Exponential Sums of the Titan Polynomial: Numerical Evidence for Non-Generic Monodromy

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Abstract

We present a numerical study of the exponential sums $S_p = \sum_{n=0}^{p-1} e^{2\pi i Q(n)/p}$ associated with the Titan polynomial $Q(n) = n^{47} - (n-1)^{47}$, a degree-46 cyclotomic norm form satisfying the palindromic symmetry $Q(n) = Q(1-n)$. The Weil bound gives $|S_p| \leq 45\sqrt{p}$, and the sum decomposes into 45 Frobenius eigenvalues. We restrict attention to the 111 primes $p \equiv 1 \pmod{47}$ up to 50,000, for which the polynomial retains its full root structure modulo p. The observed mean normalized magnitude is $\overline{|S_p/\sqrt{p}|} \approx 1.50$ and the maximum is ≈ 8.65 (at $p = 283$). We compare these statistics with three reference models: a Gaussian random walk of 45 unit vectors (expected mean ≈ 5.97), the compact symplectic group $USp(44)$ (expected mean ≈ 3.74), and the Sato–Tate distribution of $SU(2)$ (applicable to elliptic curves, included as a baseline). The observed mean lies significantly below all higher-dimensional predictions, suggesting that the geometric monodromy group is a proper subgroup of $USp(44)$, constrained by the cyclotomic origin of $Q(n)$.

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1 Introduction

For a polynomial $f(x)$ of degree d over $\mathbb{F}_p$, the additive exponential sum

$$S_p(f) = \sum_{n=0}^{p-1} \exp\left(\frac{2\pi i f(n)}{p}\right)$$

encodes deep arithmetic information. By the Weil–Deligne theorem [2], if $\gcd(d, p) = 1$, then

$$|S_p(f)| \leq (d-1)\sqrt{p}.$$

The sum decomposes as $S_p = -1 + \sum_{j=1}^{d-1} \alpha_j(p)$, where the Frobenius eigenvalues satisfy $|\alpha_j(p)| = \sqrt{p}$.

The Katz–Sarnak philosophy [1] predicts that as $p \to \infty$, the conjugacy class of normalized Frobenius $(\alpha_1/\sqrt{p}, \ldots, \alpha_{d-1}/\sqrt{p})$ becomes equidistributed in a compact group G (the *geometric monodromy group*) according to Haar measure. The distribution of $S_p/\sqrt{p}$ is then determined by the trace distribution of G.

For the Titan polynomial $Q(n) = n^{47} - (n-1)^{47}$ of degree $d = 46$, we have $d - 1 = 45$ Frobenius eigenvalues. A key structural feature is the *palindromic symmetry* $Q(n) = Q(1-n)$, which implies that the exponential sum satisfies $S_p = \overline{S_p}$ when -1 is a 47-th power modulo p. More precisely, by a theorem of Katz [1], the geometric monodromy group of the exponential sum family of a polynomial f of degree d is generically $Sp(d-2)$ when f has odd degree, or lies in $O(d-1)$ when f has even degree and certain additional symmetries hold. Since $\deg(Q) = 46$ is even but Q arises from the 47th cyclotomic polynomial Φ_{47} (of odd prime order), the natural functional equation symmetry is symplectic: the monodromy group is expected to lie in $USp(44)$ rather than $O(45)$ or $SU(45)$. This paper presents numerical evidence characterizing the distribution of $S_p/\sqrt{p}$ and its implications for the monodromy group.

2 Setup and Effective Primes

2.1 The Exponential Sum

We study the normalized variable

$$x_p = \frac{S_p}{\sqrt{p}}, \quad \text{where} \quad S_p = \sum_{n=0}^{p-1} \exp\left(\frac{2\pi i\, Q(n)}{p}\right).$$

The Weil bound gives $|x_p| \le 45$.

2.2 Restriction to $p \equiv 1 \pmod{47}$

For primes $p \not\equiv 1 \pmod{47}$ (with $p \neq 47$), the map $n \mapsto n^{47}$ is a bijection on $\mathbb{F}_p$ (since $\gcd(47, p-1) = 1$). While the sum S_p is *not* trivial in this case, the bijection creates additional symmetries in the Frobenius eigenvalue structure that complicate the equidistribution analysis.

For $p \equiv 1 \pmod{47}$, the 47th-power map has a non-trivial kernel of order 47, and $Q(n)$ has $\omega_Q(p) = 46$ roots modulo p [4]. All 45 Frobenius eigenvalues are "fully active" without degeneracy imposed by the power map.

We therefore restrict to

$$\mathcal{P}_{\text{eff}} = \{p \le 50{,}000 : p \equiv 1 \pmod{47}\},$$

which contains $N = 111$ primes.

3 Experimental Results

3.1 Statistical Summary

Table 1 reports the observed statistics of $|x_p| = |S_p|/\sqrt{p}$ and compares them with three theoretical reference distributions.

