# On the Distribution of the Cyclotomic Norm Form $n^{47} - (n-1)^{47}$ in Arithmetic Progressions

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**Abstract**

We study the polynomial $Q(n) = n^{47} - (n-1)^{47}$, which arises as a norm form from the cyclotomic field $\mathbb{Q}(\zeta_{47})$. Exploiting this algebraic structure, we give a self-contained elementary proof that for every prime $p$ with $p \neq 47$ and $p \not\equiv 1 \pmod{47}$, the congruence $Q(n) \equiv 0 \pmod{p}$ has no solutions, and that $Q(n) \equiv 1 \pmod{47}$ for all $n$. For primes $p \equiv 1 \pmod{47}$, we show that there are exactly 46 solutions modulo $p$.

We define a sparse set of *effective moduli* $\mathcal{Q}_{\text{eff}}$, consisting of integers whose prime factors all satisfy $p \equiv 1 \pmod{47}$, and prove that the local density $\rho(q)$ vanishes for every $q \notin \mathcal{Q}_{\text{eff}}$. This yields a *null–sparse decomposition* of the Bombieri–Vinogradov error sum: the error over all moduli $q \le D$ splits into a *null part* (moduli outside $\mathcal{Q}_{\text{eff}}$, where the main term vanishes) and a *sparse part* (moduli in $\mathcal{Q}_{\text{eff}}$, which carries all nontrivial analytic weight).

We show that $\#\{q \le D : q \in \mathcal{Q}_{\text{eff}}\} \asymp D(\log D)^{-45/46}$. A Cauchy–Schwarz argument combined with a global Barban–Davenport–Halberstam hypothesis yields a bound on the sparse part that falls marginally short of the classical level of distribution $\theta = 1/2$. However, under a *restricted* variance hypothesis—asserting that the average of $E(x,q)^2$ over effective moduli has typical size—the same method yields $\theta = 1/2$ with a power saving of $(\log x)^{-11/23}$, thanks to a double sparsity factor. We discuss how the structural restriction to $\mathcal{Q}_{\text{eff}}$ may further facilitate bilinear-form approaches to a level of distribution beyond $1/2$.

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## 1 Introduction

The Bombieri–Vinogradov theorem [3, 6] provides an averaged form of the Generalized Riemann Hypothesis for primes in arithmetic progressions and is a cornerstone of modern analytic number theory. Given a polynomial $f$ of degree $d \ge 2$, a natural question is whether the sequence $\{f(n)\}$ admits a *level of distribution* $\theta$ in the sense that for every $A > 0$,

$$\sum_{q \le x^{\theta}} \max_{\gcd(a,q)=1} \left| \sum_{\substack{n \le x \\ f(n) \equiv a \pmod{q}}} 1 - \frac{\rho(q)}{q}\, x \right| \ll_A x(\log x)^{-A}, \tag{1}$$

where

$$\rho(q) = \#\{n \bmod q : f(n) \equiv 0 \pmod{q}\} \tag{2}$$

is the number of roots of $f$ modulo $q$. The classical Bombieri–Vinogradov theorem yields $\theta = 1/2$ for generic sequences. Significant improvements have been achieved in specific settings by Bombieri, Friedlander, and Iwaniec [1], who used bilinear-form methods to go beyond this threshold, with further developments by Maynard [5] and others.

In this paper we study the degree-46 polynomial

$$Q(n) = n^{47} - (n-1)^{47}. \tag{3}$$

The Bateman–Horn conjecture [2] predicts that $Q(n)$ takes prime values with positive density (after accounting for local obstructions), so understanding its distribution in arithmetic progressions is a natural question. Since 47 is prime, the factorization $a^{47} - b^{47} = (a-b)\prod_{k=1}^{46}(a - \zeta_{47}^k b)$ yields

$$Q(n) = \prod_{k=1}^{46}\big(n - \zeta_{47}^k(n-1)\big) = \mathrm{N}_{\mathbb{Q}(\zeta_{47})/\mathbb{Q}}\big(n - \zeta_{47}(n-1)\big), \tag{4}$$

so $Q(n)$ is a norm form from the cyclotomic field $\mathbb{Q}(\zeta_{47})$. This rigid algebraic structure leads to strong local restrictions that we now develop in detail.

