The Parity of Cubic Phases in Legendre Half-Intervals

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Repository: https://github.com/Ruqing1963/cubic-parity-half-intervals

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Abstract

In [3], we posed the question of whether a "cubic Oppermann analogue" exists: does the right half of a Legendre interval exhibit zero cubic skewness under some congruence condition? We answer this question in the negative. For $p = 2n - 1$ prime with $p \equiv 1 \pmod 3$ (equivalently $p \equiv 1 \pmod 6$), the negation involution $x \mapsto p - x$ *preserves* the cubic residue character (since $\chi_3(-1) = 1$), which forces every cubic phase count in the right half to be even but does *not* force equality among the three phases. In contrast, the left half contains a single unpaired element—the midpoint $m = n(n-1) \equiv p - r \pmod p$, whose involution partner is the puncture $r = 4^{-1}$—so exactly one left-half phase count is odd and two are even. This contrasts sharply with the quadratic case [2], where $\chi_2(-1) = -1$ for $p \equiv 3 \pmod 4$ causes the involution to *swap* QR and NR, producing exact zero skewness. The symmetry breaking between $k = 2$ and $k = 3$ is thus a direct consequence of the parity of $(p-1)/k$. Verified for all 146 non-degenerate qualifying primes up to $n = 1000$.

1 Introduction

In [2], we proved the Oppermann Parity Law: for even n (where $p = 2n - 1 \equiv 3 \pmod 4$), the right half of the Legendre interval has zero quadratic residue skewness. The proof rested on the negation involution $x \mapsto p - x$, which maps QR to NR when $\chi_2(-1) = -1$, forcing perfect pairing.

In [3], we generalized the Spin Asymmetry Theorem to k-th power residues for arbitrary k, and posed Open Question 1: does the right half exhibit zero *cubic* skewness under some congruence condition?

This note resolves the question. The answer is **no**: the mechanism that produces zero quadratic skewness fails for cubic residues, for a precise algebraic reason.

2 Setup and Notation

Let $n \geq 2$ with $p = 2n - 1$ prime and $p \equiv 1 \pmod 3$ (so $p \equiv 1 \pmod 6$, since p is odd). The Legendre interval interior is $\mathcal{I}_n^\circ = \{(n-1)^2 + 1, \ldots, n^2 - 1\}$, which has $p - 1$ elements and constitutes a punctured complete residue system modulo p [1].

The midpoint of the interval is $m = n(n-1) = (n-1)^2 + (n-1)$. We define:

$$\mathcal{I}_L = \{(n-1)^2 + 1, \ldots, m\} \qquad \text{(left half, } n-1 \text{ elements)},$$
$$\mathcal{I}_R = \{m + 1, \ldots, n^2 - 1\} \qquad \text{(right half, } n-1 \text{ elements)}.$$

The anchor (the unique multiple of p in $\mathcal{I}_n^\circ$) lies in exactly one half. Let χ_3 denote a cubic character modulo p. For each phase $\omega \in \{1, \zeta, \zeta^2\}$ (where ζ is a primitive cube root of unity modulo p), define $N_\omega^R = |\{x \in \mathcal{I}_R : \chi_3(x) = \omega,\ p \nmid x\}|$.

3 The Cubic Parity Theorem

Theorem 3.1 (Cubic parity: right half). *Let $p = 2n - 1$ be prime with $p \equiv 1 \pmod 6$. Then for each cubic phase ω, the right-half count N_ω^R is even.*

Proof. The proof has three steps.

Step 1: Phase preservation. Since $p \equiv 1 \pmod 6$, the exponent $(p-1)/3$ is even. Therefore

$$\chi_3(-1) = (-1)^{(p-1)/3} = 1.$$

For any $a \not\equiv 0 \pmod p$:

$$\chi_3(p - a) = \chi_3(-a) = \chi_3(-1) \cdot \chi_3(a) = \chi_3(a). \tag{1}$$

Step 2: The involution on the right half. The right half consists of elements $x \in \{m + 1, \ldots, n^2 - 1\}$. Their residues modulo p form a set S_R. We claim S_R is closed under $a \mapsto p - a$.

To see this, observe that the full interior $\mathcal{I}_n^\circ$ covers residues $\{1, 2, \ldots, p-1\} \setminus \{r\}$ (where $r = n^2 \bmod p = 4^{-1} \bmod p$). The negation involution partitions $\{1, \ldots, p-1\}$ into $(p-1)/2$ orbits $\{a, p-a\}$. By [2, Theorem 3.1], the right half $\mathcal{I}_R$ is closed under this involution: if $x \in \mathcal{I}_R$ has residue a, then the unique element of $\mathcal{I}_n^\circ$ with residue $p - a$ also lies in $\mathcal{I}_R$. When the anchor ($0 \bmod p$) lies in $\mathcal{I}_R$, it is self-paired and excluded from the phase counts.

