COPERNICUS MASTER
IN DIGITAL EARTH

Computer Vision

Lecture 2: (Binary) Mathematical Morphology

September 15, 2021

Prof. Sébastien Lefèvre
sebastien.lefevre@univ-ubs.fr

History

Mathematical Morphology (MM) is a branch of image processing/analysis that has been founded in the 60s for material sciences. It offers a mathematically rigorous framework for analysing object shapes and geometric structures in the image.

Images are scanned with a structuring element (i.e. a neighbourhood, window, pattern, template). Morphological operators then rely on the comparison of the structuring element with the local image content.

The structuring element shape and size are thus of first importance.

History

Mathematical Morphology (MM) is a branch of image processing/analysis that has been founded in the 60s for material sciences. It offers a mathematically rigorous framework for analysing object shapes and geometric structures in the image.

Images are scanned with a structuring element (i.e. a neighbourhood, window, pattern, template). Morphological operators then rely on the comparison of the structuring element with the local image content.

The structuring element shape and size are thus of first importance.

Initially defined for binary images using set theory, MM has then been extended to grayscale images with lattice theory (and even later to multivariate images).

Set theory

Set operators:

- union: $A \cup B$
- intersection: $A \cap B$
- complement: $A^c = \{p : p \notin A\}$
- difference: $A - B = A \cap B^c$
- reflection: $\breve{B} = \{p : p = -b \quad \forall b \in B\}$
- translation: $(A)_z = \{p : p = a + z \quad \forall a \in A\}$

Equivalence with logical operators AND, OR, NOT, XOR

Set theory

Set operators:

- union: $A \cup B$
- intersection: $A \cap B$
- complement: $A^c = \{p : p \notin A\}$
- difference: $A - B = A \cap B^c$
- reflection: $\breve{B} = \{p : p = -b \quad \forall b \in B\}$
- translation: $(A)_z = \{p : p = a + z \quad \forall a \in A\}$

Equivalence with logical operators AND, OR, NOT, XOR

Minkowski set operators:

- addition: $A \oplus B = \bigcup_{b \in B} (A)_b$
- subtraction/difference: $A \ominus B = \bigcap_{b \in B} (A)_{-b}$

Erosion and dilation

Erosion of A (object, image) by B (structuring element)

$$\varepsilon_B(A) = \{x : (B)_x \subseteq A\} = A \ominus B$$

Erosion and dilation

Erosion of A (object, image) by B (structuring element)

$$\varepsilon_B(A) = \{x : (B)_x \subseteq A\} = A \ominus B$$

Dilation is the dual operator to erosion

$$\delta_B(A) = \{x : (\breve{B})_x \cap A \neq \emptyset\} = A \oplus \breve{B} = (A^c \ominus \breve{B})^c$$

Several definitions exist with Minkowski addition, using B or $\breve{B}$:
No difference if B is symmetric.

Properties

- commutativity: $A \oplus B = B \oplus A$
- associativity: $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
- multiplication with scalar: $t(A \oplus B) = tA \oplus tB$ and $t(A \ominus B) = tA \ominus tB$
- structuring element decomposition:

$$
\begin{aligned}
&A \ominus (B \oplus C) = (A \ominus B) \ominus C \\
&A \ominus 2B = A \ominus (B \oplus B) = (A \ominus B) \ominus B \\
&nB = B \oplus B \oplus B \ldots \oplus B
\end{aligned}
$$

Properties

- commutativity: $A \oplus B = B \oplus A$
- associativity: $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
- multiplication with scalar: $t(A \oplus B) = tA \oplus tB$ and $t(A \ominus B) = tA \ominus tB$
- structuring element decomposition:

$$
\begin{aligned}
&A \ominus (B \oplus C) = (A \ominus B) \ominus C \\
&A \ominus 2B = A \ominus (B \oplus B) = (A \ominus B) \ominus B \\
&nB = B \oplus B \oplus B \ldots \oplus B
\end{aligned}
$$

- translation invariance $\Psi(A_x) = (\Psi(A))_x$
- monotonous increasing: if $A_1 \subset A_2$ then $\Psi(A_1) \subset \Psi(A_2)$
- dual operator: $\Psi^*(A) = (\Psi(A^c))^c$
- distributivity: $(A_1 \cup A_2) \oplus B = (A_1 \oplus B) \cup (A_2 \oplus B)$ and $(A_1 \cap A_2) \ominus B = (A_1 \ominus B) \cap (A_2 \ominus B)$

