Option Pricing

Rupesh Poudel

11 March 2022

1 Introduction

Financial Derivatives: Let's define any financial derivative in a generic form:

$$y = f(x)$$

where, y is the derivative, x is the underlying, and f(x) is the function on this underlying. The underlying of a derivative can be stocks, bonds, or even cryptocurrency. The f(x) is a black-box and is different for each derivative. In general, a pre-specified unique operation is performed on the underlying and based on the movement of the underlying security, the value of the derivative is determined. Derivatives are actively traded both at the exchanges like Chicago Mercantile Exchange or Börse Frankfurt. Why would someone pay for this derivative? Let us evaluate the European Call Option to know why.

European Call Option: The black box of a European Call Option looks like this:

$$C \equiv (S_T - K)^+ = max(S_T - k, 0)$$

where y = C, $x = S_T$ and $f(x) = (S_T - K)^+$. We said that y was the derivative. So, here C, the European Call Option, is the derivative. The underlying is S_T, i.e. the market price of the stock at time T. The black box in simple term reads: there is a function that can take two possible value. A maximum of a difference term and 0 implies a non negative value associated with it. The function either takes nothing or a positive amount as its value. For $S_T > K$, the function will have a positive value and for the remaining case the function will yield a value of zero. If there is a non-zero probability in terms of expectation that a derivative will yield positive payoff, the price one would be willing to pay for this derivative should also be non-zero. Once this argument is accepted, the question of the fair price of this derivative remains to be calculated. In this seminar, I wish to calculate an initial estimate for the approximate price for this derivative.

Hull(2018) defines option as the "right to buy and sell an asset", call option as "an option to buy an asset at a certain price by a certain date", and European Option as " an option that can only be exercised at the end of its life". I will use these three concepts to define a European Call Option as a financial contract that endows the owner of the contract the right to buy an asset for a prespecified price at a prespecified date. The option expires as worthless if the right is not exercised at that particular date.

Why would an investor want to buy the European Call Option on an asset? There are many valid reasons. If an investor expects the price of the underlying behind the call option to rise in the worth in comparison to its present worth; and if the investor expects the rise in the worth to persist until the predetermined date when one is allowed to exercise their right to buy it on a prespecified low price, then one can gain from the call option because it reinforces the belief required for a buyer of a European Call Option that $S_T > K$ will be fulfilled. If an investor believes that there is underpricing of the underlying; and the market will correct itself by the date one is allowed to exercise their right to buy the underlying, there is a profit opportunity there as well. However, the rise in the worth of the underlying should bigger than the future value of what one pays for the option. If the opportunity cost of buying an option is higher than the expected payoff which accounts for the risk taken.

Let us revisit the equation from before to define the parameters and variables. The payoff equation of the long position on a European Call Option is given by:

$$C \equiv (S_T - K)^+ = max(S_T - k, 0)$$

where T represents the time in years left until the maturity of the option, the prespecified date at which the underlying asset could be bought for the prespecified price. S_T represents the price of the underlying at time T. K is referred to as the exercise price of the option, the prespecified price for which the underlying asset could be bought at the prespecified date. In case of our long position on a call option of the underlying stock, if the stock is trading above K, the difference is the realised payoff of the option.

2 Black and Scholes Equation

A formula to price an European option already exists. Fisher Black and Myron Scholes have postulated a partial differential equation for an underlying stock paying no dividend as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

We have, V as the value of the option, r as the risk free interest rate, t as the time, S as the stock price and σ as volatility of the underlying. Some important observation of the PDE above is that there is no μ term involved. In Appendix 1, the evolution of the stock process involved μ. This implies that there is no dependence on the parameter μ and the risk free rate is used instead. This replacement is required to make a discounted stock process a martingale process. Oosterlee and Grzelak (2019, Chapter 3) details the derivation of the Black Scholes PDE.

Black and Scholes (1973) also postulated a deterministic solution for pricing the European Option:

$$C = S_t(d_1) - Ke^{-rt}(d_2)$$

Where, $d_1 = \frac{ln\frac{S_t}{K} + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$, and $d_2 = d_1 - \sigma\sqrt{t}$

Here, C = the price of the call option, S_t as the spot price of the underlying at time t, K = the strike price of the option. t = time to maturity, and as the CDF of Normal Distribution. Some limitations of the Black and Scholes Equation includes the assumption of a risk free world and a constant volatility. Such assumptions fail in real life so even if they are deterministic, they are not necessarily the "true" value of the option. Normality is assumed for the asset prices, and this fails to capture heavy tails and asymmetries. Cost of rebalancing the portfolio for hedging is assumed to be 0, which is not necessarily the case. This model has some serious limitations. However, databases like Thompson Reuters and many other trading desks still report implied volatility and delta implied by the Black-Scholes.