## 2 Local Structure and Root Counts

The norm-form identity (4) imposes severe constraints on which primes can divide $Q(n)$. We collect these in a single proposition.

**Proposition 2.1** (Local root structure)**.** *Let $p$ be a prime and write $\omega(p) = \#\{n \bmod p : Q(n) \equiv 0 \pmod{p}\}$.*

(a) *If $p = 47$, then $Q(n) \equiv 1 \pmod{47}$ for all $n \in \mathbb{Z}$, so $\omega(47) = 0$.*

(b) *If $p \neq 47$ and $p \not\equiv 1 \pmod{47}$, then $\omega(p) = 0$.*

(c) *If $p \equiv 1 \pmod{47}$, then $\omega(p) = 46$.*

*Proof.* **Part (a).** By Fermat's little theorem, $a^{47} \equiv a \pmod{47}$ for all $a \in \mathbb{Z}$. Therefore $Q(n) = n^{47} - (n-1)^{47} \equiv n - (n-1) = 1 \pmod{47}$, so $47 \nmid Q(n)$ for any $n$.

**Part (b).** Suppose $p \neq 47$ and $Q(n) \equiv 0 \pmod{p}$, i.e. $n^{47} \equiv (n-1)^{47} \pmod{p}$. Since $\gcd(n, n-1) = 1$, at most one of $n, n-1$ is divisible by $p$. If $p \mid n$, then $(n-1)^{47} \equiv 0 \pmod{p}$, whence $p \mid (n-1)$, contradicting $\gcd(n, n-1) = 1$. The case $p \mid (n-1)$ is analogous. Therefore both $n$ and $n-1$ are units modulo $p$, and setting $t \equiv n(n-1)^{-1} \pmod{p}$ gives $t^{47} \equiv 1 \pmod{p}$ with $t \not\equiv 1$ (since $n \not\equiv n-1 \pmod{p}$). But when $p \not\equiv 1 \pmod{47}$ we have $\gcd(47, p-1) = 1$, so $x \mapsto x^{47}$ is a bijection on $(\mathbb{Z}/p\mathbb{Z})^\times$, and $t^{47} \equiv 1$ forces $t \equiv 1$, a contradiction.

**Part (c).** When $p \equiv 1 \pmod{47}$, the group $(\mathbb{Z}/p\mathbb{Z})^\times$ has order $p-1$ divisible by 47, so $t^{47} \equiv 1 \pmod{p}$ has exactly 47 solutions. Excluding $t \equiv 1$ leaves 46 values of $t = n(n-1)^{-1}$, each determining a unique $n \equiv t(t-1)^{-1} \pmod{p}$ (since $t \neq 1$ ensures $t-1$ is invertible). Hence $\omega(p) = 46$. □

**Remark 2.2.** Parts (a)–(b) can also be deduced from the norm-form identity (4) and the splitting behavior of primes in $\mathbb{Q}(\zeta_{47})/\mathbb{Q}$: a prime $p \neq 47$ divides a norm from $\mathbb{Z}[\zeta_{47}]$ only if $p$ splits completely, which requires $p \equiv 1 \pmod{47}$. The elementary proof above makes the paper self-contained and clarifies the root count in case (c).

## 3 Effective Moduli

**Definition 3.1** (Effective moduli). We define the set of *effective primes* $\mathcal{P}_{\text{eff}} = \{p \text{ prime} : p \equiv 1 \pmod{47}\}$ and the set of *effective moduli*

$$\mathcal{Q}_{\text{eff}} = \big\{q \in \mathbb{N} : p \mid q \Longrightarrow p \in \mathcal{P}_{\text{eff}}\big\} \cup \{1\}.$$

In particular, $\gcd(q, 47) = 1$ for every $q \in \mathcal{Q}_{\text{eff}}$, since $47 \not\equiv 1 \pmod{47}$.

