# A Conductor Census of 425,082 Goldbach–Frey Curves: Asymptotic Decomposition and Bandwidth Stability

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**Abstract**

Using the exact conductor formula $\mathrm{Cond}_{\mathrm{odd}}(\mathrm{Jac}(C)) = [\mathrm{rad}_{\mathrm{odd}}(p \cdot q \cdot N \cdot (p-q))]^2$ established in the preceding papers of this series, we compute the conductor ratio $\rho$ for every Goldbach pair $(p,q)$ with $p+q = 2N \leq 10{,}000$: a census of 425,082 curves. We decompose the mean conductor ratio into three components: a static term $\rho_{\mathrm{static}}$ depending only on $N$ (6.5% of $\langle \rho \rangle$), a boundary term $\rho_{\mathrm{boundary}}$ from $p \cdot q$ (66.0%), and a gap term $\delta$ from $|p-q|$ (27.5%). The key empirical finding is *bandwidth stability*: the standard deviation $\sigma(\rho) \approx 0.33$ is approximately constant over the entire range $2N \in [100,\ 10{,}000]$, while $\langle \rho \rangle$ grows logarithmically. This implies the coefficient of variation $\sigma/\langle \rho \rangle \to 0$ as $N \to \infty$, providing a quantitative explanation for the high $R^2 > 0.997$ of the Band Shifting Law and connecting the conductor geometry to the Hardy–Littlewood prime pair density.

## 1 Introduction

Papers [1] and [2] established the exact odd conductor formula for the Goldbach–Frey curve $C\colon y^2 = x(x^2 - p^2)(x^2 - q^2)$, with $p \neq q$ distinct odd primes and $p + q = 2N$:

$$\mathrm{Cond}_{\mathrm{odd}}\big(\mathrm{Jac}(C)\big) \;=\; \big[\mathrm{rad}_{\mathrm{odd}}(p \cdot q \cdot N \cdot (p-q))\big]^2. \tag{1}$$

The conductor ratio

$$\rho(p,q) \;=\; \frac{\log \mathrm{Cond}_{\mathrm{odd}}\big(\mathrm{Jac}(C)\big)}{\log(2N)} \;=\; \frac{2 \log \mathrm{rad}_{\mathrm{odd}}(p \cdot q \cdot N \cdot (p-q))}{\log(2N)} \tag{2}$$

drives the Band Shifting Law (BSL) with $R^2 > 0.997$. But the preceding work studied only 10 test cases. The present paper asks: what can we learn from *all* Goldbach pairs up to a given bound?

We compute $\rho(p,q)$ for every Goldbach pair with $p + q \leq 10{,}000$—a total of 425,082 pairs spanning 4,997 distinct even integers—and study the statistical distribution of $\rho$ as a function of $N$.

## 2 Three-Component Decomposition

The conductor ratio admits a natural decomposition. Since $p$ and $q$ are prime, $\mathrm{rad}_{\mathrm{odd}}(p) = p$ and $\mathrm{rad}_{\mathrm{odd}}(q) = q$. Write

$$\rho \;=\; \underbrace{\frac{2 \log \mathrm{rad}_{\mathrm{odd}}(N)}{\log(2N)}}_{\rho_{\mathrm{static}}} + \underbrace{\frac{2 \log(p \cdot q)}{\log(2N)}}_{\rho_{\mathrm{boundary}}} + \underbrace{\frac{2 \log \mathrm{rad}_{\mathrm{odd}}^{\mathrm{new}}(|p-q|)}{\log(2N)}}_{\delta}, \tag{3}$$

where $\mathrm{rad}_{\mathrm{odd}}^{\mathrm{new}}(|p-q|)$ denotes the product of odd primes in $|p-q|$ not already dividing $p \cdot q \cdot N$. The three terms have distinct roles:

- $\rho_{\mathrm{static}}$: depends *only* on $N$. This is the static conduit $\xi = 2\log\mathrm{rad}_{\mathrm{odd}}(N)/\log(2N)$ that sets the *band position* in the BSL.

- $\rho_{\mathrm{boundary}}$: depends on the specific pair $(p,q)$ but not on their difference. For "balanced" pairs $(p \approx q \approx N)$, $\rho_{\mathrm{boundary}} \approx 4\log N/\log(2N) \to 4$. For extreme pairs $(p = 3, q \approx 2N)$, it is smaller.

- $\delta$: the gap correction, a chaotic term depending on the prime factorisation of $|p-q|$. This contributes only scatter ("bandwidth") to the BSL.

