Linear equations

1. Solve the system of equations $4x + 3y = 17$ and $2x - y = 10$ by

   (a) method of substitution;

   (b) method of elimination;

   (c) method of comparison;

   (d) cross-multiplication method;

   (e) Gauss-elimination method.

   Answer: $(x, y) = \left(\frac{47}{10}, -\frac{3}{5}\right)$.

2. Compute the solution of the system of equations $5x + 4y = 12$ and $x - 2y = 10$ by method of substitution, if the solution exists.
   Answer: $(x, y) = \left(\frac{32}{7}, -\frac{19}{7}\right)$.

3. Find the values of b_1 and b_2 such that the system of equations $5x + 4y = b_1$ and $x - 3y = b_2$ has

   (a) infinitely many solution;

   (b) unique solution;

   (c) No solution.

   solution: (a) None. (b) for every pair (b_1, b_2). (c) None.

4. Compute the solution of the system of equation $2x+3y+z = 5$, $y+3z = 4$, $x+4y+4z = 7$ by Gauss elimination method, if exists.
   Answer: $(x, y, z) = \left(2, -\frac{1}{8}, \frac{11}{8}\right)$.

5. Compute the solution of the system of equation $3x + 2y + z = 5$, $2x + 3y + z = 5$, $x + y + 4z = 6$ by

   (a) method of substitution;

   (b) method of elimination;

   (c) method of comparison;

   (d) cross-multiplication method;

   (e) Gauss-elimination method,

   if exists.
   Answer: $(x, y, z) = \left(\frac{7}{9}, \frac{7}{9}, \frac{10}{9}\right)$.

Arithmetic and geometric progression

1. The sum of the three numbers in A.P. is 21 and the product of the first and third number of the sequence is 45. What are the three numbers?
   Solution: $5, 7, 9$ (or, $9, 7, 5$).

2. Find S_{15} for the following series: $1, 8, 15, \ldots$.
   Solution: 750

3. Rahul saves Rs. 100 in January 2014 and increases his saving by Rs. 51 every month over previous month. What is annual saving for Rahul in the year 2014?
   Solution: 4566

4. Find the sum of the following series: $-64, -66, -68, \ldots, -100$.
   Solution: -1558

5. Find the sum of all even numbers between 99 and 999.
   Solution: 247050

6. A progression has a first term of 12 and a fifth term of 8.
   (a) Find the sum of the first 25 terms if the progression is arithmetic.
   (b) Find the 13^{th} term if the progression is geometric.

   Answer: (a) $= 0$. (b) $\frac{32}{9}$.

7. If a rubber ball consistently bounces back $\frac{2}{3}$ of the height from which it is dropped, what fraction of its original height will the ball bounce after being dropped and bounced four times without being stopped?
   Answer: $\frac{16}{81}h$.

8. If the first term of a G.P. is 20 and the common ratio is 4. Find the 5^{th} term.
   Answer: 5120

9. The number 2048 is which term in the following Geometric sequence $2, 8, 32, 128, \ldots$.
   Answer: 6

10. The sum of the first three terms of a G.P. is $\frac{21}{2}$ and their product is 27. Find the common ratio.
    Answer: 2 and 0.5.

11. A progression has a second term of 96 and a fourth term of 54. Find the first term of the progression in each of the following cases:
    (a) the progression is arithmetic,
    (b) the progression is geometric with a positive common ratio.

    Answer: (a) $= 117$. (b) 128.

Permutation and combination

1. Show that ${}^{n}P_{r} = \ n.\ {}^{n-1}P_{r-1}$.

2. Show that ${}^{n}P_{r} = \ {}^{n-1}P_{r} + r.\ {}^{n-1}P_{r-1}$

3. Six officials of a company are to fly to a conference in Dhaka. Company policy states that no two can fly on the same plane. If there are 9 flights available, how many flight schedules can be established?
Answer: 60480.

4. In a class, there are 25 boys and 15 girls. The teacher wants to select 1 boy and 1 girl to represent the class for a function. In how many ways can the teacher make this selection?
Answer: 375.

5. Evaluate each of the following :
   (a) ${}^{3}C_{2}$.
   (b) ${}^{5}C_{2}$.
   (c) ${}^{5}C_{3}$.
   (d) ${}^{4}C_{3} + \ {}^{4}C_{2}$.
   (e) ${}^{6}C_{3}$.

