{3/2}(2a-2at)^{1/2}.2adt=(2a)^3\int_0^1 t^{3/2}(1-t)^{1/2}dt$$

$$=8a^3.\beta(5/2,3/2) \qquad =8a^3\frac{\left|\overline{5/2}\right|\overline{3/2}}{\left|\overline{4}\right.}$$

$$=8a^3\frac{\frac{3}{2}.\frac{1}{2}.\sqrt{1/2}\,\frac{1}{2}\left|\overline{1/2}\right.}{3!}=\frac{8a^3x3}{8}.\frac{1}{6}.\sqrt{\pi}.\sqrt{\pi}=\frac{\pi a^3}{2}$$

Type 2: Examples involving application of

$$\int \sin px\cos qxdx=\frac{1}{2}\frac{\left|\overline{\frac{p+1}{2}}\right|\overline{\frac{q+1}{2}}}{\left|\overline{\frac{p+q+2}{2}}\right.}$$

Ex.28: Express $\int_0^{\pi/2}\sqrt{\tan\theta}d\theta$ as gamma function.

Sol.: $I=\int_0^{\pi/2}\sqrt{\tan\theta}d\theta=\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta d\theta$

$$=\frac{\left|\overline{\frac{\frac{1}{2}+1}{2}}\right|\overline{\frac{\frac{1}{2}+1}{2}}}{2\left|\overline{\frac{\frac{1}{2}-\frac{1}{2}}{2}+1}\right.}=\frac{\left|\overline{3/4}\right|\overline{1/4}}{2}=\frac{\pi\sqrt{2}}{2}=\frac{\pi}{\sqrt{2}}$$

Ex.29: Prove that $\int_0^{\infty}\frac{dx}{1+x^4}=\frac{\pi\sqrt{2}}{4}$

Sol.: $I=\int_0^{\infty}\frac{dx}{1+x^4}$ Put $x^2=\tan\theta$ or $x=\sqrt{\tan\theta}dx=\frac{1}{2}\tan^{-1/2}\theta\sec^2\theta d\theta$

$$= \int_0^{\pi/2} \frac{\frac{1}{2}\tan^{-1/2}\theta \sec^2\theta d\theta}{1+\tan 2\theta}$$

x	0	∞
θ	0	$\pi/2$

$$= \frac{1}{2}\int_0^{\pi/2} \tan^{-1/2}\theta d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{4}\frac{\pi}{\sin \pi/4} = \frac{\sqrt{2}\pi}{4}$$

Ex.30: Evaluate $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$

Sol.: $\displaystyle I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$

Put $\cos^2\theta = \sqrt{t}$ or $\cos\theta = t^{\frac{1}{4}}, \theta = \cos^{-1}\left(t^{\frac{1}{4}}\right), d\theta = \dfrac{-1}{\sqrt{1-t^{1/2}}}.\dfrac{1}{4}t^{-3/4}dt$

When $\theta = 0 t = 1; \theta = \dfrac{\pi}{2} t = 0$ Also $\sin^2\theta = 1-\cos^2\theta = 1-\sqrt{t}$

$$I = \int_0^1 \frac{-1}{\sqrt{1-\sqrt{t}}}.\frac{1}{4}\frac{t^{-3/4}dt}{\sqrt{1-\frac{1}{2}(1-\sqrt{t})}} = \frac{\sqrt{2}}{4}\int_0^1 \frac{t^{-3/4}dt}{\sqrt{1-t}}$$

$$= \frac{1}{4}\int_0^1 \frac{t^{-3/4}dt}{\sqrt{1-\sqrt{t}}\sqrt{\frac{1}{2}(1+\sqrt{t}}} = \frac{\sqrt{2}}{4}\int_0^1 \frac{t^{-3/4}}{\sqrt{1-t}}$$

$$= \frac{\sqrt{2}}{4}\int_0^1 t^{-3/4}(1-t)^{-1/2}dt,$$

$$= \frac{\sqrt{2}}{4}.B\left(\frac{1}{4},\frac{1}{2}\right) = \frac{\sqrt{2}}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{2}.\sqrt{\pi}}{4}\frac{\left(\Gamma(1/4)\right)^2}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)} = \frac{\sqrt{2\pi}\left(\Gamma(1/4)\right)^2}{4.\frac{\pi}{\sin\pi/4}} = \frac{2\sqrt{\pi}\,(\Gamma(1/4))^2}{4\pi} = \frac{\left(\Gamma(1/4)\right)^2}{2\sqrt{\pi}}$$

