Q1) $P(\omega_1) = \frac{1}{4}$ $\quad P(\omega_2) = \frac{3}{4}$

$P(x|\omega_1) = N(2,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2}$

$P(x|\omega_2) = N(5,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-5)^2}$

(i) For zero-one loss,
choose ω_1 when

$$\frac{P(x|\omega_1)}{P(x|\omega_2)} > \frac{P(\omega_2)}{P(\omega_1)} = \frac{3}{1}$$

$e^{\frac{1}{2}((x-5)^2-(x-2)^2)} > 3$ then choose ω_1

$\Rightarrow (x-5)^2 - (x-2)^2 > 2\ln 3$

$\Rightarrow 21 - 6x > \ln 9$

when $x < \frac{21 - \ln 9}{6} \approx 3.13$

so $x = 3.13$ is the decision boundary

choose ω_1 when $x < 3.13$ and ω_2 when $x > 3.13$

(ii) For given loss function, in order to minimise the loss/error (not error rate because that is already given by zero-one loss)
choose ω_1 when

$$\frac{P(x|\omega_1)}{P(x|\omega_2)} > \frac{2-0}{3-0} \cdot \frac{\frac{3}{4}}{\frac{1}{4}} = \frac{1}{2}$$

$e^{\frac{1}{2}((x-5)^2-(x-2)^2)} > 2$

$(x-5)^2 - (x-2)^2 > 2\ln 2$

$\Rightarrow 21 - 6x > \ln 4$

when $x < \frac{21 - \ln 4}{6} \approx 3.26$ choose ω_1

$x > \frac{21 - \ln 4}{6} \approx 3.26$ choose ω_2

so $x = 3.26$ is the decision boundary

Zero-one ~~to~~ loss fn. for cancer prediction is not preferred because the consequences of a false negative is far worse than a false positive, which is not there in zero-one loss which is symmetric in nature.

Q2) $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ X_1, X_2, X_3 are three random ~~vectors~~ variables

$\mu_x = \begin{bmatrix} 5 \\ -5 \\ 6 \end{bmatrix}$ this implies

$A = [2 \ -1 \ 2]^T = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$Y = A^T X + B$

$\mu_Y = \mu(A^T X + B) = \mu(A^T X) + \underbrace{\mu(B)}_{5}$ (as B is a constant)

$\left(A^T X = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 2X_1 - X_2 + 2X_3 \right)$

$\mu_Y = \mu(2X_1 - X_2 + 2X_3) + 5$

$= \mu(2X_1) - \mu(X_2) + \mu(2X_3) + 5$

$= 2\mu(X_1) - \mu(X_2) + 2\mu(X_3) + 5$

$= 2 \cdot 5 - (-5) + 2 \cdot 6 + 5$

$= \boxed{32}$

Q3)

a) $P(x|\omega_i) = \frac{1}{\pi b}\left(\frac{1}{1+\left(\frac{x-a_i}{b}\right)^2}\right)$

For zero-one loss, ~~*~~ as $P(\omega_1) = P(\omega_2) = \frac{1}{2}$ the decision boundary can be determined by

if $\frac{P(x|\omega_1)}{P(x|\omega_2)} > 1$ choose ω_1 else ω_2 if smaller than 1

$$\frac{P(x|\omega_1)}{P(x|\omega_2)} = \frac{b^2 + (x-a_2)^2}{b^2 + (x-a_1)^2} > 1$$

$\Rightarrow$ choose ω_1 when $(x-a_2)^2 > (x-a_1)^2$

choose ω_2 when $(x-a_2)^2 < (x-a_1)^2$

$$x^2 + a_2^2 - 2a_2x > x^2 + a_1^2 - 2a_1x$$

$$\underbrace{\cancel{a_2^2 - a_1^2}}_{(a_2+a_1)} > 2x(\cancel{a_2 - a_1})$$

if $x < \frac{a_1+a_2}{2}$ choose ω_1

if $x > \frac{a_1+a_2}{2}$ choose ω_2

hence $x = \frac{a_1+a_2}{2}$ is the decision boundary

b) Using Baye's rule we know that,

$$P(\omega_1|x) = P(x|\omega_1) \cdot \frac{P(\omega_1)}{P(x)} \quad - (1)$$

also as there are only 2 states

$$\cancel{P(x|\omega_1) + P(x|\omega_2)}$$

$$P(\omega_1|x) + P(\omega_2|x) = 1 \quad - ②$$

From ① & ② and substituting values $a_1 = 3,\ a_2 = 5,\ b = 1$

$$\frac{1}{2\pi} \cdot \frac{1}{1+(x-3)^2} \cdot \frac{1}{P(x)} + \frac{1}{2\pi}\left(\frac{1}{(x-5)^2+1}\right) \cdot \frac{1}{P(x)} = 1$$

$$P(x) = \frac{1}{2\pi}\left(\frac{2+(x-3)^2+(x-5)^2}{(1+(x-3)^2)(1+(x-5)^2)}\right)$$

$$\Rightarrow P(\omega_1|x) = \frac{1}{\cancel{2\pi}}\left(\frac{1}{\cancel{1+(x-3)^2}}\right) \cdot \frac{\cancel{2\pi}\,\cancel{(1+(x-3)^2)}(1+(x-5)^2)}{2+(x-3)^2+(x-5)^2}$$

$$= \frac{(1+(x-5)^2)}{2+(x-3)^2+(x-5)^2}$$

c) To find error rate we need to integrate $P(\omega_i|x)P(x)$ over regions where $P(\omega_i|x) < 0.5$