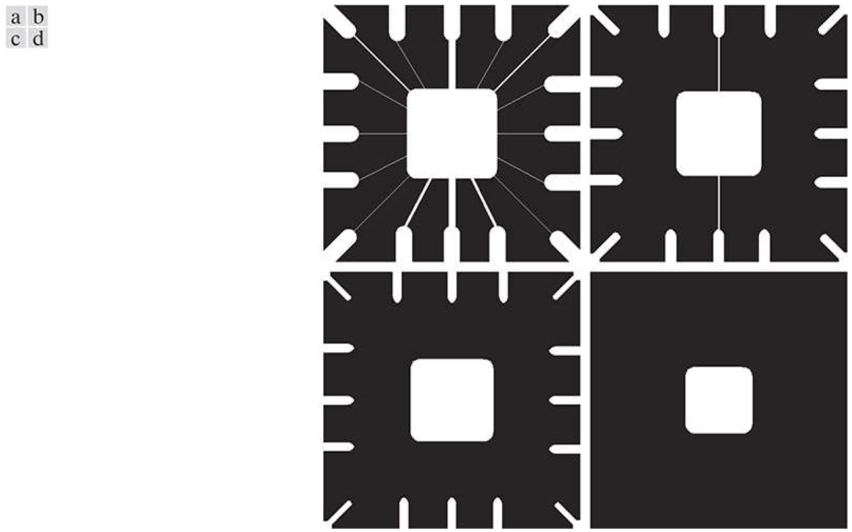


FIGURE 9.5

Using erosion to remove image components. (a) A 486×486 binary image of a wire-bond mask in which foreground pixels are shown in white. (b)–(d) Image eroded using square structuring elements of sizes 11×11, 15×15, and 45×45 elements, respectively, all valued 1.

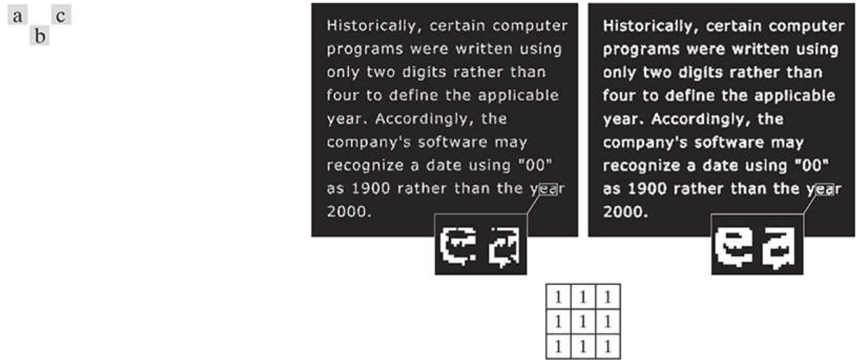


FIGURE 9.7

(a) Low-resolution text showing broken characters (see magnified view). (b) Structuring element. (c) Dilation of (a) by (b). Broken segments were joined.

Opening and closing

Opening is made of successive erosion and dilation:

$$\gamma_B(A) = A \circ B = (A \ominus B) \oplus B = \bigcup (B_x : B_x \subseteq A)$$

Opening and closing

Opening is made of successive erosion and dilation:

$$\gamma_B(A) = A \circ B = (A \ominus B) \oplus B = \bigcup (B_x : B_x \subseteq A)$$

Closing is the dual operation, made of successive dilation and erosion:

$$\phi_B(A) = A \bullet B = (A \oplus B) \ominus B = \left(\bigcup (B_x : B_x \cap A = \emptyset)\right)^c$$

Properties

- duality: $A \bullet B = (A^c \circ \breve{B})^c$
  and $A \circ B = (A^c \bullet \breve{B})^c$
- translation invariance
- opening is antiextensive $\Psi(A) \subseteq A$
- closing is extensive $A \subseteq \Psi(A)$
- idempotence $\Psi(\Psi(A)) = \Psi(A) \quad \forall A$