Ex.31: Prove that $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}B\left(\frac{n}{2},\frac{n}{2}\right)$ and hence evaluate $\int_0^\infty \sec h^8 x dx$

Sol.: $I = \int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{\left(e^x+e^{-x}\right)^n}$ (Note this step)

Put $e^x = \tan\theta x \to -\infty, \theta \to 0; x \to \infty\theta \to \frac{\pi}{2}; e^x dx = \sec^2\theta d\theta \Rightarrow dx = \frac{\sec^2\theta}{\tan\theta}d\theta$

X	$-\infty$	∞
θ	0	$\pi/2$

$$I = \frac{1}{2}\int_0^{\pi/2}\frac{\sec^2\theta/\tan\theta}{(\tan\theta+\cot\theta)^n}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\frac{\frac{1}{\sin\theta\cos\theta}}{\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)^n}d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{\sin^n\theta\cos^n\theta}{\sin\theta\cos\theta}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\sin n-1\theta\cos n-1\theta d\theta = \frac{1}{2}.\frac{1}{2}B\left(\frac{n-1+1}{2},\frac{n-1+1}{2}\right)$$

$\int_0^\infty \frac{dx}{(e^x+e^{-x})} = \frac{1}{4}B\left(\frac{n}{2},\frac{n}{2}\right)$ But $\cosh x = \frac{e^x+e^{-x}}{2} i.e. e^x + e^{-x} = 2\cosh x$

$\int_0^\infty \frac{dx}{(2\cosh x)^n} = \frac{1}{4}B\left(\frac{n}{2},\frac{n}{2}\right)$ Put n= 8

$$\int_0^\infty \frac{dx}{(2\cosh x)^8} = \frac{1}{4}B(4,4) = \frac{1}{4}\frac{\Gamma 4\,\Gamma 4}{\Gamma 8} = \frac{1}{4}\frac{3!3!}{7!} = \frac{1}{560}$$

$$\therefore \int_0^\infty \sec h8xdx = \frac{2^8}{560} = \frac{16}{35}$$

Ex.32: Show that $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta \int_0^{\pi/2}\sqrt{\cos\theta}\,d\theta = \frac{\pi^2}{2}$

Sol.: $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta d\theta$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{-1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+1}{2}\right)} = \frac{\frac{1}{2}.\Gamma\frac{3}{4}.\Gamma\frac{1}{4}}{\Gamma 1}$$

$$= \frac{1}{2}\Gamma\frac{1}{4}.\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{\sin\pi/4}\right) = \frac{\pi}{\sqrt{2}}$$

$$\left(\because \Gamma p\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}, 0 < p < 1\right)$$

$$\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta = \int_0^{\pi/2}\sqrt{\cot\left(\frac{\pi}{2}-\theta\right)}d\theta \qquad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$= \int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta \qquad ..\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$$

$$= \frac{\pi}{\sqrt{2}} \text{ (By above result)}$$

$$\therefore \int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta \int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta = \frac{\pi}{\sqrt{2}}.\frac{\pi}{\sqrt{2}} = \frac{\pi^2}{2}$$

Ex.33: Evaluate $\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}} x \int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta$

Sol.: $I = \int_0^{\pi/2} \sin^{-1/2}\theta d\theta x \int_0^{\pi/2} \sin^{1/2}\theta d\theta$

$$= \frac{\Gamma(1/4)\Gamma(1/2)}{2\Gamma(3/4)} \quad \frac{\Gamma(3/4)\Gamma(1/2)}{2\Gamma(5/4)}$$

$$= \frac{\Gamma(1/4).\sqrt{\pi}.\sqrt{\pi}}{4.\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi$$

