# Oppermann's Parity Law: Quadratic Residue Symmetry Breaking in Half-Intervals via the Negation Involution


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Repository: https://github.com/Ruqing1963/oppermann-parity-law



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**Abstract**

We refine the Spin Asymmetry Theorem of [5] by analyzing the distribution of quadratic residues in the *left* and *right halves* of Legendre intervals, motivated by Oppermann's conjecture on primes in half-intervals. For $n \geq 2$ with $p = 2n - 1$ prime, the interior of $\mathcal{I}_n = [(n-1)^2, n^2]$ splits at the midpoint $n(n-1)$ into a left half $\mathcal{L}_n$ and a right half $\mathcal{R}_n$, each containing $n-1$ integers. We prove that when $n$ is *even* (equivalently $p \equiv 3 \pmod 4$), the right half $\mathcal{R}_n$ has *exactly zero* quadratic residue skewness modulo $p$: the number of QR classes equals the number of NR classes. The proof uses the negation involution $x \mapsto p - x$ on $(\mathbb{Z}/p\mathbb{Z})^*$, which swaps QR $\leftrightarrow$ NR when $p \equiv 3 \pmod 4$, combined with the fact that $\mathcal{R}_n$ is closed under this involution. Consequently, the global skewness $N^- - N^+ = 1$ from [5] is carried *entirely* by the left half. The theorem is verified computationally for all 87 qualifying even $n \leq 500$. We also prove a complementary result for odd $n$ ($p \equiv 1 \pmod 4$): the right half skewness is always even and generically nonzero.


## 1 Introduction

### 1.1 Oppermann's conjecture

Oppermann's conjecture (1882) strengthens Legendre's conjecture by asserting that for every $n \geq 2$, there exists a prime in *each* of the two half-intervals

$$(n^2 - n,\, n^2) \quad \text{and} \quad (n^2,\, n^2 + n).$$

Equivalently, using our indexing of the Legendre interval $\mathcal{I}_n = [(n-1)^2, n^2]$, the conjecture demands primes on *both* sides of the midpoint $n(n-1)$.

In the companion paper [5], we proved the Spin Asymmetry Theorem: the interior of $\mathcal{I}_n$ (when $p = 2n - 1$ is prime) has exactly one more quadratic non-residue than quadratic residue modulo $p$. This global asymmetry was left undissected—it was not known how the skewness distributes between the two halves.

### 1.2 Main result: the Parity Law

We prove (Theorem 4.4) that for even $n$, the global skewness is distributed in the most extreme possible way: the right half carries *zero* skewness, and the left half carries *all* of the asymmetry.

The proof reveals a clean algebraic mechanism: the negation involution $\iota : x \mapsto p - x$ on $(\mathbb{Z}/p\mathbb{Z})^*$ swaps QR $\leftrightarrow$ NR precisely when $p \equiv 3 \pmod 4$ (i.e., when $n$ is even), and the residue classes of $\mathcal{R}_n$ are closed under $\iota$. This forces perfect QR/NR pairing in $\mathcal{R}_n$.

### 1.3 Context within the Titan Project

This paper continues the algebraic programme initiated in [5], building on the conductor rigidity framework of [1, 2, 3] and the function field resolution in [4].

## 2 Setup and Preliminaries

**Definition 2.1** (Half-intervals)**.** For $n \geq 2$, the *left half* and *right half* of the Legendre interval $\mathcal{I}_n = [(n-1)^2, n^2]$ are

$$\mathcal{L}_n = \{(n-1)^2 + 1, \ \ldots, \ n(n-1)\},$$
$$\mathcal{R}_n = \{n(n-1) + 1, \ \ldots, \ n^2 - 1\}.$$

Each half has cardinality $n - 1$. The full interior is $\mathcal{I}_n^\circ = \mathcal{L}_n \cup \mathcal{R}_n$.

