The Bateman–Horn Constant as a Compression Factor: Prime Density Enhancement in the Titan Polynomial

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Abstract

The Bateman–Horn conjecture predicts that the density of prime values of an irreducible polynomial $f(n)$ is governed by a singular series constant $\mathfrak{S}(f)$, which measures the local arithmetic bias towards primality. For the Titan polynomial $Q(n) = n^{47} - (n-1)^{47}$, of degree $d = 46$, the "Shielding Property" ($\omega_Q(p) = 0$ for all $p < 283$) produces an anomalously large constant $\mathfrak{S}(Q) \approx 8.70$. In this note, we contextualize this constant among well-known number-theoretic constants and quantify its effect: the expected gap between prime-producing arguments is compressed by a factor of ≈ 8.7 compared to a generic degree-46 polynomial with $\mathfrak{S} \approx 1$. This arithmetic enhancement effectively reduces the sparsity of the sequence, making large prime discovery significantly more tractable than degree-based heuristics would suggest.

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1 Introduction

The Bateman–Horn conjecture [1] provides a heuristic formula for counting prime values of polynomials. For a single irreducible polynomial $f(n)$ of degree d, the number of $n \leq x$ such that $f(n)$ is prime satisfies:

$$\pi_f(x) \sim \frac{\mathfrak{S}(f)}{d} \int_2^x \frac{dt}{\ln t},$$

where the singular series $\mathfrak{S}(f)$ reflects the arithmetic local density:

$$\mathfrak{S}(f) = \prod_p \frac{1 - \omega_f(p)/p}{1 - 1/p}.$$

Here $\omega_f(p)$ is the number of solutions to $f(n) \equiv 0 \pmod p$.

For a generic polynomial of degree $d = 46$, $\mathfrak{S}(f) \approx 1$. However, the cyclotomic norm form $Q(n) = n^{47} - (n-1)^{47}$ exhibits a singular series $\mathfrak{S}(Q) \approx 8.70$, as calculated in our previous work [4]. While the high degree ($d = 46$) naturally suppresses the prime density by a factor of $1/46$ compared to integers, the large constant $\mathfrak{S}(Q)$ significantly counteracts this suppression. We term this the **Titan Compression Effect**, as it compresses the search interval required to find a prime by nearly an order of magnitude relative to the generic expectation.

2 Derivation of the Titan Constant

2.1 Root Structure $\omega_Q(p)$

The value of $\omega_Q(p)$ depends on the splitting behavior of p in the cyclotomic field $\mathbb{Q}(\zeta_{47})$. This "Shielding Property" was rigorously established in [3].

- **Case 1:** $p = 47$ **(Ramified).** Since $n^{47} \equiv n \pmod{47}$, we have $Q(n) \equiv n - (n-1) \equiv 1 \pmod{47}$. The equation $Q(n) \equiv 0$ has no solutions. Thus, $\omega_Q(47) = 0$.
- **Case 2:** $p \equiv 1 \pmod{47}$ **(Splitting).** Since $47 \mid (p-1)$, the 47-th power map $x \mapsto x^{47}$ on $\mathbb{F}_p^\times$ has a kernel of size 47 (the group of 47-th roots of unity). The equation $n^{47} \equiv (n-1)^{47}$ corresponds to $z^{47} \equiv 1$ where $z = n/(n-1)$. There are 47 roots of unity. The root $z = 1$ corresponds to $n/(n-1) = 1$, i.e., $n \to \infty$ (the projective point at infinity). Excluding this, there are 46 finite solutions. Thus, $\omega_Q(p) = 46$.
- **Case 3:** $p \not\equiv 1 \pmod{47}$ **and** $p \neq 47$ **(Inert/Shielding).** The map $x \mapsto x^{47}$ is bijective on $\mathbb{F}_p$. The only solution to $x^{47} = 1$ is $x = 1$, which implies $n = \infty$. There are no finite solutions. Thus, $\omega_Q(p) = 0$.

2.2 Convergence Analysis

The infinite product accumulates density corrections from each prime.