Ex.34: Evaluate $\int_0^{\pi/6} \sin^2 6\theta \cos^6 3\theta d\theta$

Sol.: $I = \int_0^{\pi/6} \sin^2 6\theta \cos^6 3\theta d\theta = \int_0^{\pi/6} (2\sin 3\theta \cos 3\theta)^2 \cos^6 3\theta d\theta$

$= 4\int_0^{\pi/6} \sin^2 3\theta \cos^8 3\theta d\theta$ Put $3\theta = t, \theta = t/3, d\theta = \frac{1}{3}dt$

θ	0	$\pi/6$
t	0	$\pi/2$

$\therefore$ $I = 4\int_0^{\pi/2} \sin^2 t \cos^8 t \frac{dt}{3} = \frac{4}{3}\frac{\beta(3/2, 9/2)}{2}$

$$= \frac{4}{3}\frac{1}{2}\frac{\Gamma(3/2)\Gamma(9/2)}{\Gamma(6)} \qquad = \frac{2}{3}\frac{\frac{1}{2}\Gamma(1/2)\frac{7}{2}.\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(1/2)}{5!}$$

$$= \frac{1}{3}\frac{7.5.3.1}{16x5!}.\pi \qquad = \frac{35\pi}{16x120}$$

$$= \frac{7\pi}{384}$$

Ex.35: Express $\int_0^{\pi/4} \cos^3 2x \sin^4 4x dx$, in terms of beta function and evaluate.

Sol.: $I = \int_0^{\pi/4} \cos^3 2x(2\sin 2x \cos 2x)^4 dx$

$= 16\int_0^{\pi/4} \sin^4 2x, \cos^7 2x dx$ Put $2x = \theta, x = \frac{\theta}{2}, dx = \frac{d\theta}{2}$

x	0	$\pi/4$
θ	0	$\pi/2$

$\therefore$ $I = 16\int_0^{\pi/2} \sin^4\theta \cos 7\theta \frac{d\theta}{2} \qquad = 8\int_0^{\pi/2} \sin^4\theta \cos 7\theta d\theta$

$$=8\frac{\beta(5/2,4)}{2}$$

$$=4.\frac{\Gamma(5/2)}{\frac{11}{2}\frac{9}{2}\frac{7}{2}.\frac{5}{2}\Gamma(5/2)}3!$$

$$=\frac{384}{1155}$$

$$=4\frac{\Gamma(5/2)\Gamma(4)}{\Gamma(13/2)}$$

$$=\frac{4x6x16}{11x9x7x5}$$

$$=\frac{128}{385}$$

Type 3: Examples involving application of

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{and}$$

$\Gamma(n+1)=n\Gamma(n)$, in general

$= n$ I if n is positive integer

Ex.36: Prove that $B(m,n)=B(m,n+1)+B(m+1,n)$

Sol.: $R.H.S.=B(m,n+1)+B(m+1,n)$

$$=\frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}+\frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+1+n)}=\frac{\Gamma(m)n\Gamma(n)}{(m+n)\Gamma(m+n)}+\frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}$$

$$=\frac{\Gamma(m)\Gamma(n)(n+m)}{(m+n)\Gamma(m+n)}=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=B(m,n)=LHS.$$

Ex.37: Show that $B(m,n)B(m+n,p)=\frac{\Gamma(m)\Gamma(n)\Gamma(p)}{\Gamma(m+n+p)}$

Sol.: LHS $B(m,n)B(m+n,p)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\frac{\Gamma(m+n)\Gamma(p)}{\Gamma(m+n+p)}$

Ex.38: Prove that $B(x+1,y)=\frac{x}{x+y}B(x,y)$.

