Pricing European Call Option with Monte Carlo Simulation

Rupesh Poudel, 614779, MEMS

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Lecturer: Prof. Dr. Brenda López Cabrera

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Abstract

This seminar paper values the European Call Option of Deutsche Bank with the maturity of June 17, 2022, and the exercise price of 12€ as 49.139 cents with a standard error of 3.439 cents. Monte Carlo Simulation is used to price an option as opposed to the lattice approach. Boyle(1977), and Clewlow and Strickland(1998) are the major references used for Monte Carlo Simulation. Standard errors are reduced using two variance reduction techniques. Antithetic variate variance reduction technique brings the value of the option and the standard error to 0.4572 ± 0.02089. Delta-based control variate variance reduction technique gives the value of the same option and the standard error as 0.48279 ± 0.00125. Delta-based control variate was the effective variance reduction technique among the two in terms of reduced standard error.

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1 Introduction

Let's define any financial derivative in a generic form:

$$y = f(x)$$

where, y is the derivative security, x is the underlying security, and f(x) is the function on this underlying. The underlying of a derivative can be any other financial contract, for example: stocks, bonds, or even cryptocurrency. The f(x) can be thought of as a black-box initially and is different for each derivative. In general, the mechanics of the black box is predetermined. A pre-specified unique set of operations are performed on the underlying and based on the movement of the underlying security, the value of the derivative is determined. Financial derivatives are actively traded both at the exchanges like Chicago Mercantile Exchange and Börse Frankfurt, or OTC markets. Why would someone pay for this derivative? Let us evaluate the European Call Option to understand the motive.

European Call Option: This seminar paper will follow a European Call Option on a Deutsche Bank Stock (DB). The black box of a European Call Option, on DB looks like this:

$$C \equiv (S_T - K)^+ = max(S_T - K, 0)$$

where y = C, $x = S_T$ and $f(x) = (S_T - K)^+$. I said that y was the derivative. So, here C, the European Call Option, is the financial derivative. The underlying is S_T, i.e. the market price of the Deutsche Bank share at time T. The black box in simple terms reads: there is a function that can take two possible values. A maximum value between a difference term and zero implies a non-negative value associated with the derivative. The function either takes nothing or a positive amount as its value. For $S_T > K$, the function will have a positive value and for the remaining case, the function will yield a value of zero. I will assume that DB stock follows a Brownian motion. I will also state here that because of the Brownian motion of the stock, the stock is expected to fulfill the condition of $\exists t, T \;\; for \;\; t \leq T: \;\; S_T > K \;\; and \;\; S_t > K$ for a reasonable value of K. This will mean a non-zero probability of $S_T - K > 0$. If there is a non-zero probability in terms of expectation that a derivative will yield a positive payoff, the price one would be willing to pay for this derivative should also be non-zero. Once this argument is accepted, the option can be thought to have a value associated with it. Still, the question of the fair value of this derivative remains to be calculated. In this seminar paper, I wish to calculate an initial estimate for the approximate price or fair valuation for this derivative.

Hull(2018) defines option as the "right to buy and sell an asset", call option as "an option to buy an asset at a certain price by a certain date", and European Option as " an option that can only be exercised at the end of its life". I will use these three concepts to define a European Call Option as a financial contract that endows the owner of the contract the right to buy an asset for a prespecified price at a prespecified date. The option expires as worthless if the right is not exercised at that particular date.

Why would an investor want to buy the European Call Option on an asset? There are many valid reasons. If an investor expects the price of the underlying behind the call option to rise in the worth in comparison to its present worth; and if the investor expects the rise in the worth to persist until the predetermined date at which one is allowed to exercise their right to buy the stock on a prespecified low price, then the investor expects to gain from the purchase of the call option because it reinforces the belief required for a buyer of a European Call Option that $S_T > K$ will be fulfilled. Also, if an investor believes that there is under-pricing of the underlying; and the market will correct itself by the date one is allowed to exercise their right to buy the underlying, there is a profit opportunity present as well. However, the rise in the worth of the underlying should be bigger than the future value of what one pays for the option. If the opportunity cost of buying an option is higher than the expected payoff which accounts for the risk taken, an investor is better off investing in the alternative investment.

Let us revisit the equation from before to define the parameters and variables. The payoff equation of the long position on a European Call Option is given by:

$$C \equiv (S_T - K)^+ = max(S_T - K, 0)$$

where T represents the time in years left until the maturity of the option, the prespecified date at which the underlying asset could be bought for the prespecified price. S_T represents the price of the underlying at time T. K is referred to as the exercise price of the option, the prespecified price for which the underlying asset could be bought at the prespecified date. In the case of our long position on a call option of the underlying stock, if the stock is trading above K, the difference is the realized payoff of the option.

2 Black and Scholes Equation

A formula to price a European call option already exists. Certain simplifying assumptions are made, and these assumptions will only hold in the narrow scope of the world. Constant volatility is one of the assumptions. Still, the formula is widely used to calculate implied volatility. I use the Newton-Raphson method to figure out the volatility implied by Black Scholes Equation to value the DB European Call Option. Fisher Black and Myron Scholes have postulated a partial differential equation for an underlying stock paying no dividend as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

We have, V as the value of the option, r as the risk-free interest rate, t as the time, S as the stock price and σ as volatility of the underlying. An important observation of the PDE above is that there is no μ term involved. In Appendix 1, the evolution of the stock process involved μ. This implies that there is no dependence on the parameter μ and the risk-free rate is used instead. This replacement is required to make a discounted stock process the martingale process. Oosterlee and Grzelak (2019, Chapter 3) detail the derivation of the Black Scholes PDE and the essence of martingale process when $\mathbb{P}$-measure should be converted to $\mathbb{Q}$-measure.
Black and Scholes (1973) also postulated a deterministic solution for pricing the European Option:

$$C = S_t \Phi(d_1) - K e^{-rt} \Phi(d_2)$$

Where, $d_1 = \frac{ln \frac{S_t}{K} + (r + \frac{\sigma^2}{2})t}{\sigma \sqrt{t}}$, and $d_2 = d_1 - \sigma \sqrt{t}$

Here, C = the price of the call option, S_t as the spot price of the underlying at time t, K = the strike price of the option. t = time to maturity, and Φ as the CDF of Normal Distribution. Some limitations of the Black and Scholes Equation include the assumption of a risk-free world and constant volatility. Such assumptions fail in real life so even if they are deterministic, they are not necessarily the "true" value of the option. Normality is assumed for the asset prices, and this fails to capture heavy tails and asymmetries. The cost of rebalancing the portfolio for hedging is assumed to be 0, which is not necessarily the case. Despite these serious limitations, databases like Thompson Reuters and many other trading desks still report implied volatility and delta implied by the Black-Scholes model.

