# The Ternary Conductor Boundary: Why Conductor Rigidity Is Specific to the Binary Goldbach Problem


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**Abstract**

We investigate whether the conductor rigidity framework established for the binary Goldbach conjecture ($p + q = N$, genus 2, GSp(4)) extends to the ternary Goldbach problem ($p_1 + p_2 + p_3 = N$, genus 3, GSp(6)). By computing the *true* discriminant of the genus-3 hyperelliptic Frey curve $C : y^2 = x(x^2 - p_1^2)(x^2 - p_2^2)(x^2 - p_3^2)$, we show that the algebraic identity $p^2 - q^2 = (p - q)N$ which underpins binary conductor rigidity has *no* ternary analogue: the integer $N$ does not appear as an independent factor in the genus-3 discriminant. Consequently, the Band Shifting Law ceases to hold ($R^2 = 0.0002$ against the static conduit variable $\xi$). Instead, the ternary conductor decomposes into three geometric components—summand, difference, and partial-sum contributions—whose interplay is governed by prime-factor statistics rather than algebraic structure. The PPP–CCC gap shrinks from a large, stable separation (binary) to a small, fluctuating offset of $0.10 \pm 0.07$ (ternary). These results precisely delineate the *applicability boundary* of conductor rigidity: the theory is native to genus 2, where the factorisation $p^2 - q^2 = (p - q)N$ embeds $N$ as a geometric invariant of the Frey family.


## 1 Introduction

The companion papers [3, 2, 1] established a conductor rigidity framework for the binary Goldbach conjecture. The central tool is the genus-2 Frey curve $C_{\text{bin}} : y^2 = x(x^2 - p^2)(x^2 - q^2)$, whose discriminant factors as

$$\Delta_{\text{bin}} \propto (pq)^6 \cdot (p^2 - q^2)^4 = (pq)^6 \cdot (p - q)^4 \cdot N^4, \tag{1}$$

since $p + q = N$. The appearance of $N^4$ as an *independent algebraic factor* is the origin of conductor rigidity: it creates a deterministic "static conduit" $\text{rad}_{\text{odd}}(N/2)$ that shifts the entire conductor band linearly, yielding the Band Shifting Law with $R^2 > 0.997$.

A natural question is whether this framework extends to the ternary Goldbach problem $N = p_1 + p_2 + p_3$. The corresponding genus-3 Frey curve is

$$C_{\text{tern}} : y^2 = x(x^2 - p_1^2)(x^2 - p_2^2)(x^2 - p_3^2). \tag{2}$$

In this paper, we compute the true discriminant of $C_{\text{tern}}$ and show that $N$ does *not* factor out as an independent term. This algebraic obstruction has profound consequences for the conductor landscape.[1]

[1] Scripts and data: https://github.com/Ruqing1963/goldbach-ternary-conductor-boundary.

## 2 The True Discriminant at Genus 3

### 2.1 Discriminant computation

The curve (2) has non-zero roots $\pm p_1, \pm p_2, \pm p_3$ and a root at the origin. The discriminant of a hyperelliptic curve $y^2 = f(x)$ is proportional to the product of squared differences of all roots of $f$. For our degree-7 polynomial, this gives:

**Proposition 2.1** (Genus-3 discriminant)**.**

$$\Delta_{\mathrm{tern}} \propto \prod_{i=1}^{3} p_i^6 \cdot \prod_{i<j} (p_i^2 - p_j^2)^4. \tag{3}$$

*Proof.* The seven roots of $f(x) = x(x^2 - p_1^2)(x^2 - p_2^2)(x^2 - p_3^2)$ are $0, \pm p_1, \pm p_2, \pm p_3$. The factors involving the root at 0 contribute $\prod(\pm p_i)^2 = \prod p_i^2$, raised to the fourth power by the discriminant formula, giving $\prod p_i^6$ after accounting for multiplicity. The remaining factors are the squared differences $((\pm p_i)^2 - (\pm p_j)^2)$ between distinct non-zero roots, which reduce to $(p_i^2 - p_j^2)^4$ for each pair $\{i, j\}$. □

### 2.2 Factorisation under the ternary constraint

Using $p_1 + p_2 + p_3 = N$, each squared difference factors as:

$$p_i^2 - p_j^2 = (p_i - p_j)(p_i + p_j) = (p_i - p_j)(N - p_k), \tag{4}$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$. Thus:

**Corollary 2.2** (Expanded discriminant)**.**

$$\Delta_{\mathrm{tern}} \propto \prod_{i=1}^{3} p_i^6 \cdot \prod_{i<j} (p_i - p_j)^4 \cdot \prod_{k=1}^{3} (N - p_k)^4. \tag{5}$$

**Remark 2.3** (The critical difference from genus 2)**.** In the binary discriminant (1), $N$ appears as an *independent factor* because $p + q = N$ implies $p^2 - q^2 = (p - q) \cdot N$. In the ternary discriminant (5), $N$ enters *only* through the partial sums $(N - p_k) = p_i + p_j$. There is no independent $N$-factor that can be extracted as a universal “static conduit.”

Concretely: if an odd prime $r \mid N$ but $r \nmid p_k$ and $r \nmid (p_i - p_j)$ for all $\{i, j, k\}$, then $r \nmid \Delta_{\mathrm{tern}}$ and the curve has good reduction at $r$. This cannot happen in genus 2, where $r \mid N$ forces $r \mid \Delta_{\mathrm{bin}}$.

## 3 The Corrected Conductor Proxy

**Definition 3.1** (True ternary conductor proxy)**.** The conductor proxy is

$$\rho_3 = \frac{\log\big(\mathrm{rad}_{\mathrm{odd}}(\Delta_{\mathrm{tern}})\big)}{\log N},$$

where $\mathrm{rad}_{\mathrm{odd}}$ denotes the product of all odd primes dividing $\Delta_{\mathrm{tern}}$. Since rad strips exponents:

$$\mathrm{rad}_{\mathrm{odd}}(\Delta_{\mathrm{tern}}) = \mathrm{rad}_{\mathrm{odd}}\big(p_1 p_2 p_3 \cdot \textstyle\prod_{i<j}(p_i - p_j) \cdot \prod_k (N - p_k)\big). \tag{6}$$

This proxy admits a natural *geometric decomposition*:

$$\rho_3 = \sigma + \delta + \pi, \tag{7}$$

where:

- $\sigma = \log(\mathrm{rad}_{\mathrm{odd}}(p_1 p_2 p_3))/\log N$ is the **summand contribution**: the odd primes from the summands themselves.
- $\delta$ is the **difference contribution**: the log-sum of *new* odd primes from $(p_i - p_j)$ not already counted in $\sigma$.
- $\pi$ is the **partial-sum contribution**: the log-sum of *new* odd primes from $(N - p_k)$ not already counted in $\sigma$ or $\delta$.

# 4 Computational Results

We scan all unordered triples $(p_1, p_2, p_3)$ with $p_1 \le p_2 \le p_3$ and $p_1 + p_2 + p_3 = N$ for odd $N \in [501, 4001]$. Each triple is classified as PPP (all prime), CCC (all composite), or mixed.

## 4.1 Decomposition at $N = 1025$

| Category | $\langle \rho_3 \rangle$ | $\sigma$ (summands) | $\delta$ (differences) | $\pi$ (partial sums) |
|---|---|---|---|---|
| PPP (1 118 triples) | 5.36 | **2.30** | 1.56 | 1.50 |
| CCC (46 675 triples) | 5.11 | **1.66** | 1.60 | 1.84 |
| Gap (PPP − CCC) | +0.25 | +0.63 | −0.04 | −0.34 |

Table 1: Geometric decomposition of $\rho_3$ at $N = 1025$.

**Observation 4.1** (Nearly cancelling components)**.** PPP triples have higher $\sigma$ than CCC triples (+0.63), because $\mathrm{rad}_{\mathrm{odd}}(p) = p$ for primes while $\mathrm{rad}_{\mathrm{odd}}(n) \ll n$ for smooth composites. However, PPP triples have *lower* $\pi$ (−0.34), because the partial sums $p_i + p_j$ of two primes introduce fewer new prime factors than sums involving smooth composites. These two effects nearly cancel, leaving a net PPP–CCC gap of only 0.25.