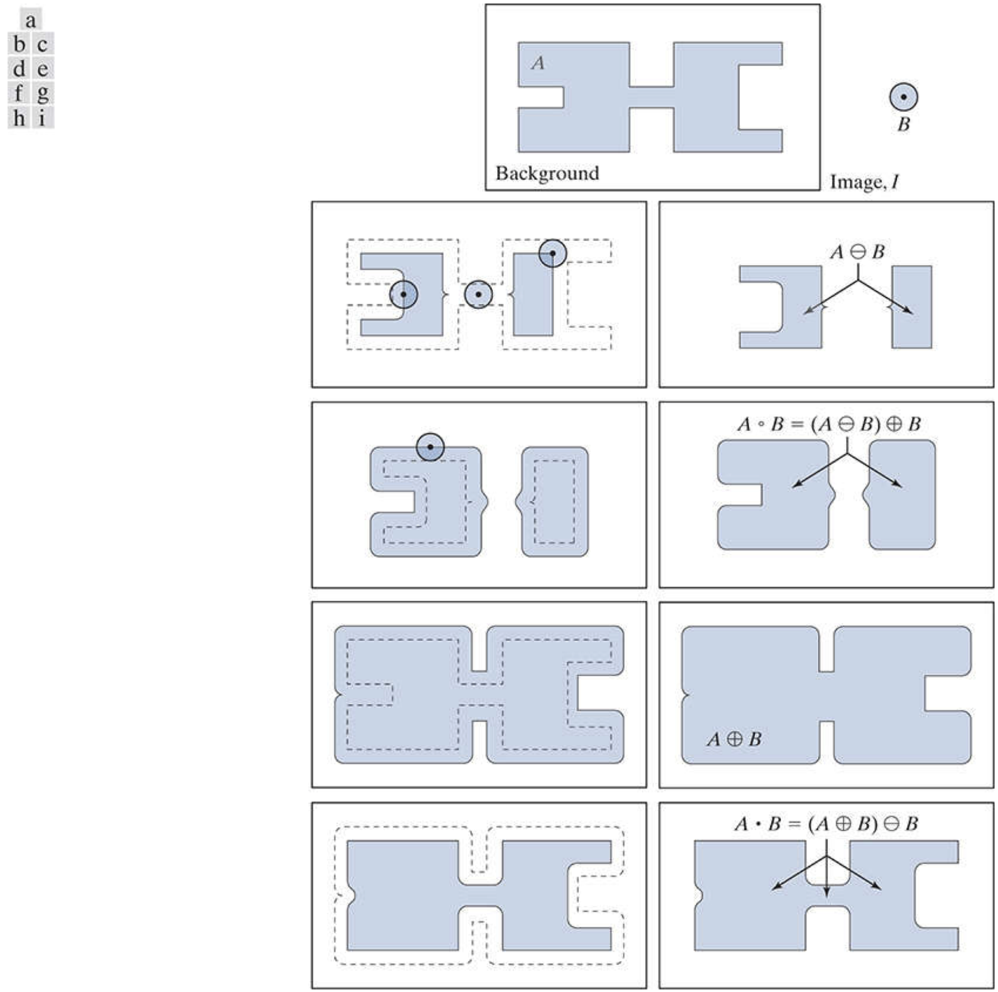


FIGURE 9.10

Morphological opening and closing. (a) Image *I*, composed of a set (object) *A* and background; a solid, circular structuring element is shown also. (The dot is the origin.) (b) Structuring element in various positions. (c)-(i) The morphological operations used to obtain the opening and closing.