**Definition 2.2** (Local skewness)**.** For a set $S$ of integers and an odd prime $p$, define

$$\mathrm{Skew}_p(S) \ = \ \#\{x \in S : \left(\frac{x}{p}\right) = -1\} \ - \ \#\{x \in S : \left(\frac{x}{p}\right) = +1\}.$$

From [5], we know $\mathrm{Skew}_p(\mathcal{I}_n^\circ) = 1$ whenever $p = 2n - 1$ is prime. By additivity:

$$\mathrm{Skew}_p(\mathcal{L}_n) + \mathrm{Skew}_p(\mathcal{R}_n) = 1. \tag{1}$$

## 3 The Anchor Position

By [5, Lemma 2.4], the interior $\mathcal{I}_n^\circ$ contains a unique multiple of $p$ (the *anchor*, with Legendre symbol 0). Its position determines which half inherits the anchor.

**Lemma 3.1** (Anchor offset)**.** *Let $n \geq 2$ with $p = 2n - 1$ prime. The unique multiple of $p$ in $\mathcal{I}_n^\circ$ is located at offset*

$$\delta \ = \ \begin{cases} \dfrac{3n-2}{2} & \text{if } n \text{ is even (i.e., } p \equiv 3 \pmod 4\text{)}, \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd (i.e., } p \equiv 1 \pmod 4\text{)}, \end{cases}$$

*from the left endpoint $(n-1)^2$. In particular:*

*(i) If $n$ is even, then $\delta \geq n$, so the anchor lies in $\mathcal{R}_n$.*

*(ii) If $n$ is odd, then $\delta < n$, so the anchor lies in $\mathcal{L}_n$.*

*Proof.* The anchor $m$ satisfies $m \equiv 0 \pmod p$ and $(n-1)^2 < m < n^2$. Its offset from $(n-1)^2$ is $\delta = m - (n-1)^2 \equiv -(n-1)^2 \equiv -n^2 \pmod p$ (using $(n-1)^2 \equiv n^2 \pmod p$ from [5]).

From $4n^2 = (2n-1)(2n+1) + 1$, we get $4n^2 \equiv 1 \pmod p$, so $n^2 \equiv 4^{-1} \pmod p$.

*Case $n$ even* ($p \equiv 3 \pmod 4$): $4^{-1} \equiv (p+1)/4 \pmod p$ since $4 \cdot (p+1)/4 = p + 1 \equiv 1$. Thus $\delta \equiv -\frac{p+1}{4} \equiv \frac{3p-1}{4} = \frac{3(2n-1)-1}{4} = \frac{3n-2}{2} \pmod p$. Since $n \leq (3n-2)/2 \leq 2n-2$ for $n \geq 2$, $\delta$ falls in the right half.

*Case $n$ odd* ($p \equiv 1 \pmod 4$): $4^{-1} \equiv (3p+1)/4 \pmod p$ since $4 \cdot (3p+1)/4 = 3p+1 \equiv 1$. Thus $\delta \equiv -\frac{3p+1}{4} \equiv \frac{p-1}{4} = \frac{n-1}{2} \pmod p$. Since $1 \leq (n-1)/2 \leq n-1$ for $n \geq 3$, $\delta$ falls in the left half. $\square$

# 4 The Negation Involution and the Parity Law

## 4.1 The involution

**Definition 4.1.** The *negation involution* on $\mathbb{Z}/p\mathbb{Z}$ is the map $\iota(x) = p - x \pmod p$. On the nonzero elements, $\iota$ is a fixed-point-free involution (since $p$ is odd, $\iota(x) = x$ implies $2x \equiv 0$, hence $x \equiv 0$).

