Option Pricing with Monte Carlo Simulation

Rupesh Poudel

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A little about me!

- Third Semester MEMS Student
- Accounting and Finance Major
- B.Sc. International Business and Economics
- Interest in Private Equity and Quantitative Finance

The Topic of the Presentation.

Here are the keywords!

- **The Technique:** Option Pricing
- **The Derivative:** European Call Option
- **The Method:** Monte Carlo Simulation

Outline

Let's talk derivatives!

A simple way to understand what financial derivatives are is the following:

$$y = f(x)$$

where x is the underlying.

- The underlying could be stocks, bonds, or even cryptocurrency.
- $f(x)$ is a black box. The inside of a black box is different for each derivative.
- Derivatives such as this are actively traded: both at exchanges or over-the-counter.
- Why would someone pay money for the derivative? Let's evaluate the European Call Option to know why.

What's inside the black-box of a European Call?

The black-box of an European Call Option looks something like this:

$$C \equiv (S_T - K)^+ = max(S_T - K, 0)$$

Here, $y = C$, $x = S_T$, and $f(x) = (S_T - K)^+$.

We have a function that takes one of the two values:

1. Either, **Nothing**
2. Or, **a positive payoff.**

If there is a non-zero probability in terms of expectation that a derivative will yield positive payoff, the price one would be willing to pay for this derivative is also non-zero.

Motivation for this presentation!

I established that the price of the financial derivatives should be non-zero. My attempt, in the presentation that will follow, is to give an initial estimate for the approximate price of this derivative.

Outline

The European Call Option

The following definitions are taken from John C Hull's standard textbook: Options, Futures, and Other Derivatives.

- **Option:** The right to buy and sell an asset.
- **Call Option:** An option to buy an asset at a certain price by a certain date.
- **European Option:** An option that can only be exercised at the end of its life.

What is the European Call Option?

- A European Call Option is a financial contract that endows the owner of the contract the right to buy an asset for a prespecified price at a predetermined date.
- The option expires as worthless if the right is not exercised at that particular date.

Why would one buy the European Call Option?

- One expects the price of the underlying behind the call option to rise in the worth in comparison to its present worth.
- One expects the rise in the worth to persist until the predetermined date when one is allowed to exercise their right to buy it on a prespecified, low price.
- One believes that there is underpricing of the underlying; and the market will correct itself by the date one is allowed to exercise their right to buy the underlying.
- The rise in the worth of the underlying better be bigger than the future value of what one pays for the option.

Let us be familiarized with the financial lingo surrounding the European Call Option!

The European Call Option

We revisit the equation from before. We name this **the payoff equation** of the **long position** on a European Call Option.

$$C = (S_T - K)^+ = max(S_T - K, 0)$$

where,
T = Time left until maturity
S_T = The price of the underlying at time T
K = Exercise Price of the option

- **Maturity:** The prespecified date at which the underlying asset could be bought for the prespecified price.
- **Exercise Price:** The prespecified price for which the underlying asset could be bought at the prespecified date.

The math driving Black Scholes

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW^{\mathbb{P}}(t)$$

$$\frac{dM_t}{M_t} = r\ dt$$

$$\Pi(t, S(t)) = V(t, S(t)) - \Delta(t)S(t)$$

$$(dW(t))^2 = dt$$

$$dg(X,t) = \left(\frac{\partial g}{\partial X}.\mu(X,t) + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}\sigma^2(X,t) + \frac{\partial g}{\partial t}\right) dt + \frac{\partial g}{\partial X}\sigma(X,t)dW$$

The Black and Scholes Equation

Fisher Black and Myron Scholes proposed the following partial differential equation for an underlying stock paying no dividend:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

We have, V as the value of the option, r as the risk free interest rate, t as the time, S as the stock price and σ as volatility of the underlying.
They also postulated a deterministic solution for pricing the European Option:

$$C = S_t\Phi(d_1) - Ke^{-rt}\Phi(d_2)$$

Where, $d_1 = \frac{ln\frac{S_t}{K} + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$, and $d_2 = d_1 - \sigma\sqrt{t}$
Here, C = the price of the call option, S_t as the spot price of the underlying at time t, K = the strike price of the option. t = time to maturity, and Φ as the CDF of Normal Distribution.

The Black and Scholes Equation

- The PDE contains no μ term. The dependence on the parameter μ is lost.
- Caution: Cost needed to rebalance the portfolio for hedging is not accounted for and is naively assumed to be 0.
- The model assumes a risk free world and constant volatility.
- Caution: Black and Scholes may be deterministic, but not necessarily the "true" value of the option.
- Caution: Normality Assumption for asset prices fails to capture fat tails and asymmetries
- Databases like Thompson Reuters and many trading desks still report implied volatility and delta implied by Black-Scholes, so it's still common despite its serious limitation.

Ways to price options

- with Partial Differential Equation approach under $\mathbb{P}$ measure e.g. Black and Scholes approach
- with Risk Neutral Probability approach with $\mathbb{Q}$ measure. e.g. Feynman-Kac approach

Outline

The essence of Random Sampling

- Without random sampling, any statistical sampling will run the risk of being biased.
- With sufficiently large sample size, and sufficiently random sampling, the central moments of the sample will converge to the central moments of the population.
- Alternatively, analyzing the sample is about the same as analyzing the population.

Law of Large Numbers!

Law of Large Numbers suggest the following:
Let's take a fair coin with two sides. The probability of heads is p.

The probability distribution closest to the coin toss experiment is Bernouli Distribution with $X_i \sim Bernouli(p)$.

The law of large numbers implies the following:

$$\frac{X_1 + X_2 + ... + X_n}{n} \to p$$

with probability 1

The boundaries set by a Random Walk

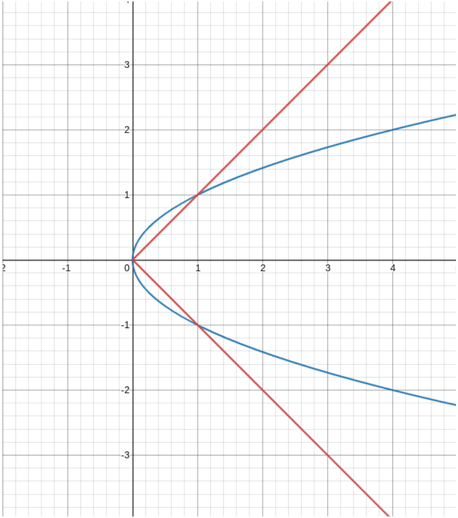

Figure: Random Walk

The Model

We aggregate the discounted payoffs:

$$V(t, S_t) = e^{-rT} \frac{\sum_{i=1}^{n} (S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T} N(0,1)} - K}{n}$$

Simulation Settings

- Number of Simulations:
- Initial Stock Price:
- Volatility:
- Risk free interest rate:
- Time to Maturity:
- Strike Price:

References

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- Oosterlee, C. W., Grzelak, L. A. (2019). Mathematical Modeling and Computation in Finance: With Exercises and Python and Matlab Computer Codes. World Scientific.