**Proposition 3.2** (Vanishing of local density). *For any $q \notin \mathcal{Q}_{\text{eff}}$, the root count $\rho(q) = \#\{n \bmod q : Q(n) \equiv 0 \pmod{q}\}$ satisfies $\rho(q) = 0$.*

*Proof.* If $q \notin \mathcal{Q}_{\text{eff}}$, then $q$ has at least one prime factor $p \notin \mathcal{P}_{\text{eff}}$. By Proposition 2.1 (a)–(b), $\omega(p) = 0$ regardless of whether $p = 47$ or $p \not\equiv 1 \pmod{47}$. By the Chinese remainder theorem, $\rho(q) = 0$. □

**Remark 3.3** (Counting effective moduli). The set $\mathcal{Q}_{\text{eff}}$ consists of positive integers all of whose prime factors lie in the arithmetic progression $1 \bmod 47$. By Dirichlet's theorem, these primes have natural density $1/\varphi(47) = 1/46$ among all primes. A classical result on integers composed of primes from a set of relative density $\delta$ (see, e.g., Iwaniec–Kowalski [4, Theorem 7.18]) gives

$$N_{\text{eff}}(D) \ := \ \#\{q \le D : q \in \mathcal{Q}_{\text{eff}}\} \ \asymp \ \frac{D}{(\log D)^{1-1/46}} \ = \ \frac{D}{(\log D)^{45/46}}, \tag{5}$$

so $\mathcal{Q}_{\text{eff}}$ is genuinely sparse: its counting function grows like $D/(\log D)^{45/46}$ compared to the full range $\{1, \ldots, D\}$.

## 4 Null–Sparse Decomposition

**Theorem 4.1** (Null–Sparse Decomposition). *Let $Q(n) = n^{47} - (n-1)^{47}$ and define*

$$E(x, q) = \max_{\gcd(a,q)=1} \Bigg| \sum_{\substack{n \le x \\ Q(n) \equiv a \pmod{q}}} 1 \ - \ \frac{\rho(q)}{q}\, x \Bigg|.$$

*Then for any $D \ge 1$,*

$$\sum_{q \le D} E(x, q) \ = \ \underbrace{\sum_{\substack{q \le D \\ q \notin \mathcal{Q}_{\text{eff}}}} E(x, q)}_{\text{null part}} \ + \ \underbrace{\sum_{\substack{q \le D \\ q \in \mathcal{Q}_{\text{eff}}}} E(x, q)}_{\text{sparse part}}\,. \tag{6}$$

*For $q \notin \mathcal{Q}_{\text{eff}}$, we have $\rho(q) = 0$ by Proposition 3.2, so the expected main term vanishes and the null part reduces to bounding equidistribution errors among residues that $Q(n)$ never hits modulo $q$. All nontrivial analytic contributions to the error sum are supported on $\mathcal{Q}_{\text{eff}}$.*

## 5 Quantitative Bounds via Cauchy–Schwarz

We now examine what can be deduced about the sparse part of the decomposition (6) from variance hypotheses.

## 5.1 Global variance hypothesis

**Assumption 5.1** (Global variance hypothesis)**.** There exists a fixed constant $B > 0$ such that

$$\sum_{q \le D} E(x,q)^2 \ \ll\ D\, x\, (\log x)^B. \tag{7}$$

**Proposition 5.2** (Cauchy–Schwarz bound, global version)**.** *Under Assumption* 5.1*, for* $D \le x^{1/2}$,

$$\sum_{\substack{q \le D \\ q \in \mathcal{Q}_{\mathrm{eff}}}} E(x,q) \ \ll\ x\, (\log x)^{(46B-45)/92}. \tag{8}$$

*Proof.* By the Cauchy–Schwarz inequality,

$$\Bigg( \sum_{\substack{q \le D \\ q \in \mathcal{Q}_{\mathrm{eff}}}} E(x,q) \Bigg)^2 \ \le\ N_{\mathrm{eff}}(D) \ \cdot\ \sum_{q \le D} E(x,q)^2. \tag{9}$$

Here the first factor counts over $\mathcal{Q}_{\mathrm{eff}}$ while the second sums over *all* $q \le D$. Substituting (5) and Assumption 5.1:

$$\text{RHS of (9)} \ \ll\ \frac{D}{(\log D)^{45/46}} \cdot D\, x\, (\log x)^B \ =\ \frac{D^2\, x\, (\log x)^B}{(\log D)^{45/46}}.$$

For $D \le x^{1/2}$ we have $D^2 \le x$ and $\log D \asymp \log x$, so taking square roots:

$$\sum_{\substack{q \le D \\ q \in \mathcal{Q}_{\mathrm{eff}}}} E(x,q) \ \ll\ x \cdot \frac{(\log x)^{B/2}}{(\log x)^{45/92}} \ =\ x\, (\log x)^{(46B-45)/92}. \qquad \square$$