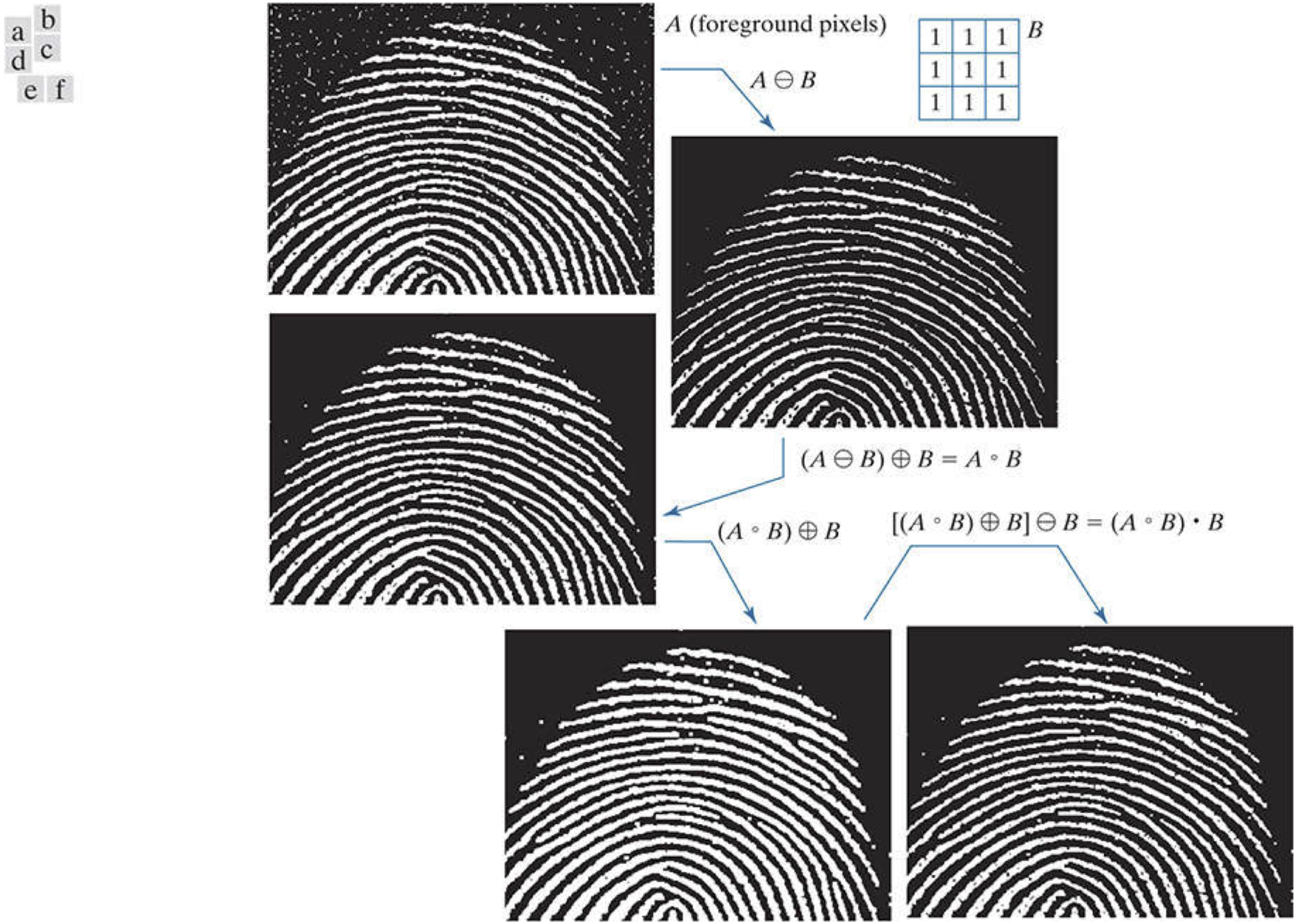


FIGURE 9.11

(a) Noisy image. (b) Structuring element. (c) Eroded image. (d) Dilation of the erosion (opening of *A*). (e) Dilation of the opening. (f) Closing of the opening.

(Original image courtesy of the National Institute of Standards and Technology.)

Hit-or-Miss Transform

Hit-or-Miss Transform (HMT) is a basic tool for template matching.

It considers two structuring elements B_1 and B_2 to analyse both the foreground and the background, assuming $B_1 \subset (B_2)^c$

$$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2) = \{x : (B_1)_x \subseteq A \text{ and } (B_2)_x \subseteq A^c\}$$

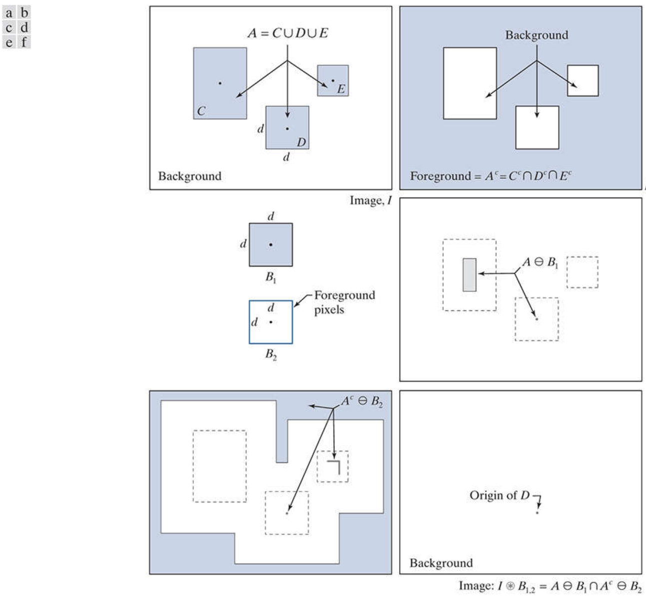


FIGURE 9.12
(a) Image consisting of a foreground (1's) equal to the union, A, of set of objects, and a background of 0's. (b) Image with its foreground defined as A^c. (c) Structuring elements designed to detect object D. (d) Erosion of A by B_1. (e) Erosion of A^c by B_2. (f) Intersection of (d) and (e), showing the location of the origin of D, as desired. The dots indicate the origin of their respective components. Each dot is a single pixel.

Morphological tools

Boundary extraction

intern gradient: $\beta(A) = A - (A \ominus B)$
extern gradient: $(A \oplus B) - A$
morphological gradient: $(A \oplus B) - (A \ominus B)$

Hole filling

$X_k = (X_{k-1} \oplus B) \cap I^c$ until convergence (with X_0 containing some predetermined hole pixels)

Extraction of connected component(s)

$X_k = (X_{k-1} \oplus B) \cap I$ until convergence (with X_0 containing some predetermined object pixels)

Skeleton

$$S(A) = \bigcup_{k=0}^{K} S_k(A) \text{ with } S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B$$

where $A \ominus kB$ indicates k successive erosions with B, starting from A
and $K = \max(k|(A \ominus kB) \neq \emptyset)$

Residue operators

Instead of filtering the images, focus on the parts that have been filtered out.