3 Applications in Finance

When there is no natural hedge, i.e. the ability to pass on the price fluctuation to the customers, a business can use the call option to lock in a specific price beforehand. For example, a European Call Option can be useful for contractors. Any contractor bidding for a massive contract but does not wish to pay for the materials before being selected for the contract has an incentive to buy a call option on materials with expected rising prices. Also, the equity of any firm behaves similar to the call option on the asset of the company where K, the exercise price, is the book value of the firm's liability. A company may choose to default on its liabilities if the present value of the firm's future payoff is going to be less than the book value of its debt.
Call Option on a company's stock is a popular incentive measure used by principals of a firm to the agents of the firm in hope of tying the agent's effort with rewards. Employee stock options are usually locked in for a specific period and can only be exercised after a prespecified time. An employee who will exercise the option on the day it is no longer locked is exercising something similar to a

European Call Option. In financial markets, options are used as a measure of hedging and speculation. A call option can endow the owner with a similar level of profit with low cash involved if the stock moves upwards, in comparison to another investor who buys the shares, and sells it at a higher price. The move is speculation, because in all likeliness, the stock could move downwards, and the owner of the option makes a loss on the entire investment whereas the owner of shares still has shares, even after the expiry of the option.

4 The method: Monte Carlo Simulation

This paper uses Monte Carlo Simulation to price options. Monte Carlo Simulation is a statistical process of repeated random sampling which uses random numbers under an appropriate distribution as a replacement of uncertain factors. Each sampling is known as a trial of a simulation and each trial simulates a process to calculate the payoff of the option. Bernoulli's law of large numbers, which is the essential backbone of Monte Carlo Simulation, can be interpreted as: for a sufficiently random sampling and a sufficiently large number of sampling, the central moments of sample starts to converge to the central moments of the population. Hsu and Robbins (1947) prove the convergence as $lim_{n \to \infty}$. As I am more interested in the accuracy up to cents, even 1000 sampling should bring us sufficiently close.

Clewlow and Strickland (1998, Chapter 4) outline the process of Monte Carlo simulation to price a European Call Option. I will briefly summarize their approach below. The value of a call option is the expected discounted payoff in a risk-neutral world. Cox and Ross (1976) show that the expected discounted value of the payoff will be the value of the option. The discounted payoff of the call option calculated once is then a trial of a simulation. The simulation is then performed for N number of trials. For example: N = 100,000 trials. Once payoffs in 100,000 scenarios are collected, an arithmetic average is taken to find the expected value of the European Call Option under consideration.

Boyle (1977) used the Monte Carlo approach to price the option for the first time. He stated that "the distribution of terminal stock prices is determined by the process generating future stock price movements." This allows me to use a stochastic differential equation, called Geometric Brownian Motion. The underlying asset is assumed to follow Geometric Brownian Motion (GBM) process, represented in the following way:

$$dS_t = (r - \delta)S_t dt + \sigma S_t dZ_t$$

where r is the risk-free rate and δ is the dividend rate. σ is the volatility and S_t is the price of the stock at time t. A GBM process does not entertain negative values, so it is suitable for stocks. The returns are independent of stocks' value and the fluctuations shown by GBM resemble the seemingly random movement of the stocks' movement. This representation is visible in Cox and Ross (1976), in Hull and White (1988), and a more complicated version involving jumps is presented by Boyle (1977). As Black and Scholes (1973) assume the lognormal distribution of the underlying, we suppose $X_t = ln(S_t)$ and reparametrize the process as the following:

$$dX_t = \nu dt + \sigma dz_t \quad where \;\; \nu = r - \delta - \frac{1}{2}\sigma^2$$

The Euler Maruyama Discretization scheme, see Kloeden and Platen (1992, Chapter 9), is used to change from the d (the infinitesimals) to Δ (the small changes). Performing discretization of dz, dx, and dt to their respective Δ version gives $\Delta X_t = \nu\Delta t + \sigma\Delta z_t$ which can alternately be written as $X_{t+\Delta t} = X_t + \nu\Delta t + \sigma(Z_{t+\Delta t} - Z_t)$. In terms of the original GBM process, this now becomes: $S_{t+\Delta t} = S_t exp(\nu\Delta t + \sigma(Z_{t+\Delta t} - Z_t))$. From here, we assume that $(Z_{t+\Delta t} - Z_t)$ is a random increment in the process with an expectation of 0 and dispersion of $\sqrt{\Delta t}\varepsilon$ where ε is drawn from the standard normal distribution, i.e. $\varepsilon \sim N(0, 1)$. This property is drawn from the Brownian motion.
The Δt is then time T distributed in N intervals, i.e. $\Delta t = T/N$. We wish to simulate S_t from 0 to T. We want to generate the value of the stock at the end of each interval $t_i = i\Delta t, where\ i = 1, ...N$. The rephrased equation will now become as follows:

$$S_{t_i} = e^{(X_{t_i})} \quad where\ X_{t_i} = X_{t_{i-1}} + \nu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_i$$