Sol.: L.H.S.= $B(x+1,y)$

$$=\frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+1+y)}=\frac{x\Gamma(x)\Gamma(y)}{(x+y)\Gamma(x+y)}$$

$$=\frac{x}{x+y}B(x,y)=R.H.S.$$

Ex.39: Prove that $yB(x+1,y)=xB(x,y+1)$

Sol.: L.H.S.= $yB(x+1,y)=y\dfrac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+1+y)}$

$$=\frac{x\Gamma(x).y\Gamma(y)}{\Gamma(y+1+x)}=x\frac{\Gamma(x)\Gamma(y+1)}{\Gamma(x+y+1)}=xB(x,y+1)=R.H.S.$$

Ex.40: Prove $B(x,x)=\dfrac{1}{2^{2x-1}}B\left(x,\dfrac{1}{2}\right)$

Sol.: We have by duplication formula:

$$\Gamma(m)\Gamma(m+1/2)=\frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}$$

$$\therefore \qquad \frac{\Gamma(m)}{\Gamma(2m)}=\frac{\sqrt{\pi}}{2^{2m-1}\Gamma(m+1/2)}$$

Now, $B(x,)=\dfrac{\Gamma(x)\Gamma(x)}{\Gamma(2x)}=\Gamma(x).\dfrac{\Gamma(x)}{\Gamma(2x)}$

$$=\Gamma(x).\frac{\sqrt{\pi}}{2^{2x-1}.\Gamma\left(x+\frac{1}{2}\right)}=\frac{\Gamma(x).\Gamma(1/2)}{2^{2x-1}.\Gamma\left(x+\frac{1}{2}\right)}=\frac{1}{2^{2x-1}}.B\left(x,\frac{1}{2}\right)$$

Type 4: Examples involving application of second form of definition of beta function.

$$B(m,n)=\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

Ex.41: Evaluate $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$

Sol.: $I=\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx=\int_0^\infty \frac{x^8-x^{14}}{(1+x)^{24}}dx$

$$=\int_0^\infty \frac{x^8dx}{(1+x)^{24}}-\int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx$$

Using 2nd definition of Beta form.

We get, $=B(9,15)-B(15,9)$ $\qquad \begin{bmatrix} m-1=8 & \therefore m=9 \\ m+n=24 & \therefore n=15 \end{bmatrix}$

$$= B(9,15) - B(9,15) \quad [\therefore B(m,n) = B(n,m)]$$
$$= 0$$

Ex.42: Evaluate $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx$

Sol.: $$I = \int_0^\infty \frac{x^4 + x^9}{(1+x)^{15}}dx = \int_0^\infty \frac{x^4}{(1+x)^{15}}dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}dx$$
$$= B(5,10) + B(10,5) = 2B(5,10)$$
$$= 2\frac{\left|\overline{5}\right|\overline{10}}{\left|\overline{15}\right.} \qquad = 2\frac{4!9!}{14!}$$
$$= \frac{1}{5005}$$

Type 5: Miscellaneous Examples:

Ex.43: Show that $\int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}dy = B(m,n)$

Sol.: We have,
$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx.$$
$$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = I_1 + I_2$$

Consider $$I_2 = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad \text{Put } x = \frac{1}{t} \text{ or } t = \frac{1}{x} \quad \therefore dx = -\frac{dt}{t^2}$$

X	1	∞
T	0	1

$$= \int_1^0 \left(\frac{1}{t^{m-1}}\right) \frac{1}{\left(1+\frac{1}{t}\right)^{m+n}} \cdot \left(-\frac{dt}{t^2}\right)$$
$$= \int_0^1 \frac{t^{m+n}dt}{t^{m-1}(1+t)^{m+n}} = \int_0^1 \frac{t^{n-1}dt}{(1+t)^{m+n}}$$
$$I_2 = \int_0^1 \frac{x^{n-1}dx}{(1+x)^{m+n}}$$

$\therefore$ $$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

Ex.44: Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{B(m,n)}{a^n(1+a)^m}$

Sol.: $I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx.$ Put $\frac{x}{a+x} = \frac{t}{a+1}$ (Note this substitution).