**Lemma 4.2** (Spin-flip lemma). *For $a \not\equiv 0 \pmod p$:*

$$\left(\frac{p-a}{p}\right) = \left(\frac{-1}{p}\right) \cdot \left(\frac{a}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & \text{if } p \equiv 3 \pmod 4, \\ +\left(\frac{a}{p}\right) & \text{if } p \equiv 1 \pmod 4. \end{cases}$$

*Proof.* $\left(\frac{p-a}{p}\right) = \left(\frac{-a}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{a}{p}\right)$. By the classical evaluation, $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$, which is $-1$ when $p \equiv 3$ and $+1$ when $p \equiv 1$. □

## 4.2 Involution closure of the right half

**Lemma 4.3.** *Let $n \geq 2$ with $p = 2n - 1$ prime. Let $S_R$ denote the set of residues modulo $p$ of elements of $\mathcal{R}_n$, and let $S_R^* = S_R \setminus \{0\}$. Then $\iota(S_R^*) = S_R^*$; that is, $S_R^*$ is closed under the negation involution.*

*Proof.* The elements of $\mathcal{R}_n$ have offsets $n, n+1, \ldots, 2n-2$ from $(n-1)^2$. Since $(n-1)^2 \equiv r \pmod p$ (where $r = n^2 \bmod p$), the residues of $\mathcal{R}_n$ are $\{r+n,\ r+n+1,\ \ldots,\ r+2n-2\} \pmod p$, a set of $n-1$ consecutive residue classes.

For $a = r + j$ with $n \leq j \leq 2n-2$, we have $\iota(a) = p - r - j$. From the proof of Lemma 3.1: $r \equiv n^2 \equiv 4^{-1} \pmod p$, so $p - 2r \equiv p - 2 \cdot 4^{-1} \equiv p - (p+1)/2 \equiv (p-1)/2 = n - 1 \pmod p$ (when $p \equiv 3 \bmod 4$; the $p \equiv 1$ case is similar).

Thus $\iota(a) = (n-1) - j + r \pmod p$, and we need to check that $(n-1) - j$ maps $\{n, \ldots, 2n-2\}$ to itself modulo $p$. Indeed, as $j$ ranges from $n$ to $2n-2$, $(n-1) - j$ ranges from $-1$ to $-(n-1)$, which modulo $p = 2n-1$ is $2n-2$ down to $n$. This is exactly the same set.

The argument is identical for $p \equiv 1 \pmod 4$ (using the corresponding value of $4^{-1}$). □

## 4.3 The main theorem

**Theorem 4.4** (Oppermann Parity Law). *Let $n \geq 2$ be an* even *integer with $p = 2n - 1$ prime. Then:*

*(i) The right half $\mathcal{R}_n$ contains $(n-2)/2$ quadratic residues, $(n-2)/2$ quadratic non-residues, and 1 multiple of $p$:*

$$\mathrm{Skew}_p(\mathcal{R}_n) = 0.$$

*(ii) The left half $\mathcal{L}_n$ contains $(n-2)/2$ quadratic residues and $n/2$ quadratic non-residues:*

$$\mathrm{Skew}_p(\mathcal{L}_n) = 1.$$

*(iii) The global skewness $N^- - N^+ = 1$ is carried entirely by the left half.*

*Proof.* Since $n$ is even, $p \equiv 3 \pmod 4$ (Lemma 3.1).

**(i)** By Lemma 3.1(i), the anchor lies in $\mathcal{R}_n$. The remaining $n-2$ elements of $\mathcal{R}_n$ have nonzero residues forming the set $S_R^*$. By Lemma 4.3, $S_R^*$ is closed under $\iota$. Since $n-2$ is even (as $n$ is even), $\iota$ partitions $S_R^*$ into $(n-2)/2$ orbits of size 2. By Lemma 4.2, each orbit $\{a, \iota(a)\}$ contains one QR and one NR (since $p \equiv 3 \pmod 4$). Therefore $N_R^+ = N_R^- = (n-2)/2$ and $\mathrm{Skew}_p(\mathcal{R}_n) = 0$.

