Quantitative Predictions for Prime Values of the Titan Polynomial: The Bateman–Horn Constant and Asymptotic Density

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Abstract

The polynomial $Q(n) = n^{47} - (n-1)^{47}$ is a degree-46 cyclotomic norm form whose values have no prime factor less than 283. While this algebraic rigidity does not alter the fundamental sieve dimension ($\kappa = 1$), it significantly impacts the local density of prime values. In this paper, we rigorously compute the Bateman–Horn constant C_Q associated with this sequence. We show that the exclusion of small prime factors leads to an Euler product enhancement of $C_Q \approx 8.68$ relative to the logarithmic integral for degree-46 polynomials. This large constant implies that, under the Bateman–Horn heuristic, prime values of $Q(n)$ occur with a frequency approximately 8.7 times that predicted for a generic polynomial of the same degree. We provide numerical evidence supporting the asymptotic prediction $\pi_Q(x) \sim \frac{C_Q}{46}\,\mathrm{Li}(x)$.

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1 Introduction

The Bateman–Horn conjecture [2] provides a heuristic asymptotic formula for the distribution of prime values of polynomials. For a single irreducible polynomial $f(n)$ of degree d, the number of integers $n \leq x$ such that $f(n)$ is prime is denoted by $\pi_f(x)$ and is conjectured to satisfy

$$\pi_f(x) \;\sim\; C_f \int_2^x \frac{dt}{\log f(t)} \;\sim\; \frac{C_f}{d} \int_2^x \frac{dt}{\log t} \;=\; \frac{C_f}{d}\,\mathrm{Li}(x),$$

where

$$C_f = \prod_p \frac{1 - \omega_f(p)/p}{1 - 1/p}$$

and $\omega_f(p)$ is the number of solutions to $f(n) \equiv 0 \pmod p$. For the Titan polynomial $Q(n) = n^{47} - (n-1)^{47}$—we adopt this name as a convenient label, without implying any established nomenclature—the degree is $d = 46$. The local root structure established in [1] shows that $\omega_Q(p) = 0$ for all primes $p < 283$. In this paper, we compute the specific enhancement constant C_Q resulting from this shielding property and compare the predictions with numerical data in verifiable ranges.

2 The Shielding Property and $\omega_Q(p)$

The local root structure of $Q(n)$, established in [1], is:

1. If $p < 283$, then $\omega_Q(p) = 0$.
2. If $p \geq 283$ and $p \equiv 1 \pmod{47}$, then $\omega_Q(p) = 46$.
3. If $p \geq 283$ and $p \not\equiv 1 \pmod{47}$, then $\omega_Q(p) = 0$.

Remark 2.1. The vanishing $\omega_Q(p) = 0$ for $p < 283$ arises from three distinct mechanisms: (i) for $p \in \{2, 3, 47\}$, the condition $(p-1) \mid 46$ implies $Q(n) \equiv 1 \pmod{p}$ for all n (congruence rigidity); (ii) for $p = 47$ specifically, this is also the ramified prime of $\mathbb{Q}(\zeta_{47})/\mathbb{Q}$; (iii) for all other primes $p < 283$ with $p \not\equiv 1 \pmod{47}$, the map $x \mapsto x^{47}$ is a bijection on $(\mathbb{Z}/p\mathbb{Z})^\times$, so no nontrivial 47-th root of unity exists. The smallest prime $p \equiv 1 \pmod{47}$ is 283, which is why the shielding extends up to that threshold.

This local root structure implies that for small primes, the local density factor $\frac{1-\omega_Q(p)/p}{1-1/p}$ simplifies to $\frac{p}{p-1} > 1$, thereby contributing a significant boost to C_Q.

3 Computation of the Bateman–Horn Constant

The constant C_Q is given by the infinite product:

$$C_Q = \prod_p \left(\frac{1 - \omega_Q(p)/p}{1 - 1/p} \right).$$

We split the product into two parts: small primes ($p < 283$) and large primes ($p \geq 283$).

3.1 Small Primes Product (P_{small})

For $p < 283$, $\omega_Q(p) = 0$. The partial product is:

$$P_{\text{small}} = \prod_{p<283} \frac{p}{p-1}.$$

Direct calculation over the 60 primes less than 283 yields:

$$P_{\text{small}} \approx 10.19.$$

(Note: This value is consistent with Mertens' Theorem approximation $e^\gamma \log(283) \approx 10.05$).

3.2 Large Primes Product (P_{large})

For $p \geq 283$, non-trivial terms occur only when $p \equiv 1 \pmod{47}$.

$$P_{\text{large}} = \prod_{p \geq 283} \frac{1 - \omega_Q(p)/p}{1 - 1/p}.$$

This product converges conditionally. Numerical evaluation of the total product C_Q up to large limits yields:

- $X = 10^5$: $C_Q \approx 8.70$
- $X = 10^7$: $C_Q \approx 8.68$

We adopt the best estimate $C_Q \approx 8.68$.

4 Asymptotic Prediction vs. Data

The Bateman–Horn asymptotic prediction for the prime counting function of $Q(n)$ is:

$$\pi_Q(x) \sim \frac{C_Q}{46}\,\mathrm{Li}(x) \approx \frac{8.68}{46}\,\mathrm{Li}(x) \approx 0.1887\,\mathrm{Li}(x).$$

Table 1 compares the observed prime counts with our prediction. The observed values were computed using the Miller–Rabin primality test (with deterministic bases for this range) and verified independently.

Table 1: Predicted vs. Observed Prime Counts for $Q(n)$. Prediction uses $C_Q = 8.68$.

x	Observed $\pi_Q(x)$	Predicted $\frac{C_Q}{46}\,\mathrm{Li}(x)$	Relative Error
10,000	232	235	−1.3%
20,000	429	432	−0.7%

Remark 4.1. The agreement to within 1−2% at these small ranges is consistent with expectations for the Bateman–Horn conjecture, which is asymptotic. The slight overestimate by the prediction is consistent with the usual lower-order correction terms. The verification range is limited to $x \leq 20{,}000$ due to the rapid growth of $Q(n)$: at $n = 20{,}000$, $Q(n) \approx n^{46} \approx 10^{197}$, requiring substantial computational resources for primality testing. Large-scale verification up to $n = 2\times 10^9$ (15.4 million Q-primes) is reported in a companion paper.

5 Conclusion

We have rigorously computed the Bateman–Horn constant for the Titan polynomial $Q(n) = n^{47} - (n-1)^{47}$. The value $C_Q \approx 8.68$ reflects the sequence's strong bias towards primality due to the absence of small prime factors: all 60 primes below 283 contribute a factor $p/(p-1) > 1$ to the Euler product, while only the sparse set of splitting primes $p \equiv 1 \pmod{47}$ contributes factors $(1 - 46/p)/(1 - 1/p) < 1$. The resulting enhancement means that prime values of $Q(n)$ occur approximately 8.7 times more frequently than for a generic degree-46 polynomial. This quantitative analysis complements the algebraic and analytic results established in companion papers, providing a complete picture of the prime distribution of this cyclotomic norm form.

Data availability. The LaTeX source, verification scripts, and supplementary data are available at:

`https://github.com/Ruqing1963/Q47-BatemanHorn-Constant`

References

[1] R. Chen, *Prime Values of a Cyclotomic Norm Polynomial and a Conjectural Bounded Gap Phenomenon*, Preprint (2026), `https://zenodo.org/records/18521551`.

[2] P. T. Bateman and R. A. Horn, *A heuristic asymptotic formula concerning the distribution of prime numbers*, Math. Comp. **16** (1962), 363–367.