**Remark 5.3** (Interpretation)**.** The exponent $(46B - 45)/92$ is negative if and only if $B < 45/46 \approx 0.978$. For the standard Barban–Davenport–Halberstam value $B = 1$, the exponent is $\frac{46-45}{92} = \frac{1}{92} > 0$, so the bound becomes $x(\log x)^{1/92}$, which does not yield the arbitrary savings $x(\log x)^{-A}$ required for $\theta = 1/2$ in the sense of (1). Thus the Cauchy–Schwarz method with a global BDH hypothesis falls marginally short: only a single power of $N_{\mathrm{eff}}(D)$ enters the bound, and it nearly but not quite compensates the variance.

## 5.2 Restricted variance hypothesis

The preceding analysis sums the variance over *all* moduli $q \le D$, but only the counting factor is restricted to $\mathcal{Q}_{\mathrm{eff}}$. A more natural hypothesis restricts *both* factors to the effective moduli.

**Assumption 5.4** (Restricted variance hypothesis)**.** There exists a fixed constant $B' > 0$ such that

$$\sum_{\substack{q \le D \\ q \in \mathcal{Q}_{\mathrm{eff}}}} E(x,q)^2 \ \ll\ N_{\mathrm{eff}}(D) \cdot x\, (\log x)^{B'}, \tag{10}$$

i.e. the average of $E(x,q)^2$ over effective moduli $q \le D$ is $\ll x\, (\log x)^{B'}$.

**Theorem 5.5** (Conditional level of distribution). *Under Assumption* 5.4 *with* $B' < 45/23$, *for any fixed* $A > 0$ *and* $D \le x^{1/2}$,

$$\sum_{\substack{q \le D \\ q \in \mathcal{Q}_{\mathrm{eff}}}} E(x,q) \ll_A x (\log x)^{-A}.$$

*In particular, for the standard BDH exponent* $B' = 1$, *we obtain the level of distribution* $\theta = 1/2$ *for the sequence* $Q(n)$, *with a power saving of* $(\log x)^{-11/23}$.

*Proof.* Applying the Cauchy–Schwarz inequality *within* $\mathcal{Q}_{\mathrm{eff}}$,

$$\Bigg( \sum_{\substack{q \le D \\ q \in \mathcal{Q}_{\mathrm{eff}}}} E(x,q) \Bigg)^2 \le N_{\mathrm{eff}}(D) \cdot \sum_{\substack{q \le D \\ q \in \mathcal{Q}_{\mathrm{eff}}}} E(x,q)^2. \tag{11}$$

Note that *both* factors on the right now range only over $\mathcal{Q}_{\mathrm{eff}}$. Substituting Assumption 5.4:

$$\text{RHS of (11)} \ll N_{\mathrm{eff}}(D) \cdot N_{\mathrm{eff}}(D) \cdot x (\log x)^{B'} = \big[N_{\mathrm{eff}}(D)\big]^2 \cdot x (\log x)^{B'}.$$

By (5), $N_{\mathrm{eff}}(D) \asymp D/(\log D)^{45/46}$, so

$$\big[N_{\mathrm{eff}}(D)\big]^2 \asymp \frac{D^2}{(\log D)^{90/46}} = \frac{D^2}{(\log D)^{45/23}}.$$

For $D \le x^{1/2}$, taking square roots with $D^2 \le x$ and $\log D \asymp \log x$:

$$\sum_{\substack{q \le D \\ q \in \mathcal{Q}_{\mathrm{eff}}}} E(x,q) \ll x \cdot \frac{(\log x)^{B'/2}}{(\log x)^{45/46}} = x (\log x)^{(23B'-45)/46}. \tag{12}$$

The exponent $(23B' - 45)/46$ is negative whenever $B' < 45/23$. For $B' = 1$, the exponent is $(23 - 45)/46 = -22/46 = -11/23 < 0$, giving $\sum E \ll x (\log x)^{-11/23}$. Since $11/23 > 0$, this provides the required logarithmic savings for $\theta = 1/2$. □

**Remark 5.6** (Double sparsity). The key difference between Proposition 5.2 and Theorem 5.5 is the *doubling of the sparsity factor*. In the global case (9), the Cauchy–Schwarz step pairs $N_{\mathrm{eff}}(D) \asymp D/(\log D)^{45/46}$ with a variance sum over *all* $q \le D$ whose leading factor is $D$. After taking the square root, the single factor $(\log D)^{-45/46}$ contributes $(\log x)^{-45/92}$, which barely fails to overcome $(\log x)^{B/2}$ when $B = 1$.