3 Applications in Finance

When there is no natural hedge i.e. the ability to pass on the price fluctuation to the customers, a business can use call option to lock in a specific price beforehand. For example, a European Call Option can be useful for contractors. Any contractor bidding for a massive contract, but does not wish to pay for the materials before being selected for the contract has an incentive to buy a call option on materials with expected rising prices. Also, the equity of the any firm behaves similar to the call option on the asset of the company where K, the exercise price, is the book value of the firm's liability. A company may choose to default on its liabilities if the present value of the firm's future payoff is going to be less than the book value of its debt.

Call Option on a company's stock is a popular incentive measure used by principals of a firm to the agents of the firm in hope of tying agent's effort with rewards. Employee stock options are usually locked in for a specific period and can only be exercised after a prespecified time. An employee who will exercise the option on the day it is no longer locked is exercising something similar to a European Call Option. In financial markets, options are used as a measure of hedging and speculation. A call option can endow the owner with a similar level of profit with low cash involved if the stock moves upwards, in comparison to another investor who buys the shares, and sells it at higher price. The move is a speculation, because in all likeliness, the stock could move downwards, and the owner of the option makes loss on the entire investment whereas the owner of shares still has shares, even after the expiry of the option.

4 The method: Monte Carlo Simulation

This paper uses Monte Carlo Simulation to price options. Monte Carlo Simulation is a statistical process of repeated random sampling which uses random numbers under an appropriate distribution as a replacement of uncertain factors. Each sampling is known as a trial of a simulation and each trial simulates a process to calculate the payoff of the option.

Clewlow and Strickland (1998) outline the process of Monte Carlo simulation to price a European Call Option. The value of a call option is the expected discounted payoff in a risk neutral world. The expected discounted payoff of the call option calculated once is then a trial of a simulation. The simulation is then performed for N number of trials. For example: N = 100,000 trials. Once payoffs in 100,000 scenarios are collected, an arithmetic average is taken in order to find the value of the European Call Option under consideration.

The underlying asset is assumed to follow Geometric Brownian Motion (GBM) process, represented in the following way:

$$dS_t = (r - \delta)S_t dt + \sigma S_t dZ_t$$

where r is the risk free rate and δ is the dividend rate. σ is the volatility and S_t is the price of the stock at time t. As we assume the lognormal distribution of the underlying, we suppose $X_t = ln(S_t)$ and reparametrize the process as the following:

$$dX_t = \nu dt + \sigma dz_t \quad where \;\; \nu = r - \delta - \frac{1}{2}\sigma^2$$

The Euler Maruyama Discretization scheme is used to change from the d (the infinitesimals) to Δ(the small changes). Performing discretization of dz, dx, and dt to their respective Δ version gives $dX_t = \nu\Delta t + \sigma\Delta z_t$ which can alternately be written as $X_{t+\Delta t} = X_t + \nu\Delta t + \sigma(Z_{t+\Delta t} - Z_t)$. In terms of the original GBM process, this now becomes: $S_{t+\Delta t} = S_t exp(\nu\Delta t + \sigma(Z_{t+\Delta t} - Z_t))$. From here, we assume that $(Z_{t+\Delta t} - Z_t)$ is a random increment in the process with an expectation of 0 and dispersion of $\sqrt{\Delta t}\varepsilon$ where ε is drawn from the standard normal distribution, i.e. $\varepsilon \sim N(0, 1)$.