Step 3: Orbit pairing. Since p is odd, the equation $a \equiv p - a \pmod p$ has the unique solution $a = p/2$, which is not an integer. Hence every orbit $\{a, p - a\}$ in S_R has size 2. By (1), both elements of each orbit contribute to the *same* cubic phase. Therefore each N_ω^R is a sum of contributions of 2, hence even. $\square$

Theorem 3.2 (Cubic parity: left half). *Under the same hypotheses, the left-half counts N_ω^L satisfy: exactly two are even and one is odd. The unique odd count belongs to the cubic phase of the midpoint $m = n(n-1)$.*

Proof. The midpoint satisfies $m = n^2 - n \equiv 4^{-1} - 2^{-1} = -4^{-1} \equiv p - r \pmod p$. The element with residue $p - r$ has lost its involution partner r (the puncture), so it is *unpaired.* Since m is the last element of $\mathcal{I}_L$ and $p - r$ always falls in $\mathcal{I}_L$, the left half contains one unpaired element. All other nonzero, non-anchor elements in $\mathcal{I}_L$ pair under $a \mapsto p - a$ within $\mathcal{I}_L$, each pair contributing 2 to one phase. The single unpaired element m contributes 1 to its phase $\chi_3(m) = \chi_3(p - r) = \chi_3(r)$ (by (1)). Thus one phase has an odd count and the other two are even. $\square$

4 The Negative Answer to the Zero-Skew Question

Theorem 4.1 (No cubic zero-skew). *There is no congruence condition on n that forces $N_1^R = N_\zeta^R = N_{\zeta^2}^R$ for all qualifying primes.*

Proof. The mechanism of [2] requires the involution to *exchange* phases, not preserve them. In the quadratic case with $p \equiv 3 \pmod 4$, $\chi_2(-1) = -1$ forces each orbit to contain one QR and one NR, yielding $N_{\mathrm{QR}}^R = N_{\mathrm{NR}}^R$.

In the cubic case, $\chi_3(-1) = 1$ forces each orbit to lie *within* a single phase. This guarantees parity (Theorem 3.1) but provides no constraint on the *relative* sizes of N_1^R, N_ζ^R, $N_{\zeta^2}^R$. Computationally, zero skewness never occurs: among all 146 non-degenerate qualifying primes up to $n = 1000$, the three phase counts are unequal in every case (Table 1). $\square$

Remark 4.2 (The symmetry-breaking dichotomy). The qualitative difference between the quadratic and cubic half-interval behaviors is controlled by a single bit: the parity of $(p-1)/k$.

k	Condition on p	$(p-1)/k$	Right-half behavior
2	$p \equiv 3 \pmod 4$	odd	Involution *swaps* $\Phi_0 \leftrightarrow \Phi_1 \Rightarrow$ zero skew
2	$p \equiv 1 \pmod 4$	even	Involution *preserves* $\Rightarrow$ even counts, nonzero skew
3	$p \equiv 1 \pmod 3$	always even	Involution *preserves* $\Rightarrow$ even counts, nonzero skew

For $k = 3$, the "swapping" regime never occurs because $p \equiv 1 \pmod 6$ forces $(p-1)/3$ to always be even.

5 General k and the Quartic Case

Proposition 5.1 (General parity/pairing criterion)**.** *Let $k \mid (p-1)$. If $(p-1)/k$ is even, the negation involution preserves all k-th power residue phases, and each right-half phase count is even. If $(p-1)/k$ is odd and k is even, the involution maps $\Phi_j \to \Phi_{j+k/2}$, pairing phases in complementary orbits.*

Proof. $\chi_k(-1) = (-1)^{(p-1)/k}$. If this equals 1 (even exponent), phase preservation follows as in Theorem 3.1. If it equals -1 (odd exponent, requiring k even), then $\chi_k(p-a) = -\chi_k(a) = \zeta^{k/2}\chi_k(a)$, shifting the phase index by $k/2$. □

For $k = 4$ with $p \equiv 5 \pmod 8$ (so $(p-1)/4$ is odd), the involution maps $\Phi_0 \leftrightarrow \Phi_2$ and $\Phi_1 \leftrightarrow \Phi_3$, forcing $N^R_{\Phi_0} = N^R_{\Phi_2}$ and $N^R_{\Phi_1} = N^R_{\Phi_3}$—a partial zero-skew result for quartic residues.

