The Discrete Wronskian Stability of Prime Gaps: Empirical Observations near the Cramér Limit

Ruqing Chen

GUT Geoservice Inc., Montréal, Canada

ruqing@hotmail.com

Repository: https://github.com/Ruqing1963/cramer-wronskian-stability

February 2026

Abstract

We introduce the *discrete Wronskian compression ratio* q_W for a prime gap: given a maximal gap of length g starting at a prime p, the $g-1$ consecutive composites $\{p+1, \dots, p+g-1\}$ have a collective radical $R = \operatorname{rad}\left(\prod C_i\right)$, and we define $q_W = \sum \log C_i / \log R$. This ratio measures the degree to which the composites in the gap share prime factors. We report a systematic computational survey of all 18 maximal (record-breaking) prime gaps up to 10^7, finding a striking empirical phenomenon: while the gap lengths g grow in accordance with the Cramér heuristic ($g \approx 0.65 \log^2 p$), the Wronskian compression q_W does *not* grow. Instead, q_W decays from initial values near 1.6 for small gaps and stabilizes into a narrow band $q_W \in [1.30, 1.43]$ for all record gaps with $g \geq 36$. We propose this stabilization as a new empirical phenomenon worthy of theoretical explanation, and formulate it as an open conjecture: the Wronskian compression of maximal prime gaps is bounded. No claims conditional on the ABC conjecture are made.

1 Introduction

1.1 Cramér's conjecture and the random model

Let p_n denote the n-th prime and $g_n = p_{n+1} - p_n$ the n-th prime gap. Cramér's conjecture (1936) asserts that $\limsup_{n\to\infty} g_n/(\log p_n)^2 = 1$, motivated by a probabilistic model in which each integer m is "prime" independently with probability $1/\log m$. While this model successfully predicts many statistical properties of primes, it says nothing about the *multiplicative structure* of the composites that fill a prime gap.

1.2 The Wronskian perspective

We propose a complementary viewpoint. Rather than asking "how long can a prime gap be?" (a question about *additive* structure), we ask: "how multiplicatively compressed can a long sequence of consecutive composites be?"

A sequence of composites in a prime gap shares prime factors in a structured way: every other number is even, every third is divisible by 3, and so on. The *discrete Wronskian compression ratio* q_W quantifies this sharing. If the composites were pairwise coprime and squarefree, $q_W = 1$; the more they share factors (or contain prime powers), the higher q_W becomes.

Our central empirical finding is that q_W does not grow with gap length. This is not obvious a priori: longer gaps might be expected to produce more factor-sharing and hence higher compression. Instead, q_W stabilizes, suggesting a structural equilibrium in the multiplicative fabric of the integers.

1.3 What this paper does and does not claim

This paper reports an empirical observation and poses an open problem. We do not claim to prove Cramér's conjecture, nor do we invoke the ABC conjecture. The Wronskian stabilization is presented as a new *phenomenon* in need of theoretical explanation.

1.4 Context within the Titan Project

In [1] and [2], we proved rigorous algebraic theorems about the quadratic residue structure of Legendre and Oppermann intervals. Those results concern the *additive/algebraic* structure of short intervals near n^2. The present paper shifts to the *multiplicative* structure of prime gaps, complementing the algebraic perspective with an arithmetic one. The function field analogue of Legendre's conjecture was resolved in [3].

2 Definitions

Definition 2.1 (Maximal prime gap)**.** A prime gap $g_n = p_{n+1} - p_n$ is *maximal* (or *record-breaking*) if $g_n > g_k$ for all $k < n$.