This way, we have the simulated stock process at maturity. Deducting the value that the stock process reaches at the maturity with the exercise price; and inserting it in the maximum function specified above will result in the payoff of the call option. For each trial, ε_i is expected to result in different prices on the day of the maturity; therefore different payoff of the call option is will be recorded. This payoff is to be discounted to time 0; so using the same notation from the Clewlow and Strickland (1998) which I have used in this section, the time 0 value of the option now becomes the following:

$$C_{0,j} = e^{(-rT)} C_{T,j}, \quad where\ r = constant$$

J here represents each trial. Now that I have discounted payoff, all that is left to be done is to find the expected discounted payoff. We assume a risk-free world, and the expected discounted payoff, or alternatively the value of the European call option, is given by the average of time 0 (t = 0) discounted payoffs. Mathematically:

$$\hat{C_0} = \frac{1}{M}\sum_{i=1}^{M} C_{0,j}$$

The average of the randomly generated samples is itself random so there exists an error in the final solution. The error is then termed as standard error and is calculated by dividing the sample standard deviation $C_{0,j}$ by the square root of the number of trials:

$$SE(\hat{C_0}) = \frac{SD(C_{0,j})}{\sqrt{M}} \;\; where \;\; SD(C_{0,j}) = \sqrt{\frac{1}{M-1}\sum_{j=1}^{M}(C_{0,j} - \hat{C_0})^2}$$

5 Variance Reduction Techniques

I acknowledge that Monte Carlo simulation in its basic form is computationally inefficient and it leads to a high standard error. There are popular techniques which can be used to reduce the high standard error associated with the Monte Carlo Simulation. This paper discusses two prominent variance reduction techniques, taken from Clewlow and Strickland (1998):

5.1 Antithetic variates Variance Reduction Technique

Boyle(1977), in his first Monte Carlo Option pricing, introduces antithetic variance reduction. Clewlow and Strickland (1998) include the mechanism outlined below and pseudocode to implement the algorithm. The idea behind the antithetic variate variance reduction technique is a creation of a synthetic or a hypothetical asset that is perfectly negatively correlated with the original asset. This hypothetical synthetic asset is called antithetic variate. The following example will clarify the antithetic variate:
There are two assets S1 and S2 which currently have the same price and are represented by the following processes:

$$dS_{1,t} = rS_{1,t}dt + \sigma S_{1,t}dZ_t$$

$$dS_{2,t} = rS_{2,t}dt - \sigma S_{2,t}dZ_t$$

The risk-free interest rate is r and the volatility is σ for both process. As the current price is the same for both the assets, the option should also have the same value. Imagine a portfolio consisting of an option on asset 1 and asset 2. This portfolio has less variance than the variance on a similarly scaled individual asset, i.e. Var[0.5(S1 + S2)] < Var [S1] and Var[0.5(S1 + S2)] < Var [S2]. This is because the negative correlation leads to negative co-movement of the asset prices and the resulting payoff of the sum of two assets has a very little movement, at least in principle.
The simulation of the process is also effortless. Let us look at the process outlined below:

$$C_{T,j} = max(0, Se^{(\nu T+\sigma\sqrt{T}(\varepsilon_j))} - K)$$

$$C^*_{T,j} = max(0, Se^{(\nu T+\sigma\sqrt{T}(-\varepsilon_j))} - K)$$

The only difference is a negatively assigned $\varepsilon \sim N(0,1)$. This is where Monte Carlo Simulation comes in handy to draw random samples from the standard normal distribution. We can take the mean of these two payoffs and then assign that mean as the payoff of our simulation. In addition to the accuracy of the estimate, the computational efficiency takes center stage as two samples are generated simultaneously. The mechanics by which the pair is generated ensures that the expectation is 0 and the variance is also reduced.

5.2 Delta Based Control Variate Variance Reduction

Hull and White (1988) make use of control variates for option pricing via the lattice approach. Boyle (1977) suggests using the control variates with conjunction to the Monte Carlo approach. I follow Boyle and draw upon Clewlow and Caverhill (1994), and Clewlow and Strickland (1998) to implement the control variate variance reduction technique. An important control variate variance reduction technique stems from the concept of a discretely rebalanced delta hedging. The main idea is that when the asset price suffers a change in a random direction, the change in the value of the option is then offset by the change in the value of the hedge position. Using the hedge as the control variate will lead to a smaller variance than the original case with no variance reduction technique. The methodology of a delta hedge is described below.

An investor will hold a short position in an option, i.e. an option will be written by the investor. From the writing of the option, the investor will receive proceeds. These proceeds will be deposited in the bank. $\delta C/\delta S$ amount of asset will be initially bought. At discrete intervals, the portfolio will be rebalanced, i.e. new $\delta C/\delta S$ amount of the stock will be held. If we have to sell some assets, we deposit the money in the bank account where we deposited the proceeds from the options sale. If we have to buy some stock, we use that bank account to buy the stock. This act is called rebalancing the portfolio. The more rebalancing, the up-to-date the portfolio becomes, but the more transaction cost is required. At the maturity of the option written, the hedge portfolio will closely follow the payoff of the option. The hedge will consist of the asset and the bank account. In mathematical terms:

$$C_{t_0}e^{r(T-t_0)} + \left[\sum_{i=0}^{N}\left(\frac{\delta C_{t_i}}{\delta S} - \frac{\delta C_{t_{i-1}}}{\delta S}\right)S_{t_i}e^{r(T-t_i)}\right] = C_T + \eta$$

$C_{t_0}e^{r(T-t_0)}$ is the proceed obtained from a short position on the option continuously compounded at the risk-free rate till maturity. The term in the square bracket [ ] is the net change in the bank account that occurs when money is deposited in or taken out for portfolio rebalancing. This is where our delta hedging occurs. C_T is the payoff of the option, and η is the error term associated with hedging. The expression from above is expanded for N steps and another reasonable assumption will be made. We will be liquidating our position into cash on the final step so the delta associated with the final rebalancing can be practically ignored. This will lead to a simplified expression.

$$C_{t_0}e^{r(T-t_0)} + \left[\sum_{i=0}^{N-1}(S_{t_{i+1}} - S_{t_i}e^{r\Delta t})e^{r(T-t_{i+1})}\right] = C_T + \eta$$

Clewlow and Strickland (1998) denote the term in the square bracket from the above expression as "delta based martingale control variate" or cv_1. The expectation $E[cv_1]$ is 0. The equation can therefore be rewritten as:

$$C_{t_0}e^{r(T-t_0)} = C_T - cv_1 + \eta = C_T + \beta cv_1 + \eta, \ \ with \ \ \beta = -1$$

If we simulate the payoff of the option and the hedge position on our portfolio, the average of this value will be the value of the option at the start. In terms of expectations, the η term disappears, and the average of the RHS can hopefully give us an option price with a much smaller variance.