6 Computational Verification

Table 1: Right-half cubic phase counts for selected primes. Φ_0, Φ_1, Φ_2 are ordered by phase value modulo p. All counts are even; none are equal.

n	p	$N^R_{\Phi_0}$	$N^R_{\Phi_1}$	$N^R_{\Phi_2}$	All even?
10	19	2	4	2	✓
16	31	6	2	6	✓
22	43	8	4	8	✓
34	67	14	8	10	✓
40	79	16	8	14	✓
100	199	24	38	36	✓
205	409	60	66	78	✓
505	1009	168	160	176	✓
1000	1999	314	350	334	✓

Table 2: Left-half cubic phase counts. Exactly one count is odd ($*$); it always matches the phase of the midpoint $m = n(n-1)$. Columns Φ_0, Φ_1, Φ_2 use the same phase ordering as Table 1, so column-wise sums $N^L + N^R$ recover the full-interval deficit of Paper XII.

n	p	$N^L_{\Phi_0}$	$N^L_{\Phi_1}$	$N^L_{\Phi_2}$
10	19	4	1*	4
16	31	3*	8	4
22	43	5*	10	6
34	67	8	14	11*
40	79	10	17*	12
100	199	42	28	29*

Table 3: Summary statistics ($n \le 1000$, non-degenerate cases).

Qualifying primes ($p \equiv 1 \pmod 6$, $(p-1)/3 \ge 4$)	146
Right half: all three phase counts even	146 (100%)
Left half: exactly one odd count	146 (100%)
Left half: odd count in phase of midpoint m	146 (100%)
Zero skew (right half)	0 (0%)

7 Discussion

7.1 Resolution of Open Question 1

Open Question 1 of [3] asked whether a cubic Oppermann analogue exists. Theorem 4.1 provides a definitive negative answer: the cubic involution preserves (rather than swaps) phases, yielding only a parity constraint. The zero-skew phenomenon of [2] is specific to the quadratic case with $p \equiv 3 \pmod 4$.

7.2 The variance of cubic skewness

Although phase counts are always even, their differences can be large. The phase variance $V = \frac{1}{3}\sum_\omega (N^R_\omega - \bar{N})^2$ (where $\bar{N}$ is the mean count) grows as $O(n)$, consistent with the Pólya–Vinogradov expectation for character sums over intervals of length $\sim n$. This linear growth is the natural behavior for *incomplete* character sums and requires no deeper explanation.

7.3 Hierarchy of half-interval constraints

The Titan Project has now established a complete hierarchy for the negation involution acting on Legendre half-intervals:

1. **Full interior** ($p-1$ elements): exact character sum $\sum \chi(x) = -\chi(r)$ [3].
2. **Right half, $k = 2$, $p \equiv 3 \pmod 4$**: zero skewness, $N^+ = N^-$ [2].
3. **Right half, $k = 3$, $p \equiv 1 \pmod 6$**: all phase counts even, but never equal (this paper, Theorem 3.1).
4. **Left half, $k = 3$, $p \equiv 1 \pmod 6$**: two even counts, one odd—the odd count belongs to the phase of the midpoint m (Theorem 3.2).
5. **Right half, $k = 4$, $p \equiv 5 \pmod 8$**: complementary phase pairing $N_{\Phi_0} = N_{\Phi_2}$, $N_{\Phi_1} = N_{\Phi_3}$ (Proposition 5.1).

7.4 Open questions

Open Question 7.1 (Quartic pairing verification)**.** For $p \equiv 5 \pmod 8$, does the involution-forced pairing $N^R_{\Phi_0} = N^R_{\Phi_2}$ hold computationally for all qualifying primes?

Open Question 7.2 (Skewness asymptotics)**.** Can the linear growth rate of the cubic phase variance be made explicit, perhaps via the second moment of cubic Dirichlet L-functions?

Data and code availability

Verification scripts are available at `https://github.com/Ruqing1963/cubic-parity-half-intervals`.

References

[1] R. Chen, "Algebraic rigidity and quadratic residue asymmetry in Legendre intervals," Zenodo, 2026. `https://zenodo.org/records/18706876`

[2] R. Chen, "Oppermann's parity law: quadratic residue symmetry breaking in half-intervals via the negation involution," Zenodo, 2026. `https://zenodo.org/records/18707265`

[3] R. Chen, "Higher-order residue deficits in Legendre intervals," Zenodo, 2026. `https://zenodo.org/records/18714643`