**(ii)** Follows from $\mathrm{Skew}_p(\mathcal{L}_n) = 1 - \mathrm{Skew}_p(\mathcal{R}_n) = 1$ via (1).

**(iii)** Immediate from (i) and (ii). □

### 4.4 The complementary case: odd $n$

**Theorem 4.5.** *Let $n \geq 3$ be an* odd *integer with $p = 2n-1$ prime. Then the right half skewness* $\mathrm{Skew}_p(\mathcal{R}_n)$ *is always* even *(but generically nonzero).*

*Proof.* When $n$ is odd, $p \equiv 1 \pmod 4$ and the anchor lies in $\mathcal{L}_n$ (Lemma 3.1(ii)). Thus $\mathcal{R}_n$ has $n-1$ elements, all with nonzero residues. By Lemma 4.3, $S_R^*$ is still closed under $\iota$, and since $n-1$ is even, $\iota$ partitions $S_R^*$ into $(n-1)/2$ pairs. But now $p \equiv 1 \pmod 4$, so by Lemma 4.2, $\iota$ *preserves* the Legendre symbol: each pair $\{a, \iota(a)\}$ consists of two QRs or two NRs. Hence $N_R^+$ and $N_R^-$ are both even, and $\mathrm{Skew}_p(\mathcal{R}_n) = N_R^- - N_R^+$ is even. $\square$

*Remark* 4.6. The contrast between even and odd $n$ is entirely controlled by the single arithmetic invariant $\left(\frac{-1}{p}\right)$:

| $n$ parity | $p$ mod 4 | $\left(\frac{-1}{p}\right)$ | $\iota$ action on QR/NR |
|---|---|---|---|
| even | 3 | $-1$ | swaps QR $\leftrightarrow$ NR $\Rightarrow$ pairing $\Rightarrow$ skewness $= 0$ |
| odd | 1 | $+1$ | preserves QR, NR $\Rightarrow$ no cross-pairing $\Rightarrow$ skewness even |

## 5 Computational Verification

### 5.1 Methodology

For each $n$ with $p = 2n-1$ prime, the scanner computes the Jacobi symbol $\left(\frac{x}{p}\right)$ for every integer $x$ in each half-interval, tallies positive/negative/zero spins, and records the skewness $N^- - N^+$ for each half.

### 5.2 Results for even $n$

Table 1: Half-interval spin distribution for even $n$ ($p \equiv 3 \pmod 4$). In every case, $\mathrm{Skew}(\mathcal{R}_n) = 0$ and the anchor lies in $\mathcal{R}_n$.

| $n$ | $p$ | $N_L^+$ | $N_L^-$ | $N_L^0$ | $\mathrm{Skew}_L$ | $N_R^+$ | $N_R^-$ | $\mathrm{Skew}_R$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 0 | 1 | 0 | 1 | 0 | 0 | **0** |
| 4 | 7 | 1 | 2 | 0 | 1 | 1 | 1 | **0** |
| 6 | 11 | 2 | 3 | 0 | 1 | 2 | 2 | **0** |
| 10 | 19 | 4 | 5 | 0 | 1 | 4 | 4 | **0** |
| 12 | 23 | 5 | 6 | 0 | 1 | 5 | 5 | **0** |
| 16 | 31 | 7 | 8 | 0 | 1 | 7 | 7 | **0** |
| 22 | 43 | 10 | 11 | 0 | 1 | 10 | 10 | **0** |
| 100 | 199 | 49 | 50 | 0 | 1 | 49 | 49 | **0** |

**Proposition 5.1.** *Theorem 4.4 has been verified computationally for all 87 even values of $n \leq 500$ with $2n-1$ prime. In every case,* $\mathrm{Skew}_p(\mathcal{R}_n) = 0$ *and* $\mathrm{Skew}_p(\mathcal{L}_n) = 1$.

### 5.3 Results for odd $n$

Table 2: Half-interval spin distribution for odd $n$ ($p \equiv 1 \pmod 4$). The right half skewness is always even but varies. The anchor lies in $\mathcal{L}_n$.

| $n$ | $p$ | $N_L^+$ | $N_L^-$ | $N_L^0$ | Skew$_L$ | $N_R^+$ | $N_R^-$ | Skew$_R$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 1 | 0 | 1 | $-1$ | 0 | 2 | 2 |
| 7 | 13 | 3 | 2 | 1 | $-1$ | 2 | 4 | 2 |
| 9 | 17 | 5 | 2 | 1 | $-3$ | 2 | 6 | 4 |
| 15 | 29 | 9 | 4 | 1 | $-5$ | 4 | 10 | 6 |
| 21 | 41 | 13 | 6 | 1 | $-7$ | 6 | 14 | 8 |

# 6 Structural Interpretation

## 6.1 The right half as an algebraically neutral zone

For even $n$, the right half $\mathcal{R}_n$ is in a state of *perfect algebraic equilibrium*: its quadratic residue distribution is exactly symmetric. This is not a statistical accident; it is forced by the involution $\iota$ acting as a QR/NR swap on a closed domain.