Definition 2.2 (Discrete Wronskian compression)**.** For a prime gap of length g starting at a prime p, the *composite sequence* is $\mathcal{C} = \{p+1, p+2, \ldots, p+g-1\}$ (all $g-1$ composites). The *discrete Wronskian compression ratio* is

$$q_W(p,g) = \frac{\displaystyle\sum_{c\in\mathcal{C}} \log c}{\log \operatorname{rad}\left(\displaystyle\prod_{c\in\mathcal{C}} c\right)} = \frac{\log\left(\prod_{c\in\mathcal{C}} c\right)}{\displaystyle\sum_{q\in\mathcal{P}(\mathcal{C})} \log q}, \tag{1}$$

where $\mathcal{P}(\mathcal{C}) = \{q \text{ prime} : q \mid c \text{ for some } c \in \mathcal{C}\}$ is the set of all distinct prime factors appearing in the composite sequence.

Remark 2.3 (Interpretation of q_W). $q_W = 1$ if and only if $\prod c$ is squarefree (equivalently, every prime divides at most one $c \in \mathcal{C}$ and every c is squarefree). $q_W > 1$ measures the "excess" coming from two sources:

(a) *Factor sharing:* a prime q divides multiple elements of $\mathcal{C}$ (e.g., $q = 2$ divides roughly half the composites).

(b) *Prime powers:* an element $c = q^k m$ contributes $k \log q$ to the numerator but only $\log q$ to the denominator.

Factor sharing is the dominant contributor for long gaps.

Definition 2.4 (Cramér ratio)**.** The *Cramér ratio* of a prime gap is $\rho(p,g) = g/(\log p)^2$. Cramér's conjecture predicts $\limsup \rho = 1$.

3 Elementary Bounds on q_W

Before presenting the data, we establish simple bounds to calibrate expectations.

Proposition 3.1 (Lower bound)**.** *For any composite sequence of length $g-1$ near X:*

$$q_W \;\geq\; 1.$$

Equality holds if and only if $\prod c$ is squarefree.

Proof. $\log \operatorname{rad}(N) \leq \log N$ for all $N \geq 1$, with equality iff N is squarefree. $\square$

Proposition 3.2 (Heuristic upper bound via Mertens)**.** *For a gap of length g near X, the numerator is $\sum \log c \approx (g-1)\log X$. The denominator satisfies $\sum_{q\in\mathcal{P}} \log q \geq \sum_{\substack{q\leq g\\ q \text{ prime}}} \log q \approx g$ (by Chebyshev's bound / PNT for the Chebyshev function $\theta(g) \sim g$). Hence*

$$q_W \;\lesssim\; \frac{(g-1)\log X}{g} \;\approx\; \log X$$

in the worst case. However, the denominator also includes primes $q > g$ that divide elements of $\mathcal{C}$, which significantly increases it in practice.

Remark 3.3. The crude upper bound $q_W \lesssim \log X$ is never remotely approached in practice, because large primes contribute substantially to the radical. The empirical finding (Section 4) is that q_W stays in $[1.3, 1.7]$, far below $\log X \sim 15$–17.

4 Computational Survey

4.1 Methodology

We enumerated all primes up to 10^7 using a sieve, identified all 18 maximal prime gaps in this range, and computed q_W for each by fully factoring every composite in the gap sequence.

4.2 Complete data

Table 1: All 18 maximal prime gaps up to 10^7. P_n = starting prime; g = gap length; ρ = Cramér ratio $g/\log^2 P_n$; q_W = Wronskian compression.

P_n	g	$\log^2 P_n$	ρ	q_W
89	8	20.15	0.397	1.4912
113	14	22.35	0.626	1.6408
523	18	39.18	0.459	1.5461
887	20	46.07	0.434	1.4210
1 129	22	49.41	0.445	1.3757
1 327	34	51.71	0.658	1.5407
9 551	36	83.99	0.429	1.3706
15 683	44	93.32	0.472	1.3817
19 609	52	97.69	0.532	1.3743
31 397	72	107.21	0.672	1.4240
155 921	86	142.97	0.602	1.3769
360 653	96	163.73	0.586	1.3457
370 261	112	164.40	0.681	1.3705
492 113	114	171.78	0.664	1.3428
1 349 533	118	199.24	0.592	1.3331
1 357 201	132	199.40	0.662	1.3364
2 010 733	148	210.66	0.703	1.3385
4 652 353	154	235.71	0.653	1.3039

4.3 Key observations

Observation 4.1 (q_W stabilization)**.** For all maximal gaps with $g \geq 36$ (12 out of 18 data points), the Wronskian compression lies in the band

$$q_W \in [1.3039,\ 1.4240].$$

The range narrows further for $g \geq 86$: $q_W \in [1.30, 1.38]$. Despite the gap length g increasing by a factor of 4 (from 36 to 154), q_W exhibits no upward trend.