6 Data

In this paper, I will use the European Call Option on Deutsche Bank with a Refinitiv Eikon identifier DBKE120bF2.EX. Databases like Thompson Reuters and Bloomberg report most of the input required for the valuation of the option. The maturity of the option is June 17 2022, and the exercise price is 12 Euros. The market value of the option with the strike price of €12 on December 22, 2021 was 0.48€. The evolution of the option price from 24^{th} June 2019 to 17^{th} January 2022 is given in the right side of figure 1 below. The left portion of the figure below is a Yahoo finance screenshot of the evolution of the stock price for the same period. I will take the closing price of 11€ on December 22, 2021 as the initial price for my evaluation. The date was chosen because on that day, the closing price was exactly 11€ and I also have access to an option price history with K = 11€. I will report the implied volatility associated with the option which has had a moneyness of 1. The days until the expiration of the option is 177 days. Time is measured in years, so T = (177/365) years.
I will assume a risk-free rate of 0.01. A risk-free rate, under normal circumstances, would be a government bold yield for a similar maturity. With rising inflation in the post COVID world, and with banks offering virtually zero interest on deposits, real rates are negative. Even nominal rates are essentially zero. 1 percent risk-free rate is an attempt to respect the time value of money. A covered interest rate parity can be a possible way of gaining risk-free returns, even when long-term German government bold yield is around 0.

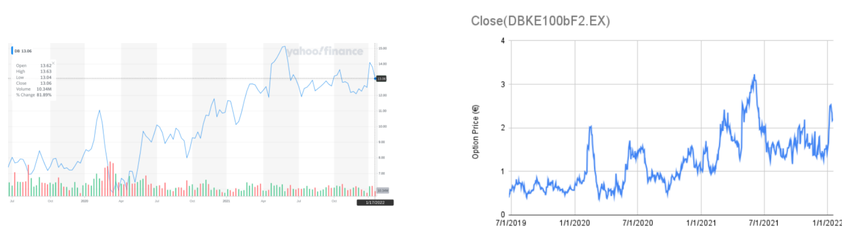

Figure 1: Evolution of the DB Stock and a selected European Call Option

The implied volatility is calculated by reverse searching the Black-Scholes equation for another option with a strike price of 13€ for the same maturity and same risk-free rate. The implied volatility was shown to be 0.2785. I will use this number in the model. Using the implied volatility associated with the strike price of 11 yields 0.2671. This is the implied volatility for the ATM call promised

above. The implied volatility associated with the strike price of 12 yields 0.2739. The implied volatility does not change much around these near strikes.
Only 1000 simulations will be performed to compute the option price. An increase in the number of simulations is also a measure to reduce the standard error. Variance reduction techniques like the ones mentioned in the paper are partly used to avoid a high number of simulations, as they are costly. A computer with low memory will have a hard time processing a million simulations. In addition, if the high simulation yields results and especially standard error particularly close to its variance reduced counterparts, the role of the variance reduction technique will be undermined. 1000 simulation will help show the contrast between variance reduced and original prices.

7 Results and Observation

First, I will start by summarizing the initial parameters for the option pricing. On December 22, 2021, the closing price of DB stock was 11€ i.e., S0 = 11. The call option maturing 177 days later was trading for 0.48€ with the strike price of 12€, i.e. K = 12, T = 177/365 years, and C0 = 0.48. N= 365 time steps are taken and M = 1000 simulations are performed. The implied volatility is 0.2739 and the risk-free rate(r) is 0.01. The European Call Option with the above parameters should have a price of 0.49139 with a standard error of 0.03439. For comparison, the Black Scholes price with volatility parameter of 0.2739 gives 0.48€; and with the volatility parameter of 0.2785, the Black Scholes closed-form solution gives the option price of 0.49€.

7.1 The price path

The following figure, Figure 2, shows the price path taken by our Deutsche Bank Stock under the simulation. The choice of $N \in \mathbb{N} : N > 0$ makes little difference in the problem I have to solve. This is only a peculiarity of the problem I have to solve. The discretization gives the exact solution without any approximation for the stochastic differential equation. That is why the choice of N is inconsequential. N time steps are used to discretize into smaller chunks of time. The dt notation needs the N. Also, the variance of the random sample drawn from Normal distribution is dependent on the choice of N. For N=1, the variance would be large. In a single step, we arrive at final values that are about as far as what it would take us with N > 1 step but with a smaller variance of movements. I use N = 365, but this does not mean the number of days in a year. It simply divides the time from December 22 2021 to June 17 2022 into 365 chunks. After the 365^{th} movement, I can collect an array of the terminal values as stated by Boyle(1977). From that terminal value, the exercise price of €12 is deducted. Then, all the negative values are replaced with 0. Thereafter, the present value of each terminal value is calculated. A new array with the present value of the last array is obtained. Then, the sum of each element is recorded. This divided by the number of simulations of 1000 gives us the point estimate for the option price.