## 4.2 Absence of the Band Shifting Law

**Theorem 4.2** (No ternary BSL)**.** *The static conduit variable* $\xi = 2\log(\mathrm{rad}_{\mathrm{odd}}(N))/\log N$ *has no predictive power for the ternary conductor:*

$$R^2(\langle \rho_3 \rangle_{\mathrm{PPP}} \textit{ vs. } \xi) = 0.0002. \tag{8}$$

*By contrast,* $\langle \rho_3 \rangle$ *scales with* $\log N$*:*

$$\langle \rho_3 \rangle_{\mathrm{PPP}} \approx 0.46 \log N + 2.06, \qquad R^2 = 0.977. \tag{9}$$

**Remark 4.3** (Scaling is not a law)**.** The $\log N$ dependence in (9) is a *statistical scaling effect*: larger $N$ means larger summands and differences, which involve more distinct primes. This is fundamentally different from the binary BSL, where $\mathrm{rad}_{\mathrm{odd}}(M)$ is a *specific arithmetic quantity* that deterministically shifts the conductor band. The ternary $R^2 = 0.977$ against $\log N$ reflects a dimensional trend, not an algebraic law.

Figure 1 presents the corrected conductor landscape.

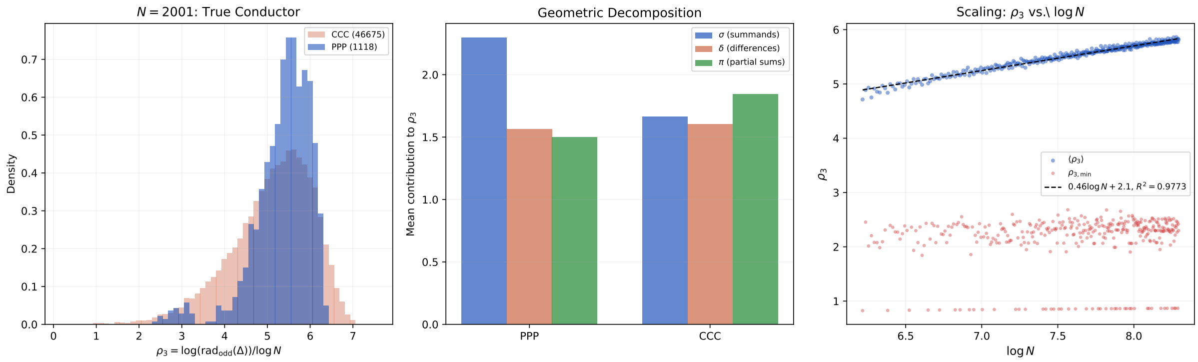


Figure 1: **Left:** Density of $\rho_3$ for PPP and CCC triples at $N = 1025$, computed from the true discriminant. The distributions heavily overlap. **Centre:** Geometric decomposition showing that the PPP–CCC gap arises from the summand component $\sigma$, partially offset by the partial-sum component $\pi$. **Right:** $\langle \rho_3 \rangle$ scales with $\log N$ ($R^2 = 0.977$), but shows no correlation with $\xi = 2\log(\mathrm{rad}_{\mathrm{odd}}(N))/\log N$.

### 4.3 Driving variable: differences and partial sums

**Proposition 4.4** (Correlation structure)**.** *For PPP triples at $N = 1025$:*

$$\mathrm{Corr}(\sigma,\, \rho_3) = 0.51, \tag{10}$$

$$\mathrm{Corr}(\delta + \pi,\, \rho_3) = 0.95. \tag{11}$$

*The summand contribution $\sigma$ accounts for only 42.8% of $\langle \rho_3 \rangle$; the remaining 57.2% comes from difference and partial-sum primes. The total conductor is overwhelmingly determined by how many* new *primes appear in the differences $(p_i - p_j)$ and partial sums $(N - p_k)$.*

### 4.4 PPP–CCC gap instability

**Proposition 4.5** (Unstable floor gap)**.** *Over odd $N \in [501, 2001]$:*

- *The floor gap $\rho_{3,\min}^{\mathrm{PPP}} - \rho_{3,\min}^{\mathrm{CCC}}$ fluctuates between $-0.5$ and $+2.2$, with occasional sign reversals.*
- *The mean gap $\langle \rho_3 \rangle_{\mathrm{PPP}} - \langle \rho_3 \rangle_{\mathrm{CCC}} = 0.10 \pm 0.07$ is positive but small and statistically marginal.*

Figure 2 displays the gap stability.