6. Show that ${}^{n}C_{r} + \ {}^{n}C_{r-1} = \ {}^{n+1}C_{r}$.

7. Find the number of subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ having 4 elements.
Answer: ${}^{11}C_{4}$.

8. A student has to answer 9 questions, choosing at least 3 from each of Parts A and B. If there are 6 questions in Part A and 7 in Part B, in how many ways can the student choose 9 questions.
Answer: 700.

9. Out of the letters P, Q, R, x, y and z, how many arrangements can be made (i) beginning with a capital; (ii) beginning and ending with a capital.
Answer: (i) 360. (ii) 144.

10. How many different words can be made out of the letters of the word 'ALLAHABAD'? In how many of these with the vowels occupy the even places?
Answer: No. of different words = $\frac{9!}{(4!)(2!)}$.
No. of words in which vowels occupy even place is 60.

11. Find the number of permutations of the word 'ENGINEERING'.
Answer:
$$\frac{11!}{(3!)(3!)(2!)(2!)}.$$

Matrix and determinant

1. If a matrix has 10 elements, what are the possible orders it can have?

2. Construct a 4×5 matrix whose elements are given by $a_{ij} = \frac{i}{3} - \frac{j}{2}$.

3. If $\begin{bmatrix} x+3 & z+4 & y-2 \\ 0 & a+1 & b-1 \\ c+5 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2z+1 & -y+1 \\ 0 & 2 & 3 \\ 1-2c & 3 & 4 \end{bmatrix}$. Find the values of a, b, c, x, y and z.

4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 3 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 4 & 1 & 2 & -1 \\ 0 & 3 & 2 & 1 \\ 1 & -2 & 3 & -1 \end{bmatrix}$. Then compute the following products

   (a) BA.

   (b) BC.

   (c) CB.

   Answer: $BA = \begin{bmatrix} 0 & 4 & -9 \\ 19 & 3 & -3 \\ 5 & 4 & -7 \\ -3 & 1 & -4 \end{bmatrix}$, $BC = \begin{bmatrix} 14 & -4 & 10 & -6 \\ 21 & 0 & 27 & -7 \\ 14 & -1 & 12 & -5 \\ 5 & -4 & 3 & -3 \end{bmatrix}$, $CB = \begin{bmatrix} 21 & -1 & 16 \\ 19 & 5 & 20 \\ 3 & -4 & -3 \end{bmatrix}$.

5. Find $A^2 - 5A + 6I_3$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$.

   Answer: $\begin{bmatrix} 1 & 1 & -3 \\ -1 & 5 & -10 \\ -1 & -4 & 10 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 4 & 1 & 0 & 3 \\ 2 & 3 & 1 & 0 \\ 0 & 2 & 4 & 3 \\ 3 & 0 & 2 & 1 \end{bmatrix}$. Evaluate A_{12}, A_{23}, A_{34}.

   Answer: $\det A = -151$, $A_{12} = -5$, $A_{23} = 1$, $A_{34} = -23$.

7. Let $A = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 2 & 3 & 1 & 0 \\ -3 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 \end{bmatrix}$. Evaluate M_{13},M_{24}, M_{43}.

   Answer: $\det(A) = 61$, $M_{13} = 4$, $M_{24} = -119$, $M_{43} = 41$.

8. Let $A = \begin{bmatrix} 4 & 2 & 3 & 5 \\ 5 & 3 & 4 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$. Evaluate $\det A$

   (a) by using first row cofactor expansion;

   (b) by using fourth row cofactor expansion;

   (c) by using third column cofactor expansion.

   Answer: $\det(A) = -87$.

9. Let $A = \begin{bmatrix} 2 & 2 & 1 & 6 \\ 5 & 3 & 2 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 3 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 1 & 2 & 3 \end{bmatrix}$.

   (a) Evaluate $\det A$, $\det B$.

   (b) Evaluate $\det(AB)$ and verify the fact that $\det(AB) = (\det A)(\det B)$.

   Answer: $\det(A) = -111$, $\det(B) = 11$, and $\det(AB) = -1221$.

10. Evaluate the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

    Answer: The eigen pairs are $\left(1, \begin{bmatrix} -0.9487 \\ 0.3162 \end{bmatrix}\right)$ and $\left(5, \begin{bmatrix} -0.7071 \\ -0.7071 \end{bmatrix}\right)$.