Table 1 and Figure 1(b) show the relative contributions at $2N = 10{,}000$.

| Component | Contribution | Role in BSL |
|---|---|---|
| $\rho_{\mathrm{static}}$ | 6.5% | Band position |
| $\langle\rho_{\mathrm{boundary}}\rangle$ | 66.0% | Pair location within band |
| $\langle\delta\rangle$ | 27.5% | Intra-band noise |

Table 1: Decomposition of $\langle\rho_{\mathrm{true}}\rangle = 5.38$ at $2N = 10{,}000$ (127 Goldbach pairs).

## 3 Census Results

### 3.1 Mean conductor ratio

Figure 1(a) shows $\langle\rho_{\mathrm{true}}\rangle$ versus $\ln(2N)$ for all 4,997 even integers. The mean grows slowly and roughly logarithmically; a linear fit in $\ln(2N)$ gives

$$\langle\rho_{\mathrm{true}}\rangle \approx 0.217\ln(2N) + 4.55 \tag{4}$$

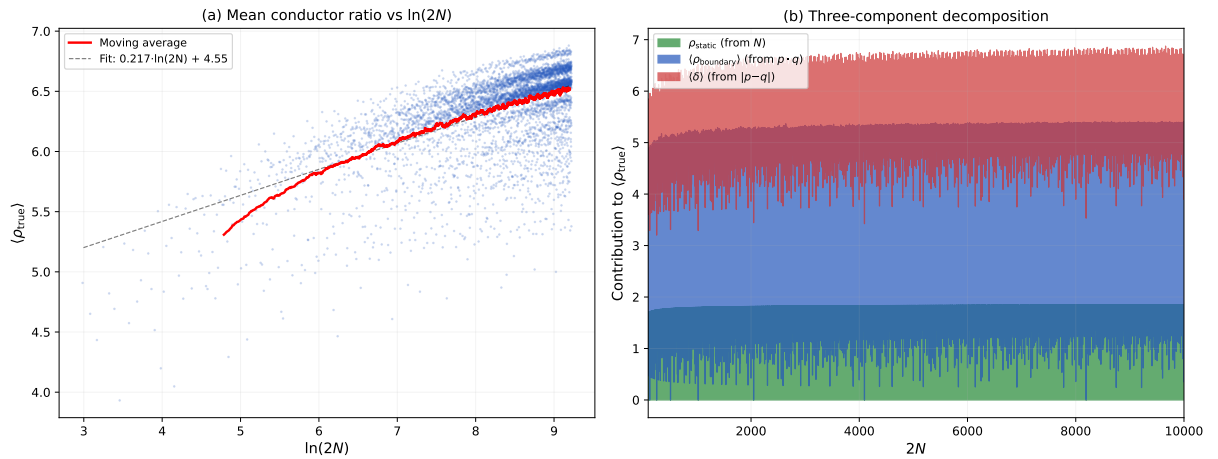but with $R^2 = 0.33$, reflecting the strong number-theoretic fluctuations in $\mathrm{rad}_{\mathrm{odd}}(N)$.



Figure 1: (a) Mean conductor ratio $\langle\rho\rangle$ vs. $\ln(2N)$ for 4,997 even numbers (blue dots), with moving average (red). (b) Stacked decomposition: static (green), boundary (blue), gap (red).

## 3.2 Bandwidth stability

The most striking finding is that the bandwidth—the standard deviation $\sigma(\rho)$ of the conductor ratio across Goldbach pairs at fixed $N$—is approximately constant:

$$\sigma(\rho) \approx 0.33 \pm 0.05 \qquad \text{for } 2N \in [100,\ 10{,}000]. \tag{5}$$

This is visible in Figure 2(a). Since $\langle \rho \rangle$ grows (slowly) while $\sigma$ stays fixed, the coefficient of variation $\text{CV} = \sigma/\langle \rho \rangle$ decreases (Figure 2(b) of the Hardy–Littlewood figure).
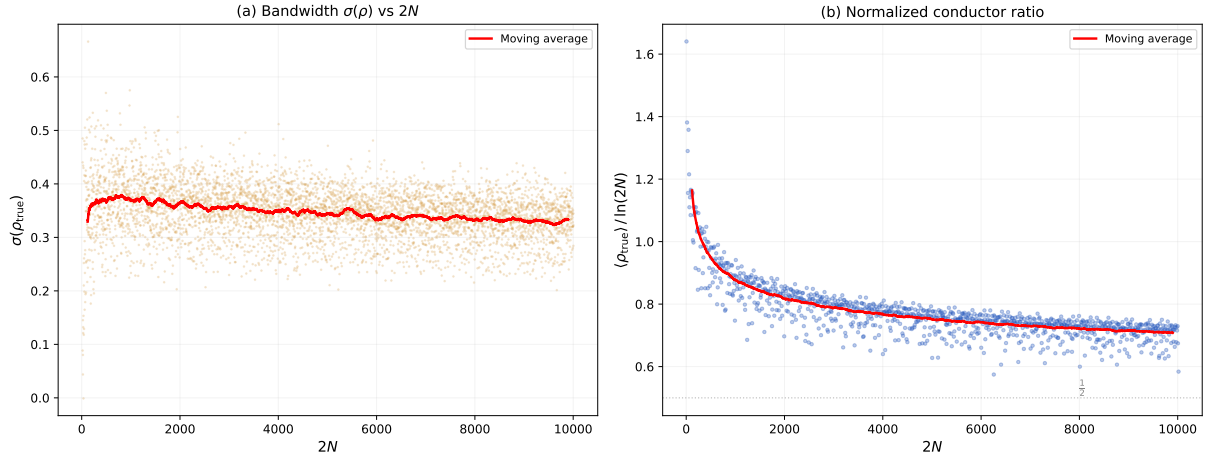