Top-hat

Top-hat by opening (or white top-hat, WTH): $A - (A \circ B)$

Bottom-hat

Top-hat by closing (or black top-hat, BTH): $(A \bullet B) - A$

Thinning (a kind of skeleton)

$A \otimes B = A - (A \circledast B) = A \cup (A \circledast B)^c$, with a sequence of structuring elements: $A \otimes B = ((\dots((A \otimes B^1) \otimes B^2)\dots) \otimes B^n)$

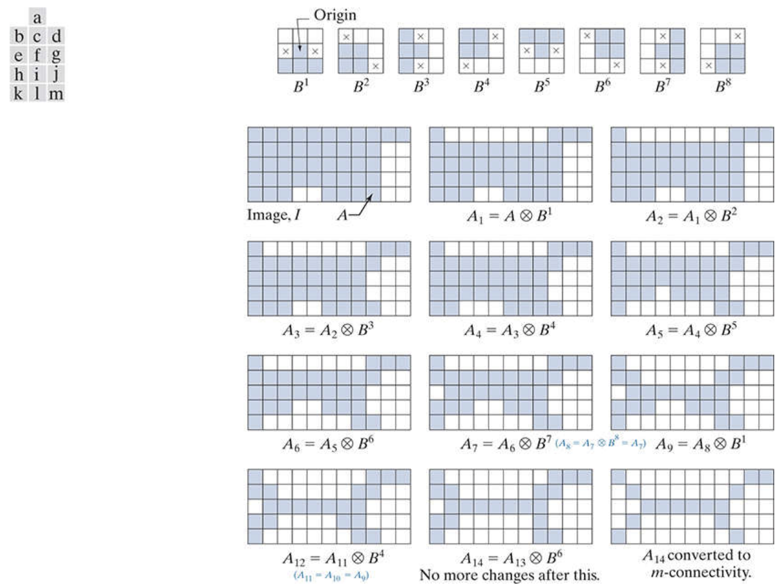


FIGURE 9.23

(a) Structuring elements. (b) Set A. (c) Result of thinning A with B^1 (shaded). (d) Result of thinning A_1 with B_2. (e)–(i) Results of thinning with the next six SEs. (There was no change between A_7 and A_8.) (j)–(k) Result of using the first four elements again. (l Result after convergence. (m) Result converted to m-connectivity.

$B^1_{\text{fg}} = \{(x-1, y+1), (x, y+1), (x+1, y+1), (x, y)\}$
$B^1_{\text{bg}} = \{(x-1, y-1), (x, y-1), (x+1, y-1)\}$

Note that $(x-1, y)$ and $(x+1, y)$ are ignored.

———

Digital image processing; Gonzalez, Woods; Pearson

Thickening

$A \odot B = A \cup (A \circledast B)$, with a sequence of structuring elements: $A \odot B = ((\ldots((A \odot B^1) \odot B^2)\ldots) \odot B^n)$

$B_{\text{thickening}} = B^c_{\text{thinning}}$ (i.e. interverting 0s and 1s).

a b
c d
e

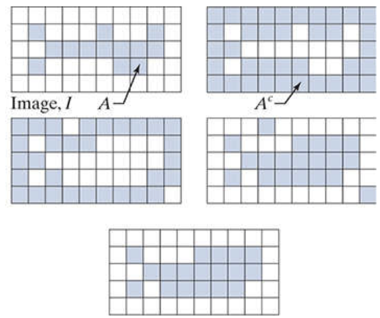


FIGURE 9.24
(a) Set *A*. (b) Complement of *A*. (c) Result of thinning the complement. (d) Thickened set obtained by complementing (c). (e) Final result, with no disconnected points.