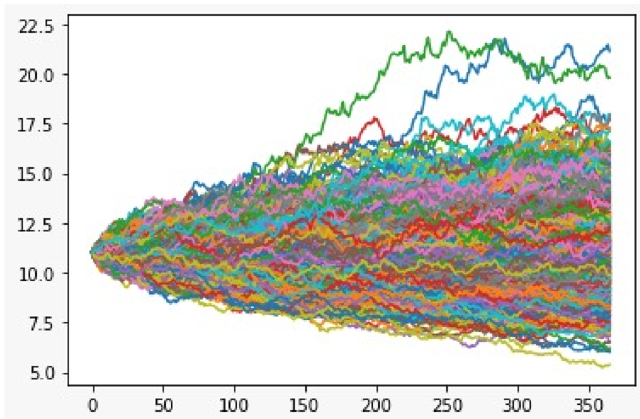

Figure 2: Price Path of the underlying

7.2 The Initial Estimate

In Figure 3, we can see how the predicted value of the option compares to the market price of that option. The red vertical line is the market price of the option equal to 0.48€. The value of the option associated with the following figure is 0.49139€. I wish to point out here that, because the Monte Carlo simulation is based on random numbers, the results will not be the same when the attached code is run. As the code requests for the random number right when it is run, it will be fed with a different set of random numbers. The difference in random numbers used to run the simulation will result in a slightly different result. Setting a seed would be a remedy for result reproducibility. However, I still choose to have them random to avoid cherry-picking a seed which gives better results.

In Figure 3, there is a light blue shade and there is a dark blue shade. The light blue shade in the middle represents 1 standard deviation or $[\mu - \sigma, \mu + \sigma]$. The darker blue shade, together with the light blue shade represents $[\mu - 3\sigma, \mu + 3\sigma]$ interval. The x-axis specifies the domain of the option price. The y-axis associates how frequent each value in the x-axis is. This is then a measure of probability. A normal distribution is said to contain around 66 percent of observation within 1 standard deviation. If the market value is within this range, this should positively reinforce the idea that according to our model, the market prices the option fairly. The farther away the market price is from the option value generated by my model, I conclude, within the scope of my model, that the market has mispriced the product and there is a possible trade scenario.

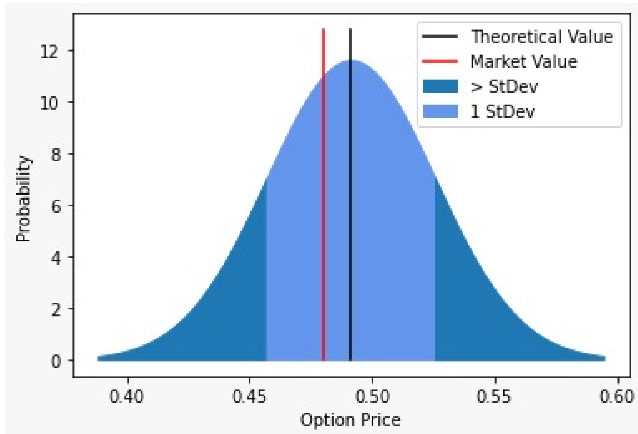

Figure 3: Initial option price

7.3 Antithetic Variance Reduction

The value of the European Call option implied after antithetic variance reduction is 0.4752€ with a standard error of 0.02089. The initial point estimate and the Black Scholes implied price both fall under 1 standard deviation. In figure 4, the red line is the market value of €0.48, the black line is the value associated with antithetic variance reduction, and the green line is the value computed without involving any variance reduction technique. Comparing the original green shade bell curve with the new blue shaded bell curve, we can make some useful observations. The green bell curve covers more space in the x- axis and less space in the y-axis. The blue bell curve covers more space in the y-axis and less space in the x-axis. The probability associated with the values around the expectation term is higher for the blue bell curve than the green bell curve. This shows that the blue bell curve, the curve generated after the antithetic variance reduction technique, does indeed have a smaller variance associated with it, in comparison to the green bell curve, the bell curve generated without any variance reduction technique.

Boyle (1977) hypothesizes that this variance reduction technique was not extremely effective in comparison to the next method I will discuss for a possible reason. The perfectly negatively correlated antithetic variable will no longer be perfectly negatively correlated after the log-normal transformation. Even though the log-normal transformation still holds the negative sign attached with it, the magnitude is not big enough. For this reason, even though this method reduces variance, the reduction is less than what is theoretically expected. Disclaimer: Similar to the procedure above, as the random samples are drawn from the normal distribution, once the code associated with this portion is run, a different value will appear for the option value and the standard error for the same reason described above. However, the results will not be distant from each other. The variance will have been reduced. As the price of the Deutsche Bank

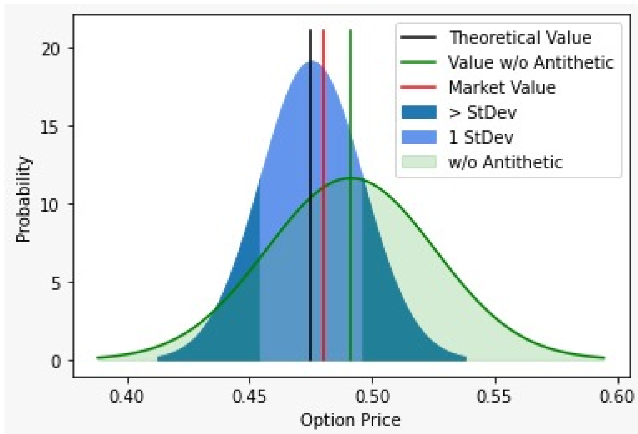

Figure 4: Variance Reduction with Antithetic Variate

European Call Option is 48 cents, a few cents movement can appear to result in more percentage change of the option price. However, the proximity of results with each set of 1000 simulations will show that the original model, in conjunction with the antithetic variance reduction, is a reasonable initial estimate for the option value. The option price is near the Black Scholes price and the price set in the market.

7.4 Delta Based Control Variate Variance Reduction

The value of the European Call option implied after delta-based control variate variance reduction is 0.48279€ with a standard error of 0.00125€. The price is within a cent of the Black Scholes implied price and the market value of that option. The standard deviation is noticeably lower than the previous two standard errors.