The Δt is then time T distributed in N intervals, i.e. $\Delta t = T/N$. We wish to simulate S_t from 0 to T. We want to generate the value of the stock at the end of each interval $t_i = i\Delta t, where\ i = 1, ...N$. The rephrased equation will now become as follows:

$$S_{t_i} = exp(X_{t_i}) \quad where X_{t_i} = X_{t_{i-1}} + \nu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_i$$

This way, we have the simulated stock process at the maturity. Deducting the value that the stock process reaches at the maturity will result in the payoff of the call option. For each trials ε_i is expected to result in different price on the day of the maturity and therefore different payoff of the call option is will be recorded. This payoff is to be discounted to time 0 so the time 0 value of the option now becomes, using the same notation from the Clewlow and Strickland, which I have used in this section, becomes the following:

$$C_{0,j} = exp(-rT)C_{T,j}, \quad where\ r = constant$$

Now that I have discounted payoff, all that is left to be done is to find the expected discounted payoff. We assume a risk free world, and the expected discounted payoff, or alternatively the value of the European call option is given by the following:

$$\hat{C_0} = \frac{1}{M}\sum_{i=1}^{M} C_{0,j}$$

The average of the randomly generated samples is itself random so there exists error in the final solution. The error is then termed as standard error and is calculated by dividing the sample standard deviation $C_{0,j}$ by the square root of the number of trials:

$$SE(\hat{C_0}) = \frac{SD(C_{0,j})}{\sqrt{M}} \;\; where \;\; SD(C_{0,j}) = \sqrt{\frac{1}{M-1}\sum_{j=1}^{M}(C_{0,j} - \hat{C_0})^2}$$

5 Variance Reduction Techniques

I acknowledge that Monte Carlo simulation in its basic form is computationally inefficient and it leads to a high standard error. There are techniques that can be used to reduce the high standard error associated with the Monte Carlo Simulation. This paper discusses two prominent variance reduction techniques, taken from Clewlow and Strickland (1998):

5.1 Antithetic variates Variance Reduction Technique

The idea behind antithetic variate variance reduction technique is creation of a synthetic or a hypothetical asset that is perfectly negatively correlated with the original asset. This hypothetical synthetic asset is called antithetic variate. The following example will clarify the antithetic variate:

There are two assets S1 and S2 which currently have the same price and are represented by the following processes:

$$dS_{1,t} = rS_{1,t}dt + \sigma S_{1,t}dZ_t$$

$$dS_{2,t} = rS_{2,t}dt - \sigma S_{2,t}dZ_t$$

The risk free interest rate is r and the volatility is σ for both process. As the current price is same for both the assets, the option should also have the same value. Imagine a portfolio consisting an option on asset 1 and asset 2. This portfolio has less variance than the variance on a similarly scaled individual asset. This is because the negative correlation leads to negative co-movement of the asset prices and the resulting payoff of the sum of two assets have very little movement.

The simulation of the process is also very simple. Let us look at the process outlined below:

$$C_{T,j} = max(0, Sexp(\nu T + \sigma\sqrt{T}(\varepsilon_j)) - K)$$

$$C^*_{T,j} = max(0, Sexp(\nu T + \sigma\sqrt{T}(-\varepsilon_j)) - K)$$

The only difference is a negatively assigned $\varepsilon \sim N(0,1)$. This is where Monte Carlo Simulation is required, to draw random samples from standard normal distribution. We can take the mean of these two payoffs and then assign that mean as the payoff of our simulation. In addition to the accuracy of the estimate, the computational efficiency takes centre stage as two samples are generated simultaneously. The mechanics by which the pair is generated ensures that the expectation is 0 and the variance is therefore reduced.

5.2 Control variates Variance Reduction Technique

6 References

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. The Journal of Political Economy: Volume 81(3) pp. 637-654.

Clewlow, L., & Strickland, C. (1998). Implementing derivative models. Wiley.

Oosterlee, C. W., & Grzelak, L. A. (2019). Mathematical Modeling and Computation in Finance: With Exercises and Python and Matlab Computer Codes. World Scientific.

7 Appendix

7.1 Appendix 1:

The following two process models the evolution of the stock in the market, and of the money at the bank respectively.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW^P(t)$$

$$\frac{dM_t}{M_t} = r\ dt\ \ and\ \ M(t) = M(t0)^{rt}$$

A portfolio of the European call option and a portion of the underlying is created which looks as follows:

$$\Pi(t, S(t)) = V(t, S(t)) - \Delta(t)S(t)$$

Quadratic variation property from the Ito's table is required to evaluate the expression in the Taylor approximation of the second order where a non negligible term appears.

$$(dW(t))^2 = dt$$

Then, Ito's Lemma can be applied as follows:

$$dg(X,t) = \left(\frac{\partial g}{\partial X}.\mu(X,t) + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}\sigma^2(X,t) + \frac{\partial g}{\partial t}\right)dt + \frac{\partial g}{\partial X}\sigma(X,t)dW$$