11. Evaluate the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

    Answer: The eigen pairs are

$$\left(\frac{490}{103}, \begin{bmatrix} -\frac{1487}{2564} \\ -\frac{1233}{1610} \\ \frac{588}{2117} \end{bmatrix}\right),$$

$$\left(\frac{167}{103} - i\frac{1420}{1461}, \begin{bmatrix} \frac{1242}{1411} \\ -\frac{215}{678} - i\frac{216}{1667} \\ -\frac{215}{678} + i\frac{216}{1667} \end{bmatrix}\right),$$

$$\left(\frac{167}{103} + i\frac{1420}{1461}, \begin{bmatrix} \frac{1242}{1411} \\ -\frac{215}{678} + i\frac{216}{1667} \\ -\frac{26}{4821} - i\frac{617}{1879} \end{bmatrix}\right).$$

Variables and functions

1. Find the domain and range for each of the following functions:

   (a) $f(x) = \frac{1}{2+\exp(x)}$.

   (b) $f(t) = \cos(\exp(-t))$.

   (c) $f(t) = \sqrt{1+3^{-t}}$.

   (d) $f(x) = \frac{3}{1-\exp(2x)}$.

2. Write the formula for $f \circ g \circ h$ for the following:

   (a) $f(x) = x+1,\ g(x) = 3x,\ h(x) = 4-x$.

   (b) $f(x) = 3x+4,\ g(x) = 2x-1,\ h(x) = x^2$.

   (c) $f(x) = \sqrt{x+1},\ g(x) = \frac{1}{x+4},\ h(x) = \frac{1}{x}$

   (d) $f(x) = \frac{x+2}{3-x},\ g(x) = \frac{x^2}{x^2+2},\ h(x) = \sqrt{2-x}$.

3. Let $f(x) = \frac{x}{x-2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

4. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x+2$.

5. Evaluate each expression using the functions $f(x) = 2-x$ and $g(x) = \begin{cases} -x, & -2 \le x < 0 \\ x-1, & 0 \le x \le 2 \end{cases}$.

   (a) $f(g(0))$,

   (b) $g(f(3))$,

   (c) $g(g(-1))$,

   (d) $f(f(2))$,

   (e) $g(f(0))$,

   (f) $f(g(\frac{1}{2}))$.

Limit and continuity

1. Compute the following:
   (a) $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$.
   (b) $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$.
   (c) $\lim_{x\to 2^+} \frac{x(2x+5)}{(x^2+x)(x+1)}$.
   (d) $\lim_{x\to 0^-} \frac{\sqrt{6}-\sqrt{5x^2+11x+6}}{x}$.
   (e) $\lim_{x\to 2^+} \frac{(x+3)|x+2|}{x+2}$.
   (f) $\lim_{x\to 2^-} \frac{(x+3)|x+2|}{x+2}$.
   (g) $lim_{x\to 0} 6x^2(\cot x)(\mathrm{cosec} 2x)$.
   (h) $lim_{x\to 0} \frac{\tan t \sec 2t}{3t}$
2. Using the Sandwich theorem solve the followings
   (a) If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.
   (b) If $2-x^2 \le g(x) \le 2\cos x$ for all x, find find $\lim_{x\to 0} g(x)$.
   (c) It can be shown that the inequalities $1-\frac{x^2}{6} \le \frac{x\sin x}{2-2\cos x} \le 1$ hold for all values of x close to zero. Compute $\lim_{x\to 0} \frac{x\sin x}{2-2\cos x}$?
3. Find the limits of the followings. Are the functions continuous at the point being approached?
   (a) $\lim_{x\to \pi} \sin(x-\sin x)$
   (b) $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(tant)\right)$
   (c) $\lim_{y\to 1} \sec\left(y\sec^2 y - \tan^2 y - 1\right)$.
4. For what values of a and b the function

$$g(x) = \begin{cases} ax+2b, & x \le 0 \\ x^2+3a-b, & 0 < x \le 2 \\ 3x-5, & x > 2 \end{cases}$$

   is continuous at every x?
5. Define $g(3)$ in a way that extends $g(x) = \frac{x^2-9}{x-3}$ to be continuous at $x=3$.
6. Define $h(2)$ in a way that extends $h(t) = \frac{t^2+3t-10}{t-2}$ to be continuous at $t=2$.
7. Define $f(1)$ in a way that extends $f(s) = \frac{s^3-1}{s^2-1}$ to be continuous at $s=1$.
8. Define $g(4)$ in a way that extends $g(x) = \frac{x^2-16}{x^2-3x-4}$ to be continuous at $x=4$.