Figure 2: (a) Bandwidth $\sigma(\rho)$ vs. $2N$: approximately constant at $\approx 0.33$. (b) Normalised ratio $\langle \rho \rangle / \ln(2N)$, decreasing toward a value near $\frac{1}{2}$.

## 3.3 Hardy–Littlewood comparison

The Goldbach pair count $G(2N)$ is consistent with the Hardy–Littlewood prediction

$$G_{\text{HL}}(2N) \ \sim \ 2C_2 \prod_{\substack{p|N \\ p>2}} \frac{p-1}{p-2} \ \cdot \ \frac{N}{(\ln N)^2}\,, \tag{6}$$

where $C_2 = \prod_{p>2}(1 - 1/(p-1)^2) \approx 0.6601$ is the twin prime constant. Figure 3(a) shows the actual–predicted comparison; the scatter around the diagonal is typical of the known fluctuations in prime pair counts.
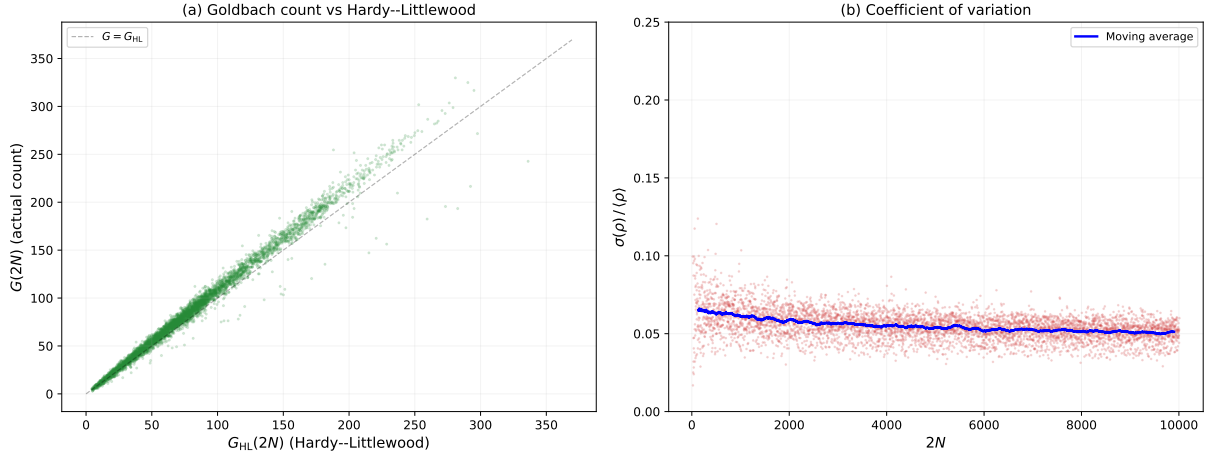
Figure 3: (a) Actual Goldbach count $G(2N)$ vs. Hardy–Littlewood prediction. (b) Coefficient of variation $\sigma(\rho)/\langle\rho\rangle$ vs. $2N$, decreasing as the signal-to-noise ratio improves.

# 4 Asymptotic Analysis

## 4.1 Why the BSL works: a quantitative explanation

The BSL models $\rho$ as a linear function of the static conduit $\xi = 2\log\mathrm{rad}_{\mathrm{odd}}(N)/\log(2N)$. From the decomposition (3), the "non-$\xi$" part of $\rho$ is $\rho_{\mathrm{boundary}} + \delta$. The BSL achieves $R^2 > 0.997$ because:

1. $\rho_{\mathrm{boundary}} = 2\log(pq)/\log(2N)$ is tightly concentrated for fixed $N$: most Goldbach pairs have $pq$ within an order of magnitude of $N^2$, so $\rho_{\mathrm{boundary}}$ varies by $\lesssim 1$.

2. The gap correction $\delta$ has bounded standard deviation ($\sigma \approx 0.33$), contributing only scatter within the band, not systematic drift.