In the figure below, the bell curve which shows the probability density function of the original function, and which corresponds to Figure 3, is shaded in green. It appears to have some noticeable y value in the vicinity of the black, blue, and red lines. The green bell curve is the same one from the Figure 3 though. We can verify this via the Y-value. In Figure 5, the range of Y-value jumps to 350 whereas, in Figure 3, the range of Y-value reaches only 12. In Figure 5 and Figure 4, the maximum of the green curve also appears to reach 12. They are the same curve shown in different Y-axis heights.

Similarly, the dark and light blue shaded curve which shows variance reduction, as it did in Figure 4, is present in Figure 5 as well. In between the red line which denotes the market price for the DB option and the green line which denotes the point estimate for the original value predicted without any variance reduction, there exists the black line, which represents the point estimate for the option value with delta based control variate variance reduction, and the blue curve,

which represents $[\mu - 3\sigma, \mu + 3\sigma]$. As the σ takes on the value of 0.00125, 3σ left and right each would only occupy 0.0075€ worth of space in the X-axis. This figure still contains the blue bell curve from above, but the very low variance has made it converge to almost a thin straight line on the peak of the curve, and a thick straight line at the base of the curve.
Delta-based control variate seems more effective in reducing variance because its 6σ spread is only 0.0075€ whereas the 6σ spread of the original distribution is 0.20634€. The ratio of the standard error of delta-based control variate to the original standard error is around 0.0036, or 0.36 percent. With delta variate, the theoretical value, in this instance, seems closer to the market value than the original prediction without delta variate. This shows a piece of the positive evidence with regards to the variance reduction capacity of this technique.
One prominent reason behind the efficiency in variance reduction is the frequency of portfolio rebalancing. For a less rebalanced portfolio, the standard error would indeed be high. In real life, rebalancing comes with a transactional cost which will eat away the profit of the investors. The 0.00125€ standard error came from N = 365. If I run the program for different N, I will get different standard errors. For N = 1, range og $\mu \pm \sigma$ is $[0.4539 \pm 0.02014]$. For N = 3, range is $[0.48897 \pm 0.01599]$.For N = 5, range is $[0.47776 \pm 0.01139]$. For N = 10, range is $[0.48634 \pm 0.0079]$. For N = 50, range is $[0.48269 \pm 0.00363]$. These values are not fixed due to different random numbers. The main takeaway is: rerunning the block of code associated with delta-based control variate for an increase in N leads to a decrease in the standard error. This pattern is consistent even if the same set of values are not present due to random numbers. The way the code is set, only the standard error of the option value from delta-based control variate, or figure 5, is affected by the choice of N. N does not affect the other two figures: Figure 3 and Figure 4.

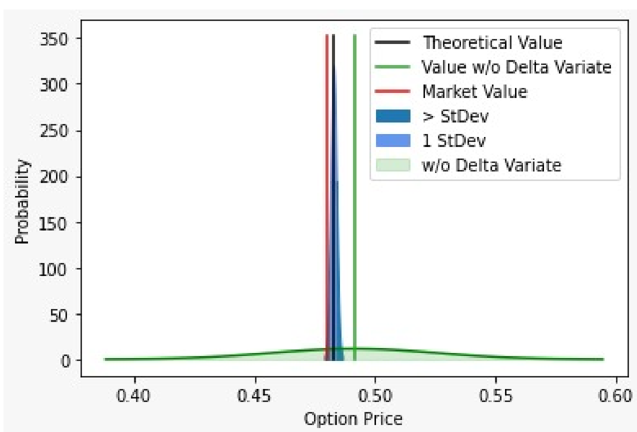

Figure 5: Variance Reduction with Delta based control Variate

8 Conclusion and Limitations

In this paper, I priced a European Call Option of Deutsche Bank AG that expires on 17 June 2022 with a strike price of 12€ as things stood on 22 December 2021. The input required for valuing the option is taken from the Refinitiv Eikon database. Two simplifying assumptions have been made. An exogenous risk-free rate of 1 percent and constant volatility are assumed but they are not necessarily realistic. Stochastic volatility models like Heston (1993), stochastic alpha, beta, rho model by Hagan et al. (2000), and General Auto-Regressive Conditional Heteroskedasticity Model can better estimate the volatility. For interest rates, Cox-Ingersoll-Ross Model is a short-rate model introduced in Cox, Ingersoll, & Ross (1985) as an extension of the model from Vasicek (1977). The HJM model by Heath Jarrow & Morton (1992), and the Ho-Lee model by Ho & Lee (1986) are other typical models dealing with the evolution of interest rates.

I used Monte Carlo Simulation to calculate the expected discounted payoff in a risk- neutral world. I used a stochastic differential equation in the form of a Geometric Brownian motion to evolve the stock till maturity and then deduct the exercise price to get the payoff of the option. For positive payoff, I discounted the payoff received at maturity to the initial time. This was a trial of my Monte Carlo Simulation. Performing the trial a thousand times gave me 1000 different time 0 payoffs. Any negative payoff was replaced with zero, and then the arithmetic average was taken to find the value of the option. The standard error of the option value was also found using the formula outlined at the end of section 4.

The model showed that there was no significant mispricing involved. The initial estimate and the variance reduced estimate gave a result which was in the vicinity of the market price of 0.48€. The standard error was around 3.5 cents. Two variance reduction techniques were used to improve the accuracy of the Monte Carlo Simulation. The standard error was decreased noticeably by variance reduction techniques. Antithetic variate reduced the standard error to around 2.1 cents. Delta variate, which is dependent on the portfolio rebalancing frequency, was the most effective for more rebalancing. In the model, even a single rebalancing yielded the reduction of the standard error to a value that is less than the standard error associated with the antithetic variance reduction technique. For 3 rebalancing, the standard error was around 1.6 cents. For 5 rebalancings, the standard error was around 1.1 cents. 0.7 cents for 10 rebalancings, 0.36 cents for 50 rebalancings, and 0.125 cents for 365 rebalancings showed that based on the tolerance of the investors for acceptable standard error and transaction cost, an optimal number of rebalancing can be determined. The computational time for 1000 simulations was on the same level of about 6 to 7 seconds for both the initial option pricing technique and the variance reduction technique. For this reason, I do not claim improvement in computation time for a smaller number of simulations.