Convex Hull

$$X_k^i = (X_{k-1}^i \circledast B^i) \cup X_{k-1}^i \quad i = 1, 2, 3, 4$$

until convergence and with $X_0 = A$, then $C(A) = \bigcup_i X_k^i$

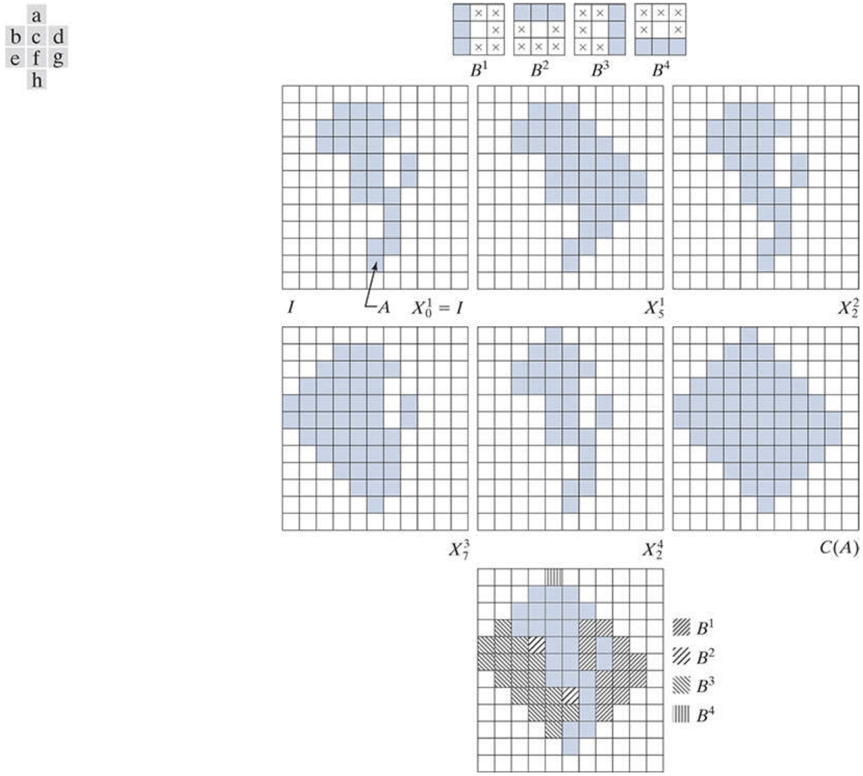


FIGURE 9.21
(a) Structuring elements. (b) Set A. (c)–(f) Results of convergence with the structuring elements shown in (a). (g) Convex hull. (h) Convex hull showing the contribution of each structuring element.

Pruning

Done in 4 steps:

1. $X_1 = A \otimes \{B\}$
2. $X_2 = \bigcup_{k=1}^{8} (X_1 \circledast B^k)$
3. $X_3 = (X_2 \oplus H) \cap A$ with H as 3×3 SE
4. $X_4 = X_1 \cup X_3$

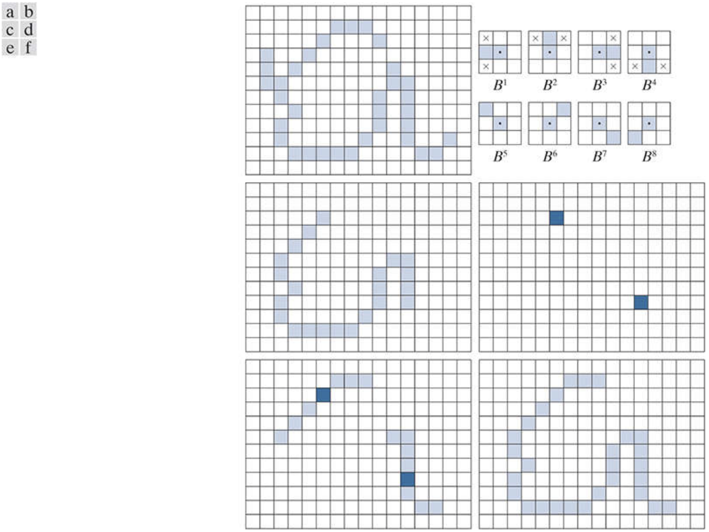


FIGURE 9.27
(a) Set *A* of foreground pixels (shaded). (b) SEs used for deleting end points. (c) Result of three cycles of thinning. (d) End points of (c). (e) Dilation of end points conditioned on (a). (f) Pruned image.

Alternate sequential filters (ASF)

ASF opening-closing

$$(((((S \bullet B) \circ B) \bullet 2B) \circ 2B) \ldots \bullet nB) \circ nB$$

ASF closing-opening

$$(((((S \circ B) \bullet B) \circ 2B) \bullet 2B) \ldots \circ nB) \bullet nB$$

Opening invariance: A is said to be B-open if $A \circ B = A$

Morphological reconstruction

Geodesic dilation of size 1 of marker F w.r.t. mask G: $D_G^{(1)}(F) = (F \oplus B) \cap G$

Geodesic dilation of size n: $D_G^{(n)}(F) = D_G^{(1)}\left(D_G^{(n-1)}(F)\right)$

Morphological reconstruction

Geodesic dilation of size 1 of marker F w.r.t. mask G: $D_G^{(1)}(F) = (F \oplus B) \cap G$

Geodesic dilation of size n: $D_G^{(n)}(F) = D_G^{(1)}\left(D_G^{(n-1)}(F)\right)$