Here:

- **Gaussian Random Walk**: The modulus of a sum of 45 independent random unit vectors in $\mathbb{C}$. Each vector has expected modulus $\sqrt{\pi/2} \approx 0.89$; the total sum has expected modulus $\sqrt{\pi/2} \times \sqrt{45} \approx 5.97$. This models maximal randomness with no correlations among the Frobenius eigenvalues.

Table 1: Observed statistics of $|S_p|/\sqrt{p}$ compared with theoretical models for 45 Frobenius eigenvalues.

Metric	**Observed**	Gaussian RW	$USp(44)$	$SU(2)^\dagger$
Mean μ	**1.50**	≈ 5.97	≈ 3.74	0.85
Max	**8.65**	(unbounded)	45.0	2.0
Weil bound	45.0	∞	45.0	2.0

$\dagger$ $SU(2)$ applies to elliptic curves (rank 1, degree 3); included as a pedagogical baseline only.

- $\boldsymbol{USp(44)}$: The expected trace distribution if the monodromy group were the full compact symplectic group of rank 22. By the Central Limit Theorem for high-rank groups, $E[|\mathrm{Tr}(U)|] \approx \sqrt{44/\pi} \approx 3.74$. The palindromic symmetry $Q(n) = Q(1-n)$ provides the structural reason to expect symplectic rather than unitary monodromy.
- $\boldsymbol{SU(2)}$: The Sato–Tate distribution [3] with $E[|a_p|/\sqrt{p}] = 8/(3\pi) \approx 0.85$ and strict bound $|a_p|/\sqrt{p} \leq 2$. This applies to elliptic curves (degree 3, single Frobenius eigenvalue pair) and is included only for reference.

Observation 3.1 (Sub-Generic Mean)**.** *The observed mean* $\mu \approx 1.50$ *lies well below the* $USp(44)$ *prediction of* ≈ 3.74*. This suggests that the geometric monodromy group of* $Q(n)$ *is a* proper subgroup *of* $USp(44)$*, whose trace distribution is more concentrated near the origin.*

Observation 3.2 (The $p = 283$ Outlier)**.** *The maximum* $|S_p|/\sqrt{p} \approx 8.65$ *occurs at* $p = 283$*, the smallest prime in* $\mathcal{P}_{\text{eff}}$*. At this prime,* $Q(n)$ *takes only 19 distinct values modulo 283 (out of 283 possible), with the value 0 occurring 46 times. This extreme concentration of values produces strong constructive interference. Excluding* $p = 283$*, the maximum drops to* ≈ 5.97 *(at* $p = 6581$*) and the mean decreases to* ≈ 1.44*.*

3.2 Distribution of $|S_p|/\sqrt{p}$

Figure 1 shows the four-panel distribution of $S_p/\sqrt{p}$ in the complex plane.

