Radical Compression in Maximal Prime Gaps: Asymptotics and the Transient Stability Illusion

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Abstract

For a prime gap of length g starting at a prime p, we define the *radical compression ratio* $q = \log(\prod C_i)/\log \operatorname{rad}(\prod C_i)$, where the product ranges over the $g-1$ consecutive composites in the gap. This ratio measures the extent to which composites in a prime gap share prime factors. We prove, using Legendre's formula and Mertens' estimates, that

$$q \;=\; 1 + \frac{\log g}{\log X} + O\left(\frac{1}{\log X}\right),$$

where X is the starting prime. In the Cramér regime $g \sim c\log^2 X$, this yields $q = 1 + \frac{2\log\log X}{\log X} + O(1/\log X) \to 1$ as $X \to \infty$. Thus, no finite "stabilization" of q occurs; the apparent stability near $q \approx 1.30$ observed for $X \leq 10^7$ is a transient artifact of the extremely slow decay of $\log\log X/\log X$. We verify this asymptotic formula against all 25 maximal prime gaps up to 10^8, finding agreement to within 3% for $g \geq 36$. The radical compression ratio provides a clean multiplicative perspective on prime gaps that complements the classical additive/probabilistic approach of Cramér.

1 Introduction

1.1 Motivation

Cramér's conjecture [4] asserts that $g_n = O((\log p_n)^2)$, where $g_n = p_{n+1} - p_n$ is the n-th prime gap. The conjecture is supported by a probabilistic model in which each integer m is "prime" independently with probability $1/\log m$. This model is purely additive: it treats primality as a coin toss and ignores the multiplicative anatomy of the composites that fill a prime gap.

In this paper, we take the complementary multiplicative viewpoint. We ask: *how much do the composites in a prime gap share their prime factors?* We quantify this via the *radical compression ratio* q, which compares the log-product of the composites to the log-radical of their collective product.

Our main result is that q admits an explicit asymptotic formula in terms of g and X (Theorem 3.1), from which $q \to 1$ as $X \to \infty$ in the Cramér regime. This resolves the initial impression from limited numerical data that q "stabilizes" around 1.3—an illusion created by the notoriously slow decay of the iterated logarithm $\log\log X/\log X$.

1.2 Context within the Titan Project

In companion papers, we proved algebraic structure theorems for Legendre and Oppermann intervals: the Spin Asymmetry Theorem [1] (global quadratic residue skewness = 1) and the Oppermann Parity Law [2] (zero right-half skewness for even n). Those results concern the additive/algebraic axis (residue classes modulo $p = 2n - 1$). The present paper addresses the multiplicative axis (radical of consecutive composites), completing a two-sided view of the arithmetic structure of prime gaps.

2 Definitions

Definition 2.1 (Radical compression ratio)**.** For a prime gap of length g starting at a prime p, the *composite sequence* is $\mathcal{C} = \{p+1, p+2, \ldots, p+g-1\}$. The *radical compression ratio* is

$$q(p,g) \;=\; \frac{N}{D} \;=\; \frac{\displaystyle\sum_{c\in\mathcal{C}} \log c}{\displaystyle\sum_{\ell\in\mathcal{P}(\mathcal{C})} \log \ell}, \tag{1}$$

where $\mathcal{P}(\mathcal{C}) = \{\ell \text{ prime} : \ell \mid c \text{ for some } c \in \mathcal{C}\}$ is the set of all distinct prime factors of the sequence.

Remark 2.2 (Relation to the ABC quality). For a single ABC triple (a,b,c) with $a+b=c$, the *quality* is $q(a,b,c) = \log c/\log \operatorname{rad}(abc)$. Our ratio $q(p,g)$ is analogous but applied to a *product* of a sequence. It is not an ABC quality, and we make no claims involving the ABC conjecture.

3 Asymptotic Analysis

Theorem 3.1 (Radical compression asymptotics)**.** *Let $\mathcal{C}$ be a sequence of $g-1$ consecutive integers near X (all elements in $[X, X+g]$), with $g \geq 2$ and $X \geq g^2$. Then*

$$q \;=\; 1 + \frac{\log g}{\log X} + O\!\left(\frac{1}{\log X}\right). \tag{2}$$

In the Cramér regime $g \sim c(\log X)^2$:

$$q \;=\; 1 + \frac{2\log\log X + \log c}{\log X} + O\!\left(\frac{1}{\log X}\right) \xrightarrow{X\to\infty} 1. \tag{3}$$

Proof. We compute the numerator N and the excess $N - D$.

Numerator. $N = \sum_{c\in\mathcal{C}} \log c = (g-1)\log X + O(g)$, since $\log(X+j) = \log X + O(g/X)$ for $0 \leq j \leq g$.

Excess via Legendre's formula. The excess $N - D$ equals $\sum_{\ell\mid\prod\mathcal{C}} (v_\ell(\prod\mathcal{C}) - 1)\log \ell$, where v_ℓ denotes the ℓ-adic valuation.

Small primes ($\ell \leq g$): By Legendre's formula, $v_\ell(\prod\mathcal{C}) = \sum_{k\geq 1} \lfloor (g-1)/\ell^k \rfloor = (g-1)/(\ell-1) + O(\log_\ell g)$. Thus the small-prime contribution to $N - D$ is

$$\begin{aligned}\sum_{\ell\leq g}\left(\frac{g-1}{\ell-1} - 1\right)\log \ell &= (g-1)\sum_{\ell\leq g}\frac{\log \ell}{\ell-1} - \theta(g) + O(g)\\ &= (g-1)(\log g + O(1)) - (g + o(g)) + O(g)\\ &= g\log g + O(g),\end{aligned}$$

using Mertens' estimate $\sum_{\ell \le x} \log \ell/(\ell-1) = \log x + O(1)$ and Chebyshev's $\theta(g) = g + o(g)$.

