Mathematical Derivation: Error range due to non-linear nature of GPS coordinates

The Delta Studio

Jiamo Liu

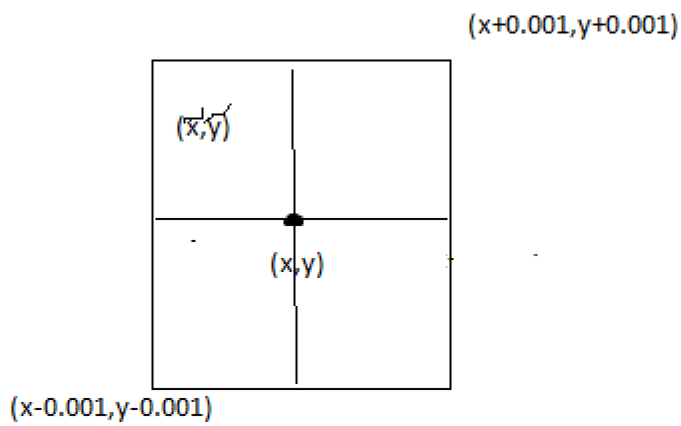
Introduction

It is known that the distance between two GPS coordinates can be obtained through Haversine's Formulae. However, due to the ecliptic nature of earth, the distance between coordinates is not linear, this paper aims to establish the distance boundary between two coordinates when the difference between two coordinates is known.

Derivation

Let's assume the user reports his location at GPS coordinate (x, y) in degrees and the GPS has an accuracy radius of R meters. Let us further assume that we cutoff all the trailing numbers after the fourth decimal place of (x, y), the coordinate therefore becomes $(\tilde{x}, \tilde{y})$ in degrees. We can then determine the space that $(\tilde{x}, \tilde{y})$ should fall in as shown in Figure 1.

Figure 1: Space that $(\tilde{x}, \tilde{y})$ falls in



It is therefore of great importance to know the maximum distance between $(\tilde{x}, \tilde{y})$ and (x, y). The Haversine formulae is given as in Equation 1, 2, 3.

$$a = \sin^2(\frac{\Delta\phi}{2}) + cos(\phi)cos(\tilde{\phi})\sin^2(\frac{\Delta\lambda}{2}) \tag{1}$$

$$c = 2atan2(\sqrt{a}, \sqrt{(1-a)}) \tag{2}$$

$$d = 6371000c \tag{3}$$

where:

$$\phi = Radians(x) \tag{4}$$

$$\tilde{\phi} = Radians(\tilde{x}) \tag{5}$$

$$\Delta\phi = \Delta\lambda = \pm 1.7453293 \times 10^{-6} = K \tag{6}$$

Let C_1 be:

$$C_1 = \sin^2(\frac{\Delta\phi}{2}) = \sin^2(\frac{\Delta\lambda}{2}) \approx 7.6154354951713931 \times 10^{-13} \tag{7}$$

Discussion

The first case is when $\tilde{x} - x = \pm 0.0001$, then a can be re-arranged as:

$$a = C_1 + \cos(\phi)\cos(\phi + K)C_1 \tag{8}$$

Using trigonometry identities, it then becomes:

$$a = C_1 + cos(\phi)(\cos(\phi)\cos(K) - \sin(\phi)\sin(K))C_1 \tag{9}$$

Finally the calculation can be re-written as:

$$a = C_1 + \cos^2(\phi)\cos(K)C_1 - \sin(\phi)\cos(\phi)\sin(K)C_1 \tag{10}$$

$$a = C_1 + \cos^2(\phi)\cos(K)C_1 - \frac{1}{2}\sin(2\phi)\sin(K)C_1 \tag{11}$$

using WolframAlpha, a has the following range:

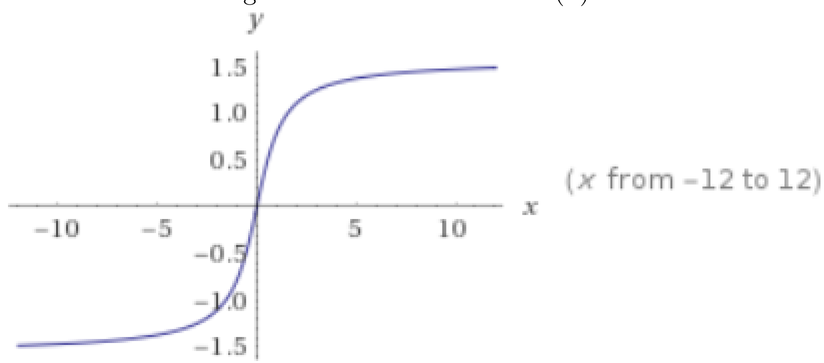
$$a \in [7.61544 \times 10^{-13}, 1.52309 \times 10^{-12}] \tag{12}$$

It is then clear that $\sqrt{1-a} > 0$ therefore, c can be re-written as:

$$c = 2(\arctan(\frac{\sqrt{a}}{\sqrt{1-a}})) \tag{13}$$

The $arctan(x)$ function is given as 2

Figure 2: Function of $arctan(x)$



It is obvious that $\sqrt{a} \approx 0$ and $\sqrt{1-a} \approx 1$, we can then conclude that the function is increasing between $a \in [C_1, C_2]$, it is then obvious that:

$$c \in [1.74533 \times 10^{-6}, 2.46827 \times 10^{-6}] \tag{14}$$

Finally d is then in range of:

$$d \in [11.11949743, 15.72534817] \tag{15}$$