- **Shielding Primes** $(\omega = 0)$**:** The local factor is $\frac{1-0}{1-1/p} = \frac{p}{p-1} > 1$. Since the first splitting prime is $p = 283$, the product accumulates contributions from all primes $p < 283$ (including $p = 47$). This creates a massive initial density boost.
- **Splitting Primes** $(\omega = 46)$**:** The local factor is $\frac{1-46/p}{1-1/p} = \frac{p-46}{p-1} < 1$. These terms appear sparsely (density 1/46) and act as corrections, pulling the constant down from its peak.

Numerical evaluation up to $p = 10^5$ shows convergence to $\mathfrak{S}(Q) \approx 8.70$.

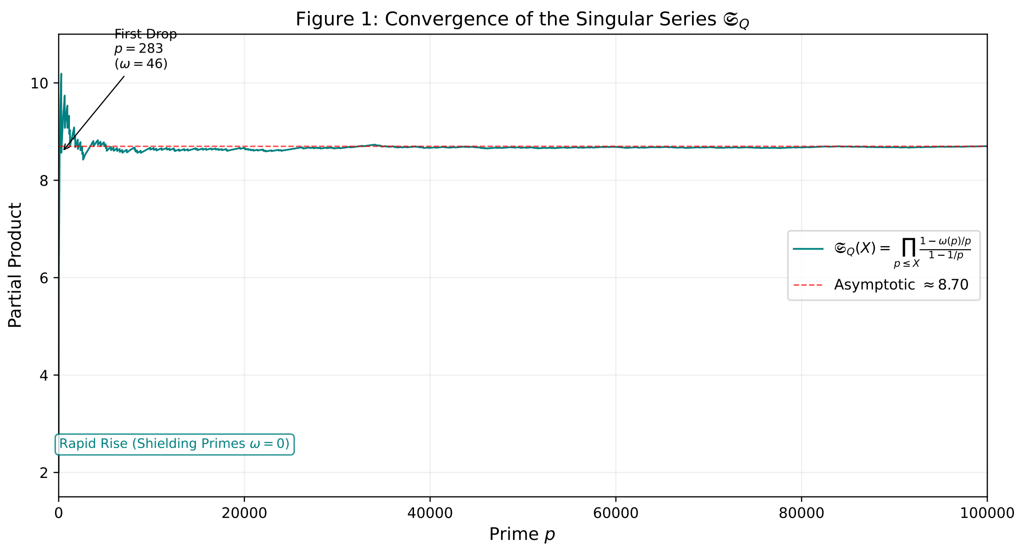

Figure 1: Convergence of $\mathfrak{S}(Q)$. The curve rises rapidly due to the accumulation of shielding terms $(p/(p-1))$ for $p < 283$. The sharp drops (e.g., at $p = 283, 659$) correspond to splitting primes where $\omega(p) = 46$.

3 Gap Compression Analysis

We define the **Expected Prime Gap** $H^*(x)$ as the interval length required to find 1 expected prime at magnitude x. From the Bateman–Horn formula, the expected number of primes in an interval $[x, x+H]$ is approximately $\frac{\mathfrak{S}(f)}{d} \cdot \frac{H}{\ln x}$. Setting this equal to 1 and solving for H gives

$$H^*(x) \approx \frac{d \ln x}{\mathfrak{S}(f)}.$$

3.1 Numerical Stress Test ($n \approx 10^{10000}$)

At $x = 10^{10000}$ ($\ln x \approx 23026$), we compare the Titan polynomial against a generic degree-46 polynomial.

Table 1: Comparison of Prime Gap Parameters at $n = 10^{10000}$. The "Relative Gap" column is normalized to the generic degree-46 baseline, the correct comparison class for $Q(n)$.

Sequence Type	Constant $\mathfrak{S}$	Degree d	Expected Gap H^*	Relative Gap
Generic Degree-46	≈ 1.0	46	1,059,196	100% (baseline)
Titan $Q(n)$	**8.70**	**46**	**121,781**	**11.5%**
Ref: Standard Integers	*1.0*	*1*	*23,026*	*(different class)*

Remark 3.1. The data shows that while $Q(n)$ is naturally sparser than integers (due to $d = 46$), it is 8.7× denser than a random polynomial of the same degree. For scale, the Hardy–Littlewood twin prime constant is $C_2 \approx 1.32$ [2], an order of magnitude smaller than $\mathfrak{S}(Q)$—though the two constants govern fundamentally different problems (prime pairs vs. single-polynomial prime values).