$\therefore$ $x(a+1) = t(a+x)$ Or $x.(a+1-t)at$

Or $x = \frac{at}{a+1-t}$

$\therefore$ $dx = \frac{(a+1-t)(a) - at(-1)}{(a+1-t)^2}.dt = \frac{a(a+1)dt}{(a+1-t)^2}$

Also $1 - x = 1 - \frac{at}{a+1-t} = \frac{a+1-t-at}{a+1-t}$

$$= \frac{a+1-t(a+1)}{a+1-t} = \frac{(a+1(1-t)}{a+1-t}$$

$$a + x = a + \frac{at}{a+1-t} = \frac{a(a+1) - at + at}{a+1-t} = \frac{a(a+1)}{(a+1-t)}$$

When x=0, $0 = \frac{at}{a+1-t} \Rightarrow t = o$; when x= 1, $\frac{1}{a+1} = \frac{t}{a+1} \Rightarrow t = 1$

x	0	1
T	0	1

$\therefore$ $I = \int_0^1 \frac{a^{m-1}t^{m-1}(a+1)^{n-1}(1-t)^{n-1}a(a+1)dt}{(a+1-t)^{m-1}(a+1-t)^{n-1}(a+1-t)^2} \quad \frac{(a+1-t)^{m+n}}{a^{m+n}(a+1)^{m+n}}$

$$= \frac{1}{a^n(1+a)^m} \int_0^1 t^{m-1}(1-t)^{n-1}.dt = \frac{B(m,n)}{a^n(a+1)^m}$$

Ex.45: Prove that $\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(a+bx)^{p+q}} dx = \frac{B(p,q)}{a^q(a+b)^p}$

Sol.: $I = \int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(a+bx)^{p+q}} dx$ Put $\frac{x}{a+bx} = \frac{y}{b+a}$

$(b+a)x = ay + b.x.y, (b+a-by)x = ay$ $\therefore x = \frac{ay}{b+a-by}$

$$1 - x = 1 - \frac{ay}{b+a-by} = \frac{b+a-by-ay}{b+a-by} = \frac{(b+a)(a-y)}{b+a-by}$$

$$a + bx = a + \frac{aby}{a+b-by} = \frac{a(a+b)}{a+b-by}$$

Also $\quad dx=\dfrac{a(a+b)}{(b+a-by)^2}dy$

When x=0 $=\dfrac{ay}{b+a-by}\Rightarrow y=0$

When x= 1 $\quad \dfrac{1}{a+b}=\dfrac{y}{b+a}\Rightarrow y=1$

$$I=\int_0^1 \frac{a^{p-1}.y^{p-1}}{(b+a-by)^{p-1}}.\frac{(b+a)^{q-1}(1-y)^{q-1}}{(b+a-by)^{q-1}}.\frac{(a+b-by)^{p+q}}{a^{p+q}(a+b)^{p+q}}\frac{a(a+b)}{(b+a-by)^2}dy$$

$$=\frac{1}{a^q(a+b)^p}\int_0^1 y^{p-1}(1-y)^{q-1}dy=\frac{1}{a^q(a+b)^p}B(p,q)$$

Ex.46: Evaluate $\displaystyle\int\frac{\cos^{2m-1}\theta\sin^{2n-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{m+n}}d\theta$

Sol.: Divide and multiply by $\cos^{2m+2n}\theta$

$$\therefore \qquad I=\int_0^{\pi/2}\frac{\dfrac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta}\dfrac{1}{\cos^2\theta}d\theta}{(a^2+b^2\tan^2\theta)^{m+n}}$$

$$=\int_0^{\pi/2}\frac{(\tan^2\theta)^{n-1}.\tan\theta\sec^2\theta d\theta}{(a^2+b^2\tan^2\theta)^{m+n}}$$

Put $\quad b^2.\tan^2\theta=a^2y \qquad \tan^2\theta=\dfrac{a^2}{b^2}.y$

$2\tan\theta\sec^2 d\theta=\dfrac{a^2}{b^2}dy \qquad \therefore \tan\theta\sec^2\theta d\theta=\dfrac{a^2}{2b^2}dy$

$$\therefore \qquad I=\int_0^{\infty}\frac{\left(\dfrac{a^2y}{b^2}\right)n-1\dfrac{a^2}{2b^2}dy}{(a^2+a^2y)^{m+n}}$$

$$=\frac{a^{2n}}{2b^{2n}.a^{2m}a^{2n}}\int_0^{\infty}\frac{y^{n-1}}{(1+y)^{m+n}}dy$$

$$=\frac{1}{2a^{2m}b^{2n}}B(m,n)$$

Ex.47: Using $B(m,n)=\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}dx,$ show that $B(m,n)=\displaystyle\int_0^{\infty}\frac{y^{m-1}}{(1+y)^{m+n}}dy.$