Monte Carlo Simulation for valuing a European Call Option might seem a complication of a simple task, especially when Black Scholes closed-form solution exists. However, Monte Carlo Simulation allows for integrating jumps by modeling a Poisson process and other complex processes of stock evolution in the model. One can modify this approach of Monte Carlo Simulation to value path-dependent options, and other complex derivatives. The results from Monte Carlo Simulation can be used in conjunction with other models such as Black and Scholes, finite difference methods, other special models. Pricing a European Call Option via the Monte Carlo Simulation is the most basic task of mathematical finance. But, building upon this model, one can easily enter the world of complex financial derivatives.

9 References

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10 Appendix

10.1 Appendix 1: Some math behind Black and Scholes

The following two process models the evolution of the stock in the market, and of the money at the bank respectively.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW^{\mathbb{P}}(t)$$

$$\frac{dM_t}{M_t} = r\ dt\ \ and\ \ M(t) = M(t0)^{rt}$$

A portfolio of the European call option and a portion of the underlying is created which looks as follows:

$$\Pi(t, S(t)) = V(t, S(t)) - \Delta(t)S(t)$$

Quadratic variation property from the Ito's table is required to evaluate the expression in the Taylor approximation of the second order where a non negligible term appears.

$$(dW(t))^2 = dt$$

Then, Ito's Lemma can be applied as follows:

$$dg(X,t) = \left(\frac{\partial g}{\partial X}.\mu(X,t) + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}\sigma^2(X,t) + \frac{\partial g}{\partial t}\right)dt + \frac{\partial g}{\partial X}\sigma(X,t)dW$$

10.2 Appendix 2: Code Used

```
# importing libraries
import numpy as np
import pandas as pd
import datetime
import scipy.stats as stats
import matplotlib.pyplot as plt

#setting up input values
S = 11              # stock price
K = 12              # strike price
vol = 0.2739        # Implied volatility (%) from Black Scholes
r = 0.01            # risk-free rate (%)
N = 365             # number of time steps
M = 1000            # number of simulations
b = 0               # Cost of carry
market_value = 0.48 #market price of option
T = ((datetime.date(2022,6,17)-datetime.date(2021,12,22)).days+1)/365
```

```
# Reverse search implied volatility

#Sidestep as to how the implied volatility was calculated
# code copied from https://kevinpmooney.blogspot.com/2017/07/
calculating-implied-volatility-from.html

from scipy.stats import norm
from math import sqrt, exp, log, pi

def d(sigma, S, K, r, t):
    d1 = 1 / (sigma * sqrt(t)) * ( log(S/K) + (r + sigma**2/2) * t)
    d2 = d1 - sigma * sqrt(t)
    return d1, d2

def call_price(sigma, S, K, r, t, d1, d2):
    C = norm.cdf(d1) * S - norm.cdf(d2) * K * exp(-r * t)
    return C

#  S  = Stock price
#  K  = strike
#  C  = price of call as predicted by Black-Scholes model
#  r  = risk-free interest rate
#  t  = time to expiration expressed in years
#  C0 = price of call option from option chain

S = 11
K = 12
r = 0.01
t = 177.0/365.0
C0 = 0.48

#  We need a starting guess for the implied volatility.  We chose 0.5
#  arbitrarily.
vol = 0.9


epsilon = 1.0          #  Define variable to check stopping conditions
abstol = 1e-4          #  Stop calculation when abs(epsilon) < this number

count = 0                  #  Variable to count number of iterations
max_iter = 1e3         #  Max number of iterations before aborting

while epsilon > abstol:
    count +=1
    #  if-statement to avoid getting stuck in an infinite loop.
    if count > max_iter:
```

```
        print ("Program failed to find a root.  Exiting.")
        break

    orig_vol = vol
    d1, d2 = d(vol, S, K, r, t)
    function_value = call_price(vol, S, K, r, t, d1, d2) - C0
    vega = S * norm.pdf(d1) * sqrt(t)
    vol = -function_value/vega + vol
    #epsilon = abs(function_value)
    epsilon = abs((vol - orig_vol) / orig_vol)

print ('Implied volatility = ',  vol)
print ('Code required', count, 'iterations.')
```