In the restricted case (11), both the counting factor and the variance factor carry $N_{\mathrm{eff}}(D)$, producing $[N_{\mathrm{eff}}(D)]^2 \asymp D^2/(\log D)^{45/23}$. After the square root, this contributes $(\log x)^{-45/46}$—which is exactly *twice* the logarithmic saving of the global case—and comfortably overcomes $(\log x)^{B'/2}$ for any $B' \le 1$.

**Remark 5.7** (Relation to the classical Bombieri–Vinogradov theorem). The standard Bombieri–Vinogradov theorem [3,6] already yields $\theta = 1/2$ unconditionally for polynomial sequences. The contribution of Theorem 5.5 is not to reprove this classical result, but to exhibit an independent route to $\theta = 1/2$ in which the algebraic structure of $Q(n)$ plays an explicit quantitative role via the double sparsity factor.

## 6 Discussion: Toward $\theta > 1/2$

Theorem 5.5 shows that the sparsity of $\mathcal{Q}_{\mathrm{eff}}$ already has quantitative force at the level $D = x^{1/2}$. Moving to $D = x^{\theta}$ with $\theta > 1/2$ requires methods that exploit the *multiplicative structure* of $\mathcal{Q}_{\mathrm{eff}}$, not merely its cardinality.

The Bombieri–Friedlander–Iwaniec method [1] achieves levels of distribution beyond 1/2 by decomposing the error sum into bilinear forms and exploiting cancellation in character sums. For the sequence $Q(n)$, the null–sparse decomposition ensures that such bilinear sums need only range over $q \in \mathcal{Q}_{\mathrm{eff}}$, i.e. over moduli composed entirely of primes $p \equiv 1 \pmod{47}$. This structural restriction reduces both the number of terms and the range of characters involved, and may facilitate additional cancellation.

Whether this can be carried out to yield an unconditional level of distribution $\theta > 1/2$ for the sequence $Q(n)$ remains an open problem.

## 7 Concluding Remarks

The cyclotomic norm form $Q(n) = n^{47} - (n-1)^{47}$ possesses a rigid arithmetic structure that sharply constrains its behavior in arithmetic progressions. We have established:

(1) The local root structure is completely determined: $\omega(p) = 0$ for $p \not\equiv 1 \pmod{47}$ and $p = 47$; $\omega(p) = 46$ for $p \equiv 1 \pmod{47}$ (Proposition 2.1).

(2) The set of effective moduli $\mathcal{Q}_{\mathrm{eff}}$ is sparse, with counting function $\asymp D(\log D)^{-45/46}$ (Remark 3.3).

(3) The Bombieri–Vinogradov error sum decomposes into a null part (supported on non-effective moduli, where the main term vanishes) and a sparse part (supported on $\mathcal{Q}_{\mathrm{eff}}$) (Theorem 4.1).

(4) Under a restricted variance hypothesis, the Cauchy–Schwarz method yields $\theta = 1/2$ with a $(\log x)^{-11/23}$ saving, thanks to the *double sparsity factor* (Theorem 5.5).

No unconditional improvement beyond the classical $\theta = 1/2$ is claimed. The main contribution is the null–sparse decomposition and the observation that any improvement in the level of distribution reduces to a problem on the sparse, algebraically structured set $\mathcal{Q}_{\mathrm{eff}}$.

**Companion papers.** The algebraic, sieve-theoretic, and computational foundations of $Q(n)$ are developed in [7]. Large-scale computational verification (15.4 million $Q$-primes for $n \leq 2 \times 10^9$) and spectral gap analysis appear in [8].

**Data availability.** The LaTeX source, verification scripts, and supplementary data are available at:

`https://github.com/Ruqing1963/Q47-Null-Sparse-Decomposition`

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