Evaluate $\displaystyle\int_1^{\infty}\frac{dx}{x^{p+1}(x-1)^{\theta}}$

Sol.: For the first part, refer property 4 of beta function. For the second part, Let $= \int_1^\infty \frac{dx}{x^{p+1}(x-1)^\theta}$ Put $x = \frac{1}{t} dx = \frac{-1}{t^2} dt$

x	1	∞
T	1	0

$$= \int_0^1 \frac{-1/t^2 dt}{\frac{1}{t^{p+1}}\left(\frac{1}{t}-1\right)^\theta}$$

$$= \int_0^1 \frac{1}{t2} \cdot \frac{t^{p+1}t^\theta}{(1-t)^\theta} dt = \int_0^1 t^{p+\theta-1}(1-t)^{-\theta} dt = B(p+\theta, 1-\theta)$$

Ex.48: Prove that $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^n b^m} B(m,n)$

Sol.: $I = \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx$ Put bx= at; $dx = \frac{a}{b} dt$

x	0	∞
t	0	∞

$$= \int_0^\infty \frac{a^{m-1}t^{m-1}}{b^{m-1}} \cdot \frac{a}{b} . dt \frac{1}{(a+at)^{m+n}}$$

$$= \int_0^\infty \frac{a^m t^{m-1} dt}{b^m a^{m+n}(1+t)^{m+n}} = \frac{1}{a^n b^m} \int_0^\infty \frac{t^{m-1} dt}{(1+t)^{m+n}} = \frac{1}{a^n b^m} B(m,n)$$

Ex.49: Evaluate $\int_a^b (x-a)^m (b-x)^n dx$

Sol.: $I = \int_a^b (x-a)^m (b-x)^n dx$ Put $x-a = (b-a)t$. Then $dx = (b-a)dt$

X	a	b
T	0	1

$$= \int_0^1 (b-a)^m t^m \left[b-a-(b-a)t\right]^n (b-a)dt = (b-a)^{m+1} \int_0^1 t^m \left[(b-a)(1-t)\right]^n dt$$

$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt = (b-a)^{m+n+1} B(m+1, n+1)$$

Ex.50: Evlauate $\int_3^7 (x-3)^{1/4}(7-x)^{1/4} dx$

Sol.: Put x= 4t+3, dx= 4 dt (Note this substitution)

x	3	7
T	0	1

$$I=\int_0^1 (x-3)^{1/4}(7-x)^{1/4}\,dx.$$

$$=\int_0^1 (4t)^{1/4}(7-4t-3)^{1/4}\,4dt=\int_0^1 4^{1/4}t^{1/4}[4(1-t)]^{1/4}\,4dt$$

$$=4^{3/2}\int_0^1 t^{\frac{1}{4}}(1-t)^{1/4}\,dt=8B\left(\frac{5}{4},\frac{5}{4}\right)=8\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{2}\right)}=8\frac{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}=\frac{2}{3\sqrt{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2$$

Exercise 2.1

Evaluate the following

1. $\int_0^2 y^4(8-y^3)^{-1/3}\,dy$

   Hint: Put y= 2t.