The left half, by contrast, carries the full topological defect: it is the sole repository of the global skewness $+1$. This asymmetry between the halves echoes the asymmetry in Oppermann's conjecture itself, which treats the left and right half-intervals as structurally distinct.

## 6.2 Obstruction interpretation

A hypothetical "prime vacuum" in $\mathcal{R}_n$ (all elements composite) would require $n-1$ consecutive composites maintaining *exact* QR/NR balance modulo $p$. While the Parity Law proves this balance is algebraically mandated (regardless of whether primes are present), the *multiplicative* realization of such a balanced configuration by composites alone faces additional constraints not captured by the residue structure.

We emphasize that the Parity Law does not by itself prove Oppermann's conjecture. It establishes a structural framework—the algebraic neutrality of $\mathcal{R}_n$ and the defect-carrying role of $\mathcal{L}_n$—within which the conjecture can be studied.

## 6.3 Limitations

The Parity Law applies only when $p = 2n - 1$ is prime, which restricts to roughly half of all $n$ values. When $p$ is composite, the Jacobi symbol replaces the Legendre symbol, the QR/NR counts are no longer simply $(p-1)/2$ each, and the involution argument does not directly apply. Extending the framework to composite $p$ is an open problem.

# 7 Discussion

## 7.1 Relation to character sums

The Parity Law can be restated as a character sum identity. For even $n$ with $p = 2n - 1$ prime:

$$\sum_{x \in \mathcal{R}_n} \left(\frac{x}{p}\right) \;=\; N_R^+ - N_R^- \;=\; 0.$$

This is a *partial character sum* over $n-1$ consecutive integers. While partial character sums are typically bounded by $O(\sqrt{p}\log p)$ via the Pólya–Vinogradov inequality, here we obtain an *exact* evaluation of zero—a much stronger result, albeit for a specific sub-interval determined by the arithmetic of $n$.

### 7.2 Connection to Paper IX

The Spin Asymmetry Theorem of [5] established the global identity $N^- - N^+ = 1$. The present paper decomposes this identity into local contributions:

$$\underbrace{1}_{\text{left}} + \underbrace{0}_{\text{right}} = 1 \qquad (\text{even } n), \qquad\qquad \underbrace{1-R}_{\text{left}} + \underbrace{R}_{\text{right}} = 1 \qquad (\text{odd } n,\ R \text{ even}).$$

The even case is maximally rigid; the odd case distributes the skewness in a manner that remains to be fully understood.

### 7.3 Open questions

(1) *Odd $n$ skewness.* For odd $n$, the right half skewness $\mathrm{Skew}_p(\mathcal{R}_n)$ is always a positive even integer in our data. Is it always positive? Is there a closed-form expression in terms of class numbers or character sums?

(2) *Sub-interval character sums.* Can the exact vanishing of the partial character sum over $\mathcal{R}_n$ be derived from classical results on Gauss sums or Jacobi sums?

(3) *Composite lengths.* When $p = 2n-1$ is composite, can a modified involution argument yield analogous (approximate) parity constraints?

(4) *Multiplicative refinement.* Among the $(n-2)/2$ QR–NR pairs in $\mathcal{R}_n$ (for even $n$), how many pairs consist of a prime and a composite? This would connect the algebraic structure to the distribution of primes in the half-interval.

### Data and code availability

Source code and the LaTeX manuscript are available at `https://github.com/Ruqing1963/oppermann-parity-l`
The companion Paper IX repository is at `https://github.com/Ruqing1963/legendre-spin-asymmetry`.

## References


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