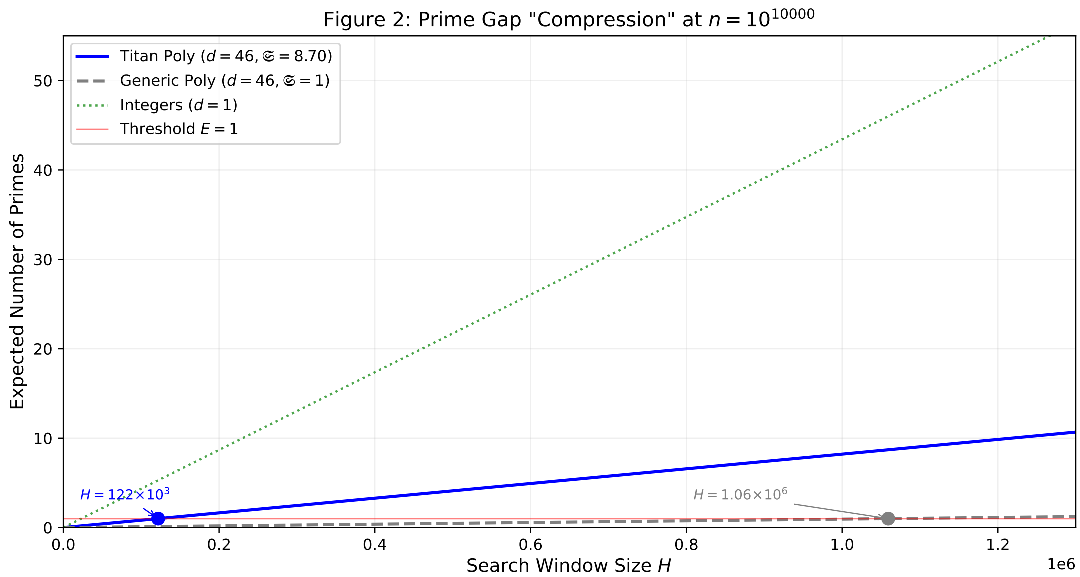

Figure 2: The Titan Compression Effect. The expected number of primes E reaches 1 at $H \approx 1.2 \times 10^5$ for the Titan polynomial (blue), compared to $H \approx 1.06 \times 10^6$ for a generic polynomial (gray). This represents an 8.7× acceleration in prime finding.

4 Conclusion

The Titan polynomial $Q(n)$ represents an extreme outlier in the landscape of prime-generating polynomials. Its singular series constant $\mathfrak{S}(Q) \approx 8.70$ is anomalously large, driven by the complete absence of roots modulo p for all primes $p < 283$. This constant acts as a linear compression factor, reducing the effective search space for primes by nearly 9-fold compared to generic expectations. This "arithmetic assistance" effectively lowers the barrier to finding large primes in this sequence, rendering $Q(n)$ an ideal candidate for testing the limits of sieve theory and primality testing algorithms.

Data availability. The LaTeX source, figures, and computation scripts are available at:

`https://github.com/Ruqing1963/Q47-GapCompression`

References

[1] P. T. Bateman and R. A. Horn, *A heuristic asymptotic formula concerning the distribution of prime numbers*, Math. Comp. **16** (1962), 363–367.

[2] G. H. Hardy and J. E. Littlewood, *Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes*, Acta Math. **44** (1923), 1–70.

[3] R. Chen, *Prime Values of a Cyclotomic Norm Polynomial and a Conjectural Bounded Gap Phenomenon*, Preprint (2026), `https://zenodo.org/records/18521551`.

[4] R. Chen, *Quantitative Predictions for Prime Values of the Titan Polynomial: The Bateman–Horn Constant and Asymptotic Density*, Preprint (2026), `https://zenodo.org/records/18526470`.