Observation 4.2 (Initial decay)**.** For small gaps ($g < 36$), q_W fluctuates between 1.38 and 1.64, with the maximum $q_W = 1.6408$ occurring at $P_n = 113$, $g = 14$. The early high values are driven by the dominance of small primes $(2, 3, 5)$ in short sequences, where factor sharing is proportionally large relative to the number of distinct prime factors introduced.

Observation 4.3 (Cramér ratio stability)**.** The Cramér ratio $\rho = g/\log^2 P_n$ fluctuates in $[0.40, 0.70]$ with no clear trend toward 1. This is consistent with the known expectation that the lim sup is approached very slowly (cf. Granville's correction [5], which predicts $\limsup \rho \geq 2e^{-\gamma} \approx 1.123$).

5 Why Does q_W Stabilize?

The stabilization of q_W is not obvious. A naive expectation might be that longer composite sequences share more prime factors (hence higher q_W), but the data shows the opposite. We identify two competing effects:

5.1 The compression force: factor sharing

In a sequence of $g-1$ consecutive integers near X, the number divisible by a prime $q \leq g$ is $\lfloor (g-1)/q \rfloor \geq 1$. Each such prime contributes $\lfloor (g-1)/q \rfloor \cdot \log q$ to the numerator but only $\log q$ to the denominator. By Mertens' theorem, the total "shared" contribution from primes $q \leq g$ is:

$$\text{(numerator excess)} \approx \sum_{q \leq g} \left(\frac{g-1}{q} - 1 \right) \log q \approx (g-1)\log g - g \sim g \log g.$$

5.2 The decompression force: large prime factors

Each composite $c \in \mathcal{C}$ has at least one prime factor $q > \sqrt{c} > \sqrt{X-g}$. These large primes are almost certainly *unique* to their composite (they divide no other element of $\mathcal{C}$). The number of such unique large primes grows roughly as $\alpha \cdot (g-1)$ for some proportion $\alpha > 0$. Each contributes $\log q \approx \frac{1}{2} \log X$ to the denominator without any sharing penalty.

The total "unique" contribution to the denominator is:

$$\alpha(g-1) \cdot \tfrac{1}{2} \log X.$$

5.3 The equilibrium

The compression ratio is approximately:

$$q_W \approx \frac{(g-1)\log X}{\alpha(g-1) \cdot \frac{1}{2}\log X + \text{(small prime contributions)}} \approx \frac{\log X}{\frac{\alpha}{2}\log X + \theta(g)} \approx \frac{1}{\alpha/2 + \theta(g)/\log X}.$$

As $X \to \infty$ with $g \sim c \log^2 X$ (Cramér regime), $\theta(g)/\log X \sim g/\log X \sim c \log X \to \infty$, which would suggest $q_W \to 0$. But $\theta(g)/\log X = O(\log X)$ grows *much* slower than $\alpha(g-1)\log X/2$, so the large-prime term dominates, and $q_W \to 2/\alpha$. The observed $q_W \approx 1.3$ is consistent with $\alpha \approx 1.5$, meaning roughly 2/3 of the composites in a gap contribute a large unique prime factor.

This heuristic explains the stabilization but does not constitute a proof. Making it rigorous would require quantitative control over the distribution of large prime factors of consecutive composites, which is closely related to deep questions in sieve theory.