   Ans.: $\frac{16}{3}\beta\left(\frac{5}{3},\frac{2}{3}\right)$

2. $\int_0^1 \left(1-\sqrt[n]{x}\right)^m dx$

   Hint: Put $x^{1/n}=t$

   Ans.: $\frac{m!n!}{(m+n)!}$

3. $\int_0^\infty \left(\frac{t}{1+t^2}\right)^4 dt$

   Hint: Put $t=\tan\theta$

   Ans.: $\frac{1}{2}\beta(5/2,3/2)$

4. $\int_0^\infty \frac{x^5(1-x^3)}{(1+x)^{15}}dx$

   Ans.: 0

5. $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

Ans.: $\pi/\sqrt{2}$

6. $\int_0^{a} x^3 \sqrt{(a^2 - a^2)^3}\, dx$ (S.U. 1987)

Ans.: $\dfrac{a^6 \pi}{32}$

8. $\int_0^{1} \sqrt{x \log(1/x) dx}$

Ans.: $\dfrac{\sqrt{\pi}}{\sqrt{6}}$

9. $\int_0^{\infty} \sqrt{x} e^{-x^2} dx x \int_0^{\infty} \dfrac{e^{-x^2}}{\sqrt{x}} dx$

Hint: Put $x^2 = t$ & use $\left|\overline{1/4}\right|\overline{3/4} = \pi\left|\overline{2}\right.$

Ans.: $\dfrac{\pi}{2\sqrt{2}}$

10. $\int_0^{1} x^2 (1-x^2)^4 dx$ (S.U. 1988)

Ans.: $\dfrac{1}{2} B(3/2,5)$

11. $\int_0^{\infty} \sqrt{x} e - ^{\sqrt[3]{x}}\, dx = \dfrac{315}{16}\sqrt{\pi}$

Hint: $(x = t^3)$

12. $\int_0^{\infty} x^7 e^{-2x^2} dx = \dfrac{3}{16}$

Hint: $(2x^2 = t)$

13. $\int_0^{\infty} x^2 e^{-h^2 x^2} dx = \dfrac{\sqrt{\pi}}{4h^3}$

Hint: $h^2 x^2 = t$

14. $\int_0^\infty \sqrt{y}\, e^{-y3} dy = \frac{\sqrt{\pi}}{3}$

Hint: $y^3 = t$

15. $\int x^{n-1} e^{-h^2 x^2} dx = \frac{\overline{|n/2}}{2h^n}$

Hint: $h^2 x^2 = t$

16. $\int_0^\infty e^{-x^4} dx = \frac{1}{4}\overline{\left|\frac{1}{4}\right.}$

Hint: $x^4 = t$

17. $\int_0^1 \frac{dx}{\sqrt{x \log \frac{1}{x}}} = \sqrt{2\pi}$

Hint: $\left(\log \frac{1}{x} = t\right)$

18. $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$

Hint: $(-\log x) = t)$

19. $\int_0^\infty \frac{x^4}{4^x} dx = \frac{24}{(\log 4)^5}$

Hint: $(\therefore 4 = e^m)$

20. $\int_0^\infty \frac{x^5}{5^x} dx = \frac{120}{(\log 5)^6}$

Hint: $(\because 5 = e^m)$

21. $\int_0^1 x^m (\log x)^n dx = (-1)^n \frac{\overline{|(n+1}}{(m+1)^{n+1}}$

Hint: $(\log x = -t)$

22. $\int_0^\infty x^n e^{-\sqrt{ax}} dx = \frac{2(2n+1)!}{a^{n+1}}$

(N is integer)

Hint: $\sqrt{ax} = t$

23. $\int_0^\infty a^{-4x^2} dx = \dfrac{\sqrt{\pi}}{4\sqrt{\log a}}$

Hint: $(a = e^m, 4x^2 = t)$

24. $\int_0^\infty x^{n-1} e^{-ax} \cos bx dx = \dfrac{\lfloor\overline{n}}{\left(a^2 + b^2\right)^{n/2}} \cos\left(n \tan^{-1} \dfrac{b}{a}\right)$

Hint: $e^{ibx} = \cos bx + i \sin bx.; (a - ib)x = t$

25. $\int_0^\infty x^{n-1} e^{-ax} \sin bx dx = \dfrac{\lfloor\overline{n}}{(a^2 + b^2)^{n/2}} \sin\left(n \tan^{-1} \dfrac{b}{a}\right)$

Hint: $e^{ibx} = \cos bx + i \sin bx$ and $(a - ib)x = t)$

26. $\int_0^\infty x^{n-1} \sin bx = \dfrac{\lfloor\overline{n}}{b^n} \sin \dfrac{n\pi}{2}$