$P(\omega_1|x) < 0.5 \ \forall\ x > 4$ and hence as $P(\omega_2|x) = 1 - P(\omega_1|x)$

$P(\omega_2|x) < 0.5 \ \forall\ x < 4$

$$\therefore P(\text{error}) = \int_{-\infty}^{4} P(\omega_2|x)P(x)\,dx + \int_{4}^{\infty} P(\omega_1|x)P(x)\,dx$$

$$= \int_{-\infty}^{4} \frac{1}{2\pi}\left(\frac{1}{1+(x-5)^2}\right)dx + \int_{4}^{\infty} \frac{1}{2\pi}\left(\frac{1}{1+(x-3)^2}\right)dx$$

$$= \frac{1}{2\pi}\left[\tan^{-1}(x-5)\right]_{-\infty}^{4} + \frac{1}{2\pi}\left[\tan^{-1}(x-3)\right]_{4}^{\infty}$$

$$= \frac{1}{2\pi}\left(-\frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= \frac{1}{2\pi} \cdot \frac{\pi}{2} = \frac{1}{4}$$

Hence error rate = 25%.

Q4 Random vector $X = [a \ b]$

As b is a ~~Bernoulli~~ Gaussian random variable

$$p(b) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{b-m}{\sigma}\right)^2}$$

And as 'a' is Bernoulli random variable

$$p(a) = \theta^{a}(1-\theta)^{1-a}$$

a) In covariance matrix of X it can be seen that $\mathrm{cov}(a,b) = 0$ hence

$$P(X) = p(a,b) = p(a) \times p(b)$$

$$= \frac{\theta^{a}(1-\theta)^{1-a}}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{b-m}{\sigma}\right)^2}$$

b) As a and b are ~~a~~ independent random variables, N iid variables of x is identical to taking N iid samples of a & b

so joint probability: ~~of~~

$$P(x_1, x_2 \ldots x_N) = P(a_1, a_2, a_3 \ldots a_N) \cdot P(b_1, b_2 \ldots b_N)$$

Since we need to find a value for θ to maximise $P(x_1, x_2 \ldots x_N)$ we can treat joint probability of the N iid Gaussian 'b' variables as a constant (let us take it as k)

$$q(x) = P(x_1, x_2 \ldots x_N) = P(a_1, a_2 \ldots a_N) \cdot k$$

$$P(\bar{a}) = P(a_1, a_2 \ldots a_N) = P(a_1) \cdot P(a_2) \ldots P(a_N)$$

$$P(a) = \theta^{a} \cdot (1-\theta)^{1-a} \qquad a = \{0, 1\}$$

$$P(\bar{a}) = \prod_{i=1}^{N} P(a_i)$$

$$= \prod_{i=1}^{N} \theta^{a_i} (1-\theta)^{1-a_i}$$

$$\ln(P(\bar{a})) = \sum_{i=1}^{N} \ln\left(\theta^{a_i} (1-\theta)^{1-a_i}\right)$$

$$= \ln\theta \sum_{i=1}^{N} a_i + \ln(1-\theta) \sum_{i=1}^{N} (1-a_i)$$

The expected value of $\sum_{i=1}^{N} a_i$ and $\sum_{i=1}^{N} (1-a_i)$ is a function of θ as $E[a_i] = \theta$

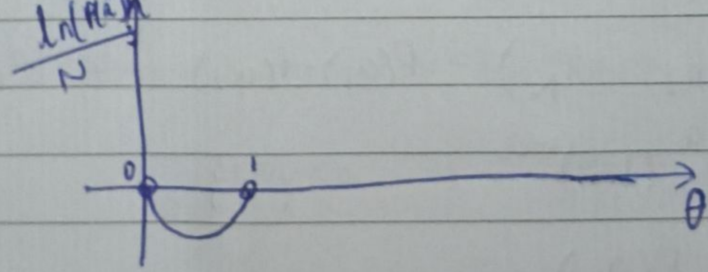
Therefore $\ln(P(\bar{a})) = \ln\theta\,(N\theta) + \ln(1-\theta)(N - N\theta)$
$= N\,(\theta\ln\theta + (1-\theta)\ln(1-\theta))$

Now

$\ln(q(x)) = \ln(P(\bar{a})) + \ln k$

To maximise $q(x)$, $\ln(q(x))$ must be maximised (as $\ln()$ is a monotonically increasing function). Hence $\ln(P(\bar{a}))$ must be maximised.

The plot for $\theta\ln\theta + (1-\theta)\ln(1-\theta)$ vs θ is as follows:



Hence maximum value occurs at 2 points, one just before 1 and just after 0. θ can have ~~two~~ values to maximise $\ln(q(x))$ ie infinitesimally small ~~value~~ positive value and a value just smaller than 1 (something like 0.9999...)

$\therefore \theta = h$ & $1-h$ where h tends to 0 and $h>0$.