Large primes ($\ell > g$): Each such prime divides at most one element of $\mathcal{C}$, so the excess is nonzero only from prime powers $\ell^k \mid c$ with $k \ge 2$. This contributes $O(g)$ in total.

Combining. $N - D = g \log g + O(g)$, so $D = g \log X - g \log g + O(g) = g(\log X - \log g) + O(g)$. Therefore

$$q = \frac{N}{D} = \frac{g \log X + O(g)}{g(\log X - \log g) + O(g)} = \frac{\log X}{\log X - \log g + O(1)} = 1 + \frac{\log g}{\log X} + O\left(\frac{1}{\log X}\right).$$

Substituting $g = c \log^2 X$ gives (3). □

Corollary 3.2. *If the Cramér conjecture holds, then the radical compression of maximal prime gaps satisfies* $\lim_{n\to\infty} q = 1$.

Remark 3.3 (Rate of decay). The decay $2 \log\log X/\log X$ is notoriously slow:

X	$1 + 2\log\log X/\log X$
10^6	1.380
10^{12}	1.240
10^{50}	1.082
10^{100}	1.047

The passage from $q \approx 1.30$ to $q \approx 1.05$ requires X to increase from 10^7 to 10^{100}. This explains why limited numerical experiments create an illusion of "stabilization."

4 Computational Verification

4.1 Methodology

We enumerated all primes up to 10^8 via sieve, identified all 25 maximal (record-breaking) prime gaps, and computed q by fully factoring every composite.

4.2 Data

Table 1: Maximal prime gaps with $g \geq 20$ up to 10^8. q_{obs} = observed; $q_{\text{pred}} = 1 + \log g / \log P_n$.

P_n	g	ρ	q_{obs}	q_{pred}	Error
887	20	0.434	1.421	1.441	−1.4%
1 129	22	0.445	1.376	1.440	−4.5%
1 327	34	0.658	1.541	1.490	+3.4%
9 551	36	0.429	1.371	1.391	−1.5%
15 683	44	0.472	1.382	1.392	−0.7%
19 609	52	0.532	1.374	1.400	−1.8%
31 397	72	0.672	1.424	1.413	+0.8%
155 921	86	0.602	1.377	1.373	+0.3%
360 653	96	0.586	1.346	1.357	−0.8%
370 261	112	0.681	1.371	1.368	+0.2%
492 113	114	0.664	1.343	1.361	−1.4%
1 349 533	118	0.592	1.333	1.338	−0.4%
1 357 201	132	0.662	1.336	1.346	−0.7%
2 010 733	148	0.703	1.339	1.344	−0.4%
4 652 353	154	0.653	1.304	1.328	−1.8%
17 051 707	180	0.649	1.294	1.312	−1.3%
20 831 323	210	0.740	1.306	1.317	−0.8%
47 326 693	220	0.704	1.294	1.305	−0.8%

Observation 4.1 (Agreement with asymptotics)**.** For all $g \geq 36$ (15 data points), the prediction $q_{\text{pred}} = 1 + \log g / \log X$ agrees with q_{obs} to within 3.5%. The systematic negative bias is consistent with the $O(1/\log X)$ correction term.

Observation 4.2 (Steady decline)**.** q_{obs} decreases from 1.54 at $P_n = 1327$ to 1.29 at $P_n =$ 47,326,693, consistent with the proven asymptotic decay to 1. The apparent plateau near 1.30 for $X \in [10^6, 10^7]$ is a transient artifact.

5 Discussion

5.1 The decompression mechanism

The formula $q = 1 + \log g / \log X + O(1/\log X)$ has a clean interpretation. The excess $N - D \approx g \log g$ is generated by *small* primes ($\ell \leq g$), each dividing multiple composites in the gap. But the denominator D is dominated by *large* prime factors ($\ell > g$), each unique to a single composite and contributing $\sim \frac{1}{2} \log X$ to D. As X grows, these large primes increasingly dilute the small-prime compression, driving $q \to 1$.

5.2 Relation to the ABC conjecture

The ABC conjecture bounds the quality of individual triples $a+b=c$. The radical compression q measures the collective compression of a *sequence*. The boundedness of q follows from elementary analysis (Theorem 3.1), not from the ABC conjecture.

5.3 Open questions

Open Question 5.1 (Explicit error term)**.** Can the $O(1/\log X)$ in (2) be made explicit, e.g., $q = 1 + \log g/\log X + C/\log X + o(1/\log X)$ for an identifiable constant C?

Open Question 5.2 (Oscillation structure)**.** The residual $q_{\mathrm{obs}} - q_{\mathrm{pred}}$ oscillates between -4.5% and $+3.4\%$. Is there a secondary term capturing these fluctuations?

Open Question 5.3 (Non-record gaps)**.** The proof of Theorem 3.1 applies to any $g - 1$ consecutive integers, not just composites. For typical (non-record) prime gaps, how does q behave?

Data and code availability

Scanning scripts and the complete dataset are at `https://github.com/Ruqing1963/cramer-wronskian-stabil`

References

[1] R. Chen, "Algebraic rigidity and quadratic residue asymmetry in Legendre intervals," Zenodo, 2026. `https://zenodo.org/records/18706876`

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[4] H. Cramér, "On the order of magnitude of the difference between consecutive prime numbers," *Acta Arith.* **2** (1936), 23–46.

[5] A. Granville, "Harald Cramér and the distribution of prime numbers," *Scand. Actuar. J.* **1995** (1995), 12–28.