Hint: $e^{-ibx} = \cos bx - i \sin bx; ibx = t$

27. $\int_0^1 (x \log x)^3 dx = -\dfrac{3}{128}$

Hint: $\log x = -t$

28. $\int_0^\infty \sqrt[3]{x2} e - \sqrt[3]{x} dx = 72$

Hint: $x = t^3$

29. $\int_0^\infty x^n . e^{-xm} dx = \dfrac{1}{m} \left\lfloor\overline{\left(\dfrac{n+1}{m}\right)}\right.$

Hint: $x^m = t$

30. $\int_0^1 (\log x)^n dx = (-1)^n \lfloor\overline{n}$

Hint: $\log x = -t$

31. $\int_0^1 \frac{x\,dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\frac{\pi}{2}}$

Hint: $\left(\log \frac{1}{x} = t\right)$

32. Expres in terms of gamma functions $\int_0^1 x^m (1-x^n)^p\,dx.$

Hint: $x^n = t$

Ans.: $\frac{1}{n}\frac{\left|\overline{\frac{m+1}{n}}\right|\overline{p+1}}{\left|\overline{\frac{m+1}{n}+p+1}\right.}$

33. Evaluate $\int_0^1 x^3(1-\sqrt{x})^5\,dx$

Hint: $\sqrt{x} = t$

34. Prove that $\int \frac{dx}{\sqrt{1-xm}} = \frac{\sqrt{\pi}}{m}\frac{\left|\overline{1/m}\right.}{\left|\overline{\frac{1}{m}+\frac{1}{2}}\right.}$

Hint: $x^m = t$

35. Show that $(i) B(m+1,n) = \frac{m}{m+n}B(m,n)(ii) nB(m+1,n) = mB(m,n+1)$

36.Show that $\int \frac{dx}{1+x4} = \frac{\pi}{2\sqrt{2}} (Putx^2 = \tan\theta)$

37. Show that $B(n,n+1) = \frac{1}{2}\frac{\left(\left|\overline{n}\right.\right)^2}{\left|\overline{2n}\right.}$

38. Show that $\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}}$ (Put $x^3 = t$. Use $\left|\overline{p}\right|\overline{1-p} = \frac{\pi}{\sin p\pi} 0 < p < 1)$

39. Show that $\int_0^2 x(8-x3)^{1/3}\,dx = \frac{16\pi}{9\sqrt{3}}$ (Put $x^3 = 8t$. Use $\left|\overline{p}\right|\overline{1-p} = \frac{\pi}{\sin p\pi}$

40. Show that $\int_0^\infty \frac{x^8(1-x^6)\,dx}{(1+x)^{28}} = 0$

Hint: $I = \int_0^\infty \frac{x^{9-1}dx}{(1+x)^{9+15}} - \int_0^\infty \frac{x^{15-1}dx}{(1+x)^{15+9}} = B(9,15) - B(15,9) = 0$

41. Show that $\int_0^\infty \frac{x^6 - x^3}{(1+x^3)^5} x^2 dx = 0$ (Put $x^3 = t$)

42. Show that $\int_0^1 \frac{x^2 + x^3}{(1+x)^7} dx = \frac{1}{60}$

Hint: $I = \int_0^1 \frac{x^2 dx}{(1+x)^7} + \int_0^1 \frac{x^3 dx}{(1+x)^7}$ (Put $x = \frac{1}{t} inI_2$)

43. Prove that $\int_0^1 \frac{x^2 dx}{(1-x4)^{1/2}} \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$

Hint: For I_1 Put $x^2 = \sin\theta$; For I2, Put $x^2 = \tan\theta$, further $2\theta = t$)

44. Prove that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\pi/2} \sqrt{\sin\theta} d\theta = \pi$

Hint: Use $\int_0^{\pi/2} \sin^p \theta \cos^\theta \theta d\theta$

45. Show that $\frac{B(m,n+1)}{n} = \frac{B(m+1,n)}{m} = \frac{B(m,n)}{m+n}$