3. Together, the systematic part ($\rho_{\mathrm{static}} + \langle\rho_{\mathrm{boundary}}\rangle$) accounts for $\sim 72.5\%$ of $\langle\rho\rangle$, and this systematic part is a deterministic function of $N$.

## 4.2 The normalised ratio

Define the normalised conductor ratio

$$R(N) \;=\; \frac{\langle\rho_{\mathrm{true}}\rangle(N)}{\ln(2N)}.$$

The census data (Figure 2(b)) shows $R(N)$ decreasing from $\sim 1.15$ at $2N = 50$ to $\sim 0.70$ at $2N = 10{,}000$. We conjecture:

**Conjecture 4.1.** *As $N \to \infty$,*

$$\langle\rho_{\mathrm{true}}\rangle(N) \;=\; 4 + \frac{2\log\mathrm{rad}_{\mathrm{odd}}(N)}{\log(2N)} + O\!\left(\frac{\log\log N}{\log N}\right).$$

The leading term 4 arises because for a "typical" Goldbach pair, $\log(pq) \sim 2\log N$, so $\rho_{\mathrm{boundary}}$ converges to 4. The second term is exactly the static conduit $\xi$. The gap correction $\langle\delta\rangle$ should be $O(\log\log N/\log N)$ on average, since by the Erdős–Kac theorem the average number of distinct prime factors of a random integer near $N$ is $\sim \log\log N$, and each contributes $O(1/\log N)$ to the normalised ratio.

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## 4.3 Bandwidth and the Erdős–Kac connection

The observed constancy of $\sigma(\rho) \approx 0.33$ can be heuristically explained as follows. The dominant source of variance in $\rho$ (for fixed $N$) is the variation in $\log(pq)$ and $\log \mathrm{rad}_{\mathrm{odd}}^{\mathrm{new}}(|p - q|)$ across Goldbach pairs. For the boundary term, the variance comes from the distribution of $\log(pq)$, which for $p + q = 2N$ is essentially $\log(p(2N - p))$. This is a concave function with bounded variance regardless of $N$. For the gap term, the Erdős–Kac theorem implies $\omega(|p-q|) \sim \log \log N$ with standard deviation $\sim \sqrt{\log \log N}$, and each prime factor contributes $2 \log r / \log(2N) = O(1/\log N)$ to $\delta$. The net variance in $\delta$ is thus $O(\log \log N/(\log N)^2)$, which is negligible. The total bandwidth is therefore dominated by the boundary term and stays bounded.

## 5 Summary of Census Statistics

| $2N$ | $G(2N)$ | $\langle \rho \rangle$ | $\sigma(\rho)$ | $\langle \rho \rangle / \ln(2N)$ | CV |
|---:|---:|---:|---:|---:|---:|
| 50 | 4 | 4.517 | 0.130 | 1.155 | 0.029 |
| 100 | 6 | 4.992 | 0.349 | 1.084 | 0.070 |
| 200 | 8 | 4.845 | 0.448 | 0.914 | 0.092 |
| 500 | 13 | 5.144 | 0.305 | 0.828 | 0.059 |
| 1000 | 28 | 5.160 | 0.369 | 0.747 | 0.072 |
| 2000 | 37 | 5.299 | 0.338 | 0.697 | 0.064 |
| 5000 | 76 | 5.349 | 0.301 | 0.628 | 0.056 |
| 10000 | 127 | 5.377 | 0.323 | 0.584 | 0.060 |

Table 2: Census statistics at selected values of $2N$. $G(2N)$ is the number of Goldbach pairs; CV is the coefficient of variation $\sigma/\langle \rho \rangle$.

## 6 Discussion

The census reveals three structural properties of the conductor landscape:

*Signal dominance.* The systematic part of $\rho$ (static conduit + mean boundary) accounts for $\sim 72.5\%$ of the total, explaining the BSL's high $R^2$.

*Noise boundedness.* The bandwidth $\sigma(\rho) \approx 0.33$ is bounded, so the signal-to-noise ratio improves with $N$. This is a necessary (though not sufficient) condition for the conductor proxy to remain faithful in the asymptotic regime.

*Hardy–Littlewood consistency.* The Goldbach pair counts follow the Hardy–Littlewood density with expected fluctuations. The conductor framework introduces no anomalies: curves with "extreme" conductor ratios do not cluster at particular $N$.

These properties do not prove the Goldbach conjecture, but they establish that the arithmetic geometry of the Frey curves is smoothly compatible with the conjectured prime pair statistics. Any approach to Goldbach through conductor methods must ultimately show that the *existence* of at least one pair forces a non-trivial conductor, connecting the density picture to an existence argument. This remains an open problem.

## Acknowledgments

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