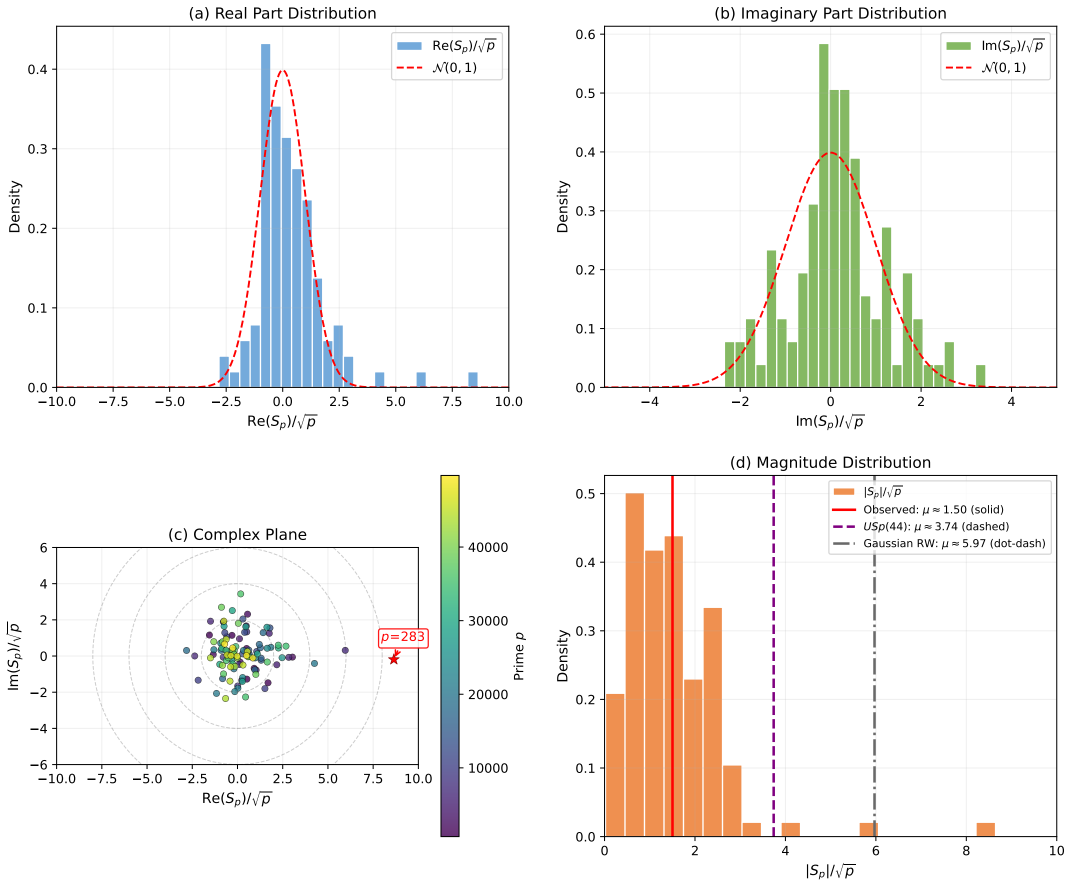

Figure 1: Distribution of normalized exponential sums $S_p/\sqrt{p}$ for the 111 primes $p \equiv 1 \pmod{47}$, $p \le 50{,}000$. **(a)** Histogram of $\mathrm{Re}(S_p)/\sqrt{p}$ with standard Gaussian overlay (red dashed). **(b)** Histogram of $\mathrm{Im}(S_p)/\sqrt{p}$ with Gaussian overlay. **(c)** Scatter plot in the complex plane, color-coded by prime size; dashed circles at $|S_p|/\sqrt{p} = 2, 4, 6, 8$; the $p = 283$ outlier is labelled. **(d)** Histogram of $|S_p|/\sqrt{p}$ with vertical lines marking the observed mean (≈ 1.50, solid), $USp(44)$ prediction (≈ 3.74, dashed), and Gaussian random walk prediction (≈ 5.97, dot-dash). The concentration of the distribution well below both higher-dimensional predictions is clearly visible. (Note: with $N = 111$ data points, the histogram shape should not be over-interpreted; the vertical line positions—i.e. the mean—are the robust quantities.)

4 Discussion

4.1 Monodromy Group Constraints

For a "generic" polynomial of degree 46, Katz's theorem [1] predicts that the geometric monodromy group is either $Sp(44)$ or $SU(45)$, depending on the functional equation symmetry. The palindromic property $Q(n) = Q(1-n)$ selects the symplectic case, so $USp(44)$ is the natural generic baseline for Q.

The observed mean of ≈ 1.50 is significantly below both the $USp(44)$ prediction (≈ 3.74) and the Gaussian random walk prediction (≈ 5.97 for 45 independent eigenvalues). This indicates a degree of cancellation among the Frobenius eigenvalues far exceeding what either model produces.

A natural explanation is that the associated algebraic variety undergoes a *Jacobian de-*

composition: because $Q(n) = \Phi_{47}(n/(n-1))$ is built from the 47th cyclotomic polynomial, the Jacobian of the associated curve may split into lower-dimensional abelian varieties with complex multiplication (CM) by $\mathbb{Q}(\zeta_{47})$. If the Jacobian decomposes into k independent sub-varieties, the exponential sum becomes a superposition of k lower-rank traces, each with smaller variance, reducing the overall mean magnitude.

Conjecture 4.1 (Jacobian Splitting)**.** *The geometric monodromy group of the exponential sum family $\{S_p(Q)\}$ is a proper subgroup of $USp(44)$, arising from a nontrivial decomposition of the associated Jacobian variety into CM abelian sub-varieties indexed by the characters of $(\mathbb{Z}/47\mathbb{Z})^\times$.*

4.2 Relation to the Shielding Property

The shielding property ($\omega_Q(p) = 0$ for $p < 283$) established in [4] is a statement about the *zeros* of $Q(n) \pmod p$. The exponential sum S_p encodes all values of $Q(n) \pmod p$, not just the zeros. Nevertheless, the two phenomena share a common origin: the factorization $Q(n) = \Phi_{47}(n/(n-1))$ (in projective terms) ties the arithmetic of Q to the 47th cyclotomic field, imposing constraints on both the root structure and the Frobenius eigenvalues.

5 Conclusion

Numerical computation of exponential sums for the Titan polynomial reveals a distribution that is neither Sato–Tate ($SU(2)$) nor generic symplectic ($USp(44)$). The observed mean $\overline{|S_p|/\sqrt{p}} \approx 1.50$ lies well below the $USp(44)$ prediction (≈ 3.74) and the uncorrelated random walk prediction (≈ 5.97), suggesting a monodromy group that is a proper, arithmetically constrained subgroup of the expected symplectic group.

We note that the sample size ($N = 111$ primes) limits the precision of distributional shape analysis; the mean and maximum statistics are more robust. Extending the computation to larger prime ranges, and determining the precise monodromy group from the cyclotomic structure of $Q(n)$, are natural directions for future work.

Data availability. The LaTeX source, computed data, and SageMath/Python scripts are available at:

`https://github.com/Ruqing1963/Q47-ExponentialSums`

References

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