The rest is copied from ASX Portfolio website "https://asxportfolio.com/options-monte-carlo-intro-to-valuing-financial-derivatives"

```
#Initial Price estimation

dt = T/N
nudt = (r - b - 0.5*vol**2)*dt
volsdt = vol*np.sqrt(dt)
lnS = np.log(S)
# Monte Carlo Method
Z = np.random.normal(size=(N, M))
delta_lnSt = nudt + volsdt*Z
lnSt = lnS + np.cumsum(delta_lnSt, axis=0)
lnSt = np.concatenate( (np.full(shape=(1, M), fill_value=lnS), lnSt ) )

ST = np.exp(lnSt)
CT = np.maximum(0, ST - K)C0w = np.exp(-r*T)*np.sum(CT[-1])/M
sigma = np.sqrt( np.sum( (CT[-1] - C0w)**2) / (M-1) )
SEw = sigma/np.sqrt(M) # and here the standard error

print("Call value is €{0} with SE +/- {1}".format(np.round(C0w,5),np.round(SEw,5)))

#Bell curve for initial price

x1 = np.linspace(C0w-3*SEw, C0w-1*SEw, 100) # from -3 to -1
x2 = np.linspace(C0w-1*SEw, C0w+1*SEw, 100) #from -1 to +1
x3 = np.linspace(C0w+1*SEw, C0w+3*SEw, 100) #from +1 to +3 standard deviation

#s1, s2, and s3 sets the boundaries
s1 = stats.norm.pdf(x1, C0w, SEw)
s2 = stats.norm.pdf(x2, C0w, SEw)
s3 = stats.norm.pdf(x3, C0w, SEw)
```

```
plt.fill_between(x1, s1, color='tab:blue',label='> StDev')
plt.fill_between(x2, s2, color='cornflowerblue',label='1 StDev')
plt.fill_between(x3, s3, color='tab:blue')

plt.plot([C0w,C0w],[0, max(s2)*1.1], 'k', label='Theoretical Value')
plt.plot([market_value,market_value],[0, max(s2)*1.1], 'r', label='Market Value')

plt.ylabel("Probability")
plt.xlabel("Option Price")
plt.legend()
plt.show()

# Antithetic variance reduction

N = 365
dt = T/N
nudt = (r- 0.5*vol*vol)*dt
volsdt = vol*np.sqrt(dt)
lnS = np.log(S)

Z = np.random.normal(size=(N,M))
delta_lnSt1 = nudt + volsdt * Z
delta_lnSt2 = nudt - volsdt * Z

lnSt1 = lnS + np.cumsum(delta_lnSt1, axis=0)
lnSt2 = lnS + np.cumsum(delta_lnSt2, axis=0)

ST1 = np.exp(lnSt1)
ST2 = np.exp(lnSt2)

CT = 0.5*(np.maximum(0, ST1[-1] - K)+ np.maximum(0, ST2[-1] - K) )
C0 = np.exp(-r*T) *np.sum(CT) / M

sigma = np.sqrt (np.sum((CT-C0)**2)/(M-1))
SE = sigma / np.sqrt(M)

print("Call value is €{0} with SE +/- {1}".format(np.round(C0,5),np.round(SE,5)))

# Bell curve for antithetic variance reduction
x1 = np.linspace(C0-3*SE, C0-1*SE, 100)
x2 = np.linspace(C0-1*SE, C0+1*SE, 100)
x3 = np.linspace(C0+1*SE, C0+3*SE, 100)
xw = np.linspace(C0w-3*SEw, C0w+3*SEw, 100)

s1 = stats.norm.pdf(x1, C0, SE)
```

```
s2 = stats.norm.pdf(x2, C0, SE)
s3 = stats.norm.pdf(x3, C0, SE)
sw = stats.norm.pdf(xw, C0w, SEw)

plt.fill_between(x1, s1, color='tab:blue',label='> StDev')
plt.fill_between(x2, s2, color='cornflowerblue',label='1 StDev')
plt.fill_between(x3, s3, color='tab:blue')
plt.plot(xw, sw, 'g-')
plt.fill_between(xw, sw, alpha=0.2, color='tab:green', label='w/o Antithetic')

plt.plot([C0,C0],[0, max(s2)*1.1], 'k', label='Theoretical Value')
plt.plot([C0w,C0w],[0, max(s2)*1.1], color='g', label='Value w/o Antithetic')
plt.plot([market_value,market_value],[0, max(s2)*1.1], 'r', label='Market Value')

plt.ylabel("Probability")
plt.xlabel("Option Price")
plt.legend()
plt.show()

#Control Variate Variance Reduction

def delta_calc(r, S, K, T, sigma, type="c"):
    "Calculate delta of an option"
    d1 = (np.log(S/K) + (r + sigma**2/2)*T)/(sigma*np.sqrt(T))
    try:
        if type == "c":
            delta_calc = stats.norm.cdf(d1, 0, 1)
        elif type == "p":
            delta_calc = -stats.norm.cdf(-d1, 0, 1)
        return delta_calc
    except:
        print("Please confirm option type, either 'c' for Call or 'p' for Put!")

N = 365
dt = T/N
nudt = (r - 0.5*vol**2)*dt
volsdt = vol*np.sqrt(dt)

erdt = np.exp(r*dt)
cv = 0
beta1 = -1

Z = np.random.normal(size=(N, M))
delta_St = nudt + volsdt*Z
ST = S*np.cumprod( np.exp(delta_St), axis=0)
ST = np.concatenate( (np.full(shape=(1, M), fill_value=S), ST ) )
```

```
deltaSt = delta_calc(r, ST[:-1].T, K, np.linspace(T,0,N), vol, "c").T
cv = np.cumsum(deltaSt*(ST[1:] - ST[:-1]*erdt), axis=0)


CT = np.maximum(0, ST[-1] - K) + beta1*cv[-1]
C0 = np.exp(-r*T)*np.sum(CT)/M

sigma = np.sqrt( np.sum( (np.exp(-r*T)*CT - C0)**2) / (M-1) )
sigma = np.std(np.exp(-r*T)*CT)
SE = sigma/np.sqrt(M)

print("Call value is €{0} with SE +/- {1}".format(np.round(C0,5),np.round(SE,5)))

# Bell curve for control variate

x1 = np.linspace(C0-3*SE, C0-1*SE, 100)
x2 = np.linspace(C0-1*SE, C0+1*SE, 100)
x3 = np.linspace(C0+1*SE, C0+3*SE, 100)
xw = np.linspace(C0w-3*SEw, C0w+3*SEw, 100)

s1 = stats.norm.pdf(x1, C0, SE)
s2 = stats.norm.pdf(x2, C0, SE)
s3 = stats.norm.pdf(x3, C0, SE)
sw = stats.norm.pdf(xw, C0w, SEw)

plt.fill_between(x1, s1, color='tab:blue',label='> StDev')
plt.fill_between(x2, s2, color='cornflowerblue',label='1 StDev')
plt.fill_between(x3, s3, color='tab:blue')
plt.plot(xw, sw, 'g-')
plt.fill_between(xw, sw, alpha=0.2, color='tab:green', label='w/o Delta Variate')

plt.plot([C0,C0],[0, max(s2)*1.1], 'k', label='Theoretical Value')
plt.plot([C0w,C0w],[0, max(s2)*1.1], color='tab:green', label='Value w/o Delta Variate')
plt.plot([market_value,market_value],[0, max(s2)*1.1], 'r', label='Market Value')

plt.ylabel("Probability")
plt.xlabel("Option Price")
plt.legend()
plt.show()
```

Please visit my Github for a full set of codes and accompanying documents by clicking: **Option Pricing with Monte Carlo Simulation**