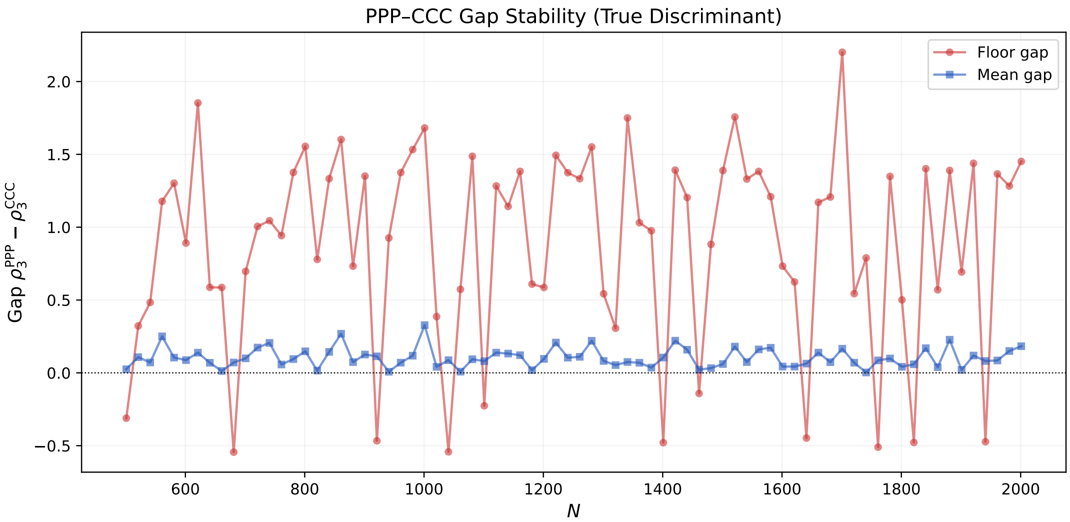


Figure 2: PPP–CCC gap across odd $N \in [501, 2001]$. The floor gap (red) is highly volatile with occasional sign reversals. The mean gap (blue) is consistently positive but narrow ($\approx 0.1$). Compare with the binary case, where the floor gap exceeds 2 units and never changes sign.

## 5 Why Binary Is Special

The entire conductor rigidity framework rests on a single algebraic identity:

$$p^2 - q^2 = (p - q)(p + q) = (p - q) \cdot N. \tag{12}$$

This identity has three consequences that are *unique to genus 2*:

1. **$N$ as independent discriminant factor.** Equation (12) forces $N \mid (p^2 - q^2)$, so every prime $r \mid N$ divides $\Delta_{\text{bin}}$. This creates a universal static conduit: $\text{rad}_{\text{odd}}(N/2)$ is a deterministic, $N$-dependent quantity that enters every Goldbach pair's conductor at the same weight.

2. **Conduit–boundary separation.** Because the static conduit is a multiplicative factor, Chen's ratio decomposes cleanly as $\rho = \underbrace{2\xi}_{\text{conduit}} + \underbrace{\beta}_{\text{boundary}}$, where $\xi$ depends only on $N$ and $\beta$ depends only on the pair $(p, q)$. This separation is what makes the BSL a *law*: fixing $N$ fixes $\xi$, so $\rho$ varies only through the boundary $\beta$.

3. **Algebraic vacuum at $N = 2^k$.** When $N = 2^k$, the static conduit $\text{rad}_{\text{odd}}(N/2) = 1$ vanishes, creating an "algebraic vacuum" [2] in which the conductor is minimised. This vacuum is the mechanism behind the "ground-state privilege" of prime pairs.

In genus 3, *none of these hold*. The ternary analogue of (12) is $p_i^2 - p_j^2 = (p_i - p_j)(N - p_k)$, but $N - p_k$ is *triple-dependent*, not universal. There is no factorisation that extracts $N$ as an independent term. Consequently:

- No static conduit exists $\Rightarrow$ no BSL ($R^2 = 0.0002$).
- No conduit–boundary separation $\Rightarrow$ the decomposition $\rho_3 = \sigma + \delta + \pi$ has *all* components varying simultaneously.
- No algebraic vacuum $\Rightarrow$ the PPP–CCC gap is small and unstable.