Geodesic erosion of size 1 of marker F w.r.t. mask G: $E_G^{(1)}(F) = (F \ominus B) \cup G$

Geodesic erosion of size n: $E_G^{(n)}(F) = E_G^{(1)}\left(E_G^{(n-1)}(F)\right)$

with $n \geq 1$ and $D_G^{(0)}(F) = E_G^{(0)}(F) = F$

Morphological reconstruction

Geodesic dilation of size 1 of marker F w.r.t. mask G: $D_G^{(1)}(F) = (F \oplus B) \cap G$

Geodesic dilation of size n: $D_G^{(n)}(F) = D_G^{(1)}\left(D_G^{(n-1)}(F)\right)$

Geodesic erosion of size 1 of marker F w.r.t. mask G: $E_G^{(1)}(F) = (F \ominus B) \cup G$

Geodesic erosion of size n: $E_G^{(n)}(F) = E_G^{(1)}\left(E_G^{(n-1)}(F)\right)$

with $n \geq 1$ and $D_G^{(0)}(F) = E_G^{(0)}(F) = F$

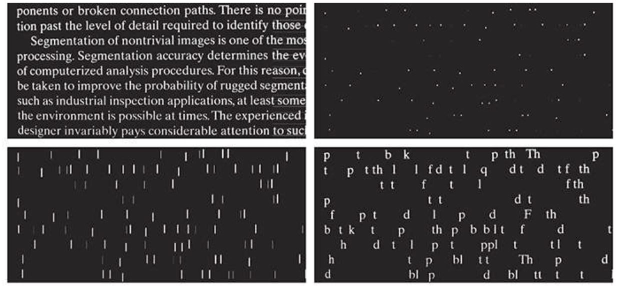
Morphological reconstruction by dilation of marker F w.r.t. mask G:
$R_G^D(F) = D_G^{(k)}(F)$ with k s.t. $D_G^{(k)}(F) = D_G^{(k+1)}(F)$

Morphological reconstruction by erosion of marker F w.r.t. mask G:
$R_E^D(F) = E_G^{(k)}(F)$ with k s.t. $E_G^{(k)}(F) = E_G^{(k+1)}(F)$

Reconstruction by erosion/dilation are duals w.r.t. set complementation.

Applications of reconstruction

- opening by reconstruction: $O_R^{(n)}(F) = R_F^D(F \ominus nB)$
- closing by reconstruction: $C_R^{(n)}(F) = R_F^E(F \oplus nB)$
- filling holes for an image I and its border-filtered $F(x,y) = 1 - I(x,y)$ if (x,y) on the border of I, and 0 otherwise: $(R_{I^c}^D(F))^c$
- border clearing for an image I and its border-filtered $F(x,y) = I(x,y)$ if (x,y) on the border of I, and 0 otherwise: $I - R_I^D(F)$



a b
c d

FIGURE 9.31

(a) Text image of size 918×2018 pixels. The approximate average height of the tall characters is 51 pixels. (b) Erosion of (a) with a structuring element of size 51×1 elements (all 1's). (c) Opening of (a) with the same structuring element, shown for comparison. (d) Result of opening by reconstruction.

a b
c d

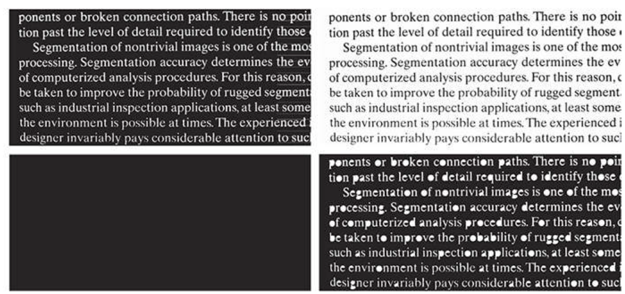


FIGURE 9.33

(a) Text image of size 918×2018 pixels. (b) Complement of (a) for use as a mask image. (c) Marker image. (d) Result of hole-filling using Eqs. (9-45) and (9-46) .

a b

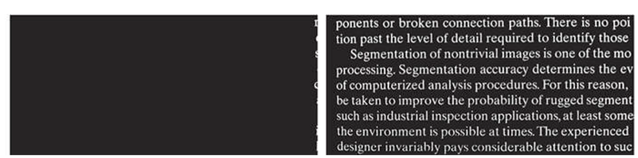


FIGURE 9.34

(a) Reconstruction by dilation of marker image. (b) Image with no objects touching the border. The original image is Fig. 9.31(a)

Granulometry

Let us consider successive applications of openings with a series of increasing SEs based on B (a.k.a. generator): $\Psi_t(A) = A \circ tB \text{ with } t > 0$

Size distribution can be measured: $\Omega(t) = v[A] - v[A \circ tB]$ with v the surface

Normalized distribution: $\Phi(t) = \frac{\Omega(t)}{v[A]}$

Pattern spectrum: $\Phi(k)$ or its derivative $d\Phi(k) = \Phi(k) - \Phi(k-1)$ and $d(0) = \mathrm{Id}$

The previous definitions also apply to series of closings, leading to $\Omega(t) = v[A \circ tB] - v[A]$

Both series/measures can be combined to focus on both black & white areas.