6 Open Problems

Open Question 6.1 (Boundedness of q_W)**.** Is $q_W(p_n, g_n)$ bounded as $n \to \infty$ for maximal prime gaps? Our data suggests $q_W \leq 1.65$ unconditionally and $q_W \leq 1.43$ for $g \geq 36$, but we have no proof.

Open Question 6.2 (Limiting value)**.** Does $\lim_{n\to\infty} q_W(p_n, g_n)$ exist for maximal gaps? The data is consistent with convergence to a value in $[1.20, 1.35]$, but the sample is too small to distinguish convergence from slow oscillation.

Open Question 6.3 (Connection to Cramér)**.** Does the boundedness of q_W imply (or follow from) Cramér's conjecture? A proof that $q_W \leq C$ for some absolute constant would imply, via the relation between the numerator and denominator, a bound of the form $g = O((\log X)^{C'})$ for some C' depending on C. Conversely, Cramér's conjecture $g = O(\log^2 X)$ would constrain the growth of the numerator, but controlling the denominator requires additional input about the radical.

Open Question 6.4 (Beyond record gaps)**.** Does q_W stabilize for *all* large prime gaps (not just maximal ones)? One expects that non-record gaps are "less compressed" (lower q_W), but this has not been systematically tested.

Open Question 6.5 (Connection to the Erdős–Pomerance conjecture)**.** The stabilization of q_W reflects the fact that $\log \operatorname{rad}(\prod C_i)$ grows proportionally to $\log(\prod C_i)$. Is this related to conjectures on the distribution of the radical function, such as those of Erdős and Pomerance on $\operatorname{rad}(n)$?

7 Discussion

7.1 What is proven vs. what is observed

Proven: $q_W \geq 1$ always (Proposition 3.1).

Observed: q_W stabilizes in $[1.30, 1.43]$ for record gaps $g \geq 36$ up to $P_n = 4{,}652{,}353$ (Table 1).

Conjectured: q_W remains bounded for all maximal prime gaps (Question 6.1).

7.2 Relation to the ABC conjecture

The ABC conjecture constrains individual triples (a, b, c) with $a + b = c$, bounding the quality $q(a, b, c) = \log c / \log \operatorname{rad}(abc)$. The Wronskian compression q_W is a *different* quantity: it measures the collective compression of a *sequence* of integers. There is no known direct implication between the ABC conjecture and the boundedness of q_W.

We emphasize this point to avoid a common confusion: the ABC conjecture does not, in any known formulation, bound the compression of a product of consecutive composites. Any future connection between q_W and ABC would require substantial new ideas.

7.3 Relation to the algebraic structure papers

In [1] and [2], we proved that Legendre intervals possess rigid quadratic residue distributions (the Spin Asymmetry Theorem and the Oppermann Parity Law). Those results concern the *additive/algebraic* axis (residues modulo $p = 2n - 1$), while the present paper concerns the *multiplicative* axis (radical of consecutive composites). A unified framework connecting these two axes—the algebraic structure of residues and the multiplicative structure of factorizations—remains a distant goal.

Data and code availability

The scanning script and complete dataset are available at `https://github.com/Ruqing1963/cramer-wronskian-stability`.

References

[1] R. Chen, "Algebraic rigidity and quadratic residue asymmetry in Legendre intervals," Zenodo, 2026. `https://zenodo.org/records/18706876`

[2] R. Chen, "Oppermann's parity law: quadratic residue symmetry breaking in half-intervals via the negation involution," Zenodo, 2026. `https://zenodo.org/records/18707265`

[3] R. Chen, "The geometry of prime vacuums: Legendre's conjecture in function fields via monodromy and Chebotarev density," Zenodo, 2026. `https://zenodo.org/records/18705744`

[4] H. Cramér, "On the order of magnitude of the difference between consecutive prime numbers," *Acta Arith.* **2** (1936), 23–46.

[5] A. Granville, "Harald Cramér and the distribution of prime numbers," *Scand. Actuar. J.* **1995** (1995), 12–28.