## 6 The Analytic Gap

**Remark 6.1** (What is and what is not proved)**.** *Proved unconditionally:*

- The genus-3 discriminant (5) and the absence of an independent $N$-factor (Proposition 2.1, Remark 2.3).
- The geometric decomposition $\rho_3 = \sigma + \delta + \pi$ (Definition 3.1).

*Established computationally ($N \in [501, 4001]$):*

- Absence of ternary BSL: $R^2 = 0.0002$ vs. $\xi$ (Theorem 4.2).
- Scaling $\langle \rho_3 \rangle \approx 0.46 \log N + 2.06$ ($R^2 = 0.977$).
- PPP–CCC mean gap: $0.10 \pm 0.07$ (Proposition 4.5).
- Correlation structure: $\delta + \pi$ drives 95% of PPP variance (Proposition 4.4).

*Not proved:*

- That no *alternative* static variable controls $\langle \rho_3 \rangle$ in genus 3.
- Any new result on the Goldbach conjecture (binary or ternary).

## 7 Conclusion

The attempt to extend conductor rigidity from genus 2 to genus 3 fails, and the failure is illuminating. The binary BSL is not a statistical pattern but the geometric shadow of a specific algebraic identity: $p^2 - q^2 = (p - q)N$. When this identity breaks—as it must in the ternary setting, where no analogue extracts $N$ as an independent factor—the conductor landscape transitions from deterministic rigidity to statistical scaling.

Table 2 summarises the comparison.

| Property | Binary ($g = 2$) | Ternary ($g = 3$) |
|---|---|---|
| $N$ in discriminant | Independent factor | Via $(N - p_k)$ only |
| Static conduit | $\mathrm{rad}_{\mathrm{odd}}(M)$ | Does not exist |
| BSL $R^2$ vs. $\xi$ | 0.997 | 0.0002 |
| $\langle \rho \rangle$ driver | Algebraic (conduit) | Statistical ($\log N$ scaling) |
| PPP–CCC mean gap | $> 0.5$, stable | $0.10 \pm 0.07$, marginal |
| PPP–CCC floor gap | Always positive | Fluctuates, sign reversals |
| Algebraic vacuum | $\mathrm{rad}_{\mathrm{odd}}(M) = 1$ at $N = 2^k$ | None |

Table 2: Binary vs. ternary conductor framework: a structural comparison.

This paper thus serves two purposes. First, it precisely delineates the *applicability boundary* of the conductor rigidity programme: the framework is native to genus 2 and cannot be directly lifted to higher genera. Second, by identifying the exact algebraic mechanism that fails ($N$ as independent discriminant factor), it reinforces the understanding of *why* binary conductor rigidity works: the identity $p + q = N$ embeds the target integer into the discriminant with a universality that has no analogue in ternary or higher-order decompositions.

The binary Goldbach conjecture occupies a privileged position not because it is the simplest additive problem, but because it is the *only* additive problem where the target integer enters the Frey discriminant as a universal geometric invariant.

## Acknowledgments

The computational scans were performed using `ternary_geometric.py`, available at https://github.com/Ruqing1963/goldbach-ternary-conductor-boundary. All results were produced with Python 3.12, NumPy 2.4, and Matplotlib 3.10. This work builds on [3, 2, 1].

An earlier version of this paper (circulated as "Ternary Conductor Rigidity in GSp(6)") contained a flawed conductor proxy that artificially injected $\mathrm{rad}_{\mathrm{odd}}(N)$ into the discriminant. The author thanks the reviewer whose analysis of the true genus-3 discriminant led to the corrected treatment presented here.

## References

[1] R. Chen, *The Goldbach mirror II: geometric foundations of conductor rigidity and the static conduit in* GSp(4), Zenodo, 2026. https://zenodo.org/records/18719056

[2] R. Chen, *The algebraic vacuum: zero-ramification conductor model for the Goldbach conjecture at* $N = 2^k$, Zenodo, 2026. https://zenodo.org/records/18720040

[3] R. Chen, *Dynamic stability of the Goldbach locus: conductor orbit propagation and the Band Shifting Law in* GSp(4), Zenodo, 2026.

[4] I. M. Vinogradov, *Representation of an odd number as a sum of three primes*, Dokl. Akad. Nauk SSSR **15** (1937), 169–172.

[5] H. A. Helfgott, *The ternary Goldbach conjecture is true*, arXiv:1312.7748, 2013.