Opening Transform

From the series of openings (by reconstruction), not only global measures, but also local information can be derived.

The opening transform returns the scale at which a given pixel has been filtered.

$$\Xi(p) = \max\{t \geq 0 \mid \Psi_t(p) \neq 0\}$$

The closing transform acts similarly:

$$\Xi(p) = \max\{t \geq 0 \mid \Psi_t(p) = 0\}$$

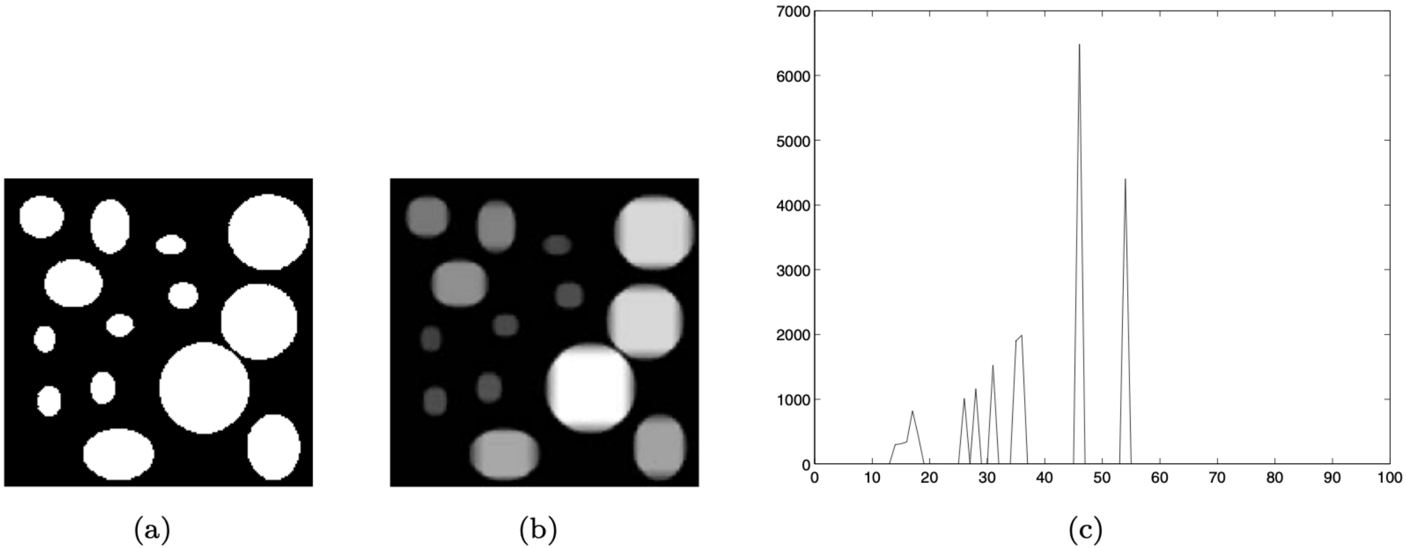


Figure 13.42: Example of binary granulometry performance. (a) Original binary image. (b) Maximal square probes inscribed—the initial probe size was 2×2 pixels. (c) Granulometric power spectrum as histogram of (b)—the horizontal axis gives the size of the object and the vertical axis the number of pixels in an object of given size. *Courtesy of P. Kodl, Rockwell Automation Research Center, Prague.*

Binary Morphology

MM is a powerful tool to process binary images:

- generated by thresholding
- or results from prior detection/classification steps

Binary Morphology

MM is a powerful tool to process binary images:

- generated by thresholding
- or results from prior detection/classification steps

Extension to grayscale images (next lecture) can be achieved with lattice theory:

- replace set union by maximum
- replace set intersection by minimum

Labs

Built-in functions available in popular toolboxes:

- `scipy.ndimage` package (the morphology component)
- `skimage.morphology` package from scikit-image

but can be recoded from scratch as well.

Practice:

1. Threshold a panchromatic (or NDVI) image or consider a binary land cover map.
2. Compare the effect of erosion, dilation, opening, closing with different structuring elements.
3. Apply the more advanced morphological operators on the image.
4. Assess (visually) the relevance of the opening transform.