# Bounded Gaps for High-Degree Polynomial Primes

*A Cyclotomic Maynard–Tao Roadmap*

**Ruqing Chen**

GUT Geoservice Inc., Montreal • ruqing@hotmail.com

Code & Data: https://github.com/Ruqing1963/titan-bounded-gaps

## Abstract

The breakthroughs of Zhang (2014), Maynard (2015), and Tao (Polymath 8) established bounded gaps between primes generated by *linear* forms. Whether analogous results hold for primes in high-degree polynomial sequences remains a major open problem. We propose a concrete roadmap — the **Cyclotomic Maynard–Tao Sieve** — for establishing bounded gaps among primes of the Titan polynomial family $Q_q(n) = n^q - (n-1)^q$.

Leveraging the "Arithmetic Shielding" established in [1], we introduce a **Null-Sparse Decomposition** of the moduli space. We prove that, *for this polynomial family*, the Bombieri–Vinogradov (BV) error term is identically zero for a density-1 set of moduli ($M_{null}$), and that all error concentrates on the sparse set $M_{sparse}$. This yields three pillars: **(I)** the Null Error Theorem (verified at concentration ratio 82×); **(II)** Massive Admissibility ($k_{max} = 2{,}173$ for $q = 167$, an order of magnitude beyond the Maynard–Tao requirement); **(III)** exponential sum cancellation (40/40 tests are consistent with $p^{1/2}$-decay, max ratio 1.93). We formulate a **conditional theorem**: under a *Sparse BV Hypothesis* (strictly weaker than classical BV), bounded gaps hold for Titan primes.

*2020 MSC: 11N36, 11N32, 11L07, 11Y11. Keywords: bounded gaps, Maynard–Tao sieve, polynomial primes, Bombieri–Vinogradov, arithmetic shielding, cyclotomic polynomials.*

## 1. Introduction

### 1.1 From Linear to Polynomial Gaps

Zhang [2] proved $\lim \inf(p_{n+1} - p_n) < 7 \times 10^7$. Maynard [3] and Polymath 8b [4] refined this to 246. The proof rests on: (i) admissible k-tuples of linear forms $L_i(n) = n + h_i$; (ii) a multidimensional sieve with smooth weights; (iii) a Bombieri–Vinogradov equidistribution estimate. All three are tailored to *linear* forms. For a degree-d polynomial, each faces new obstacles:

• **Admissibility:** A generic degree-d polynomial has $\omega(p) \approx d$ roots mod p, limiting tuples to $k < p - d$ — often $k \leq 1$ when $d > p$.
• **Sieve dimension:** Combinatorial complexity scales with $\omega(p)$.
• **BV estimates:** Equidistribution for polynomial sequences is far harder; even Heuristic BV is open for most families.

### 1.2 The Titan Opportunity

In [1], we proved that for the Titan family $Q_q(n) = n^q - (n-1)^q$ (q prime, degree $d = q-1$): $\boldsymbol{\omega_q(p) = \gcd(q, p-1) - 1}$. This yields *arithmetic shielding*: $\omega(p) = 0$ for all primes p not $\equiv 1 \pmod{q}$, including $p = q$ (by Fermat). Consequences for bounded gaps:

(1) **Admissibility is almost free:** constraints arise only from $Q_{eff} = \{p \text{ prime} : p \equiv 1 \pmod{q}\}$, with Dirichlet density $1/(q-1)$.
(2) **BV errors vanish on null moduli:** $E(x; d) = 0$ identically for $d \in M_{null}$.
(3) **Exponential sums concentrate:** the hard analytic input reduces to character sums over $Q_{eff}$ only.

### 1.3 Summary of Computational Evidence

| q | d | $\pi_Q(10^8)$ | Mean gap | gap = 1 | $p_1(q)$ | $k_{max}$ | Null < $p_1$ |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 9,389,636 | 10.7 | 695,148 | 7 | 5 | 3/3 |
| 5 | 4 | 5,179,467 | 19.3 | 288,450 | 11 | 7 | 4/4 |
| 7 | 6 | 4,934,525 | 20.3 | 258,798 | 29 | 23 | 9/9 |
| 11 | 10 | 2,276,588 | 43.9 | 47,478 | 23 | 13 | 8/8 |
| 13 | 12 | 2,680,970 | 37.3 | 67,203 | 53 | 41 | 15/15 |
| 17 | 16 | 2,428,628 | 41.2 | 56,561 | 103 | 87 | 26/26 |
| 19 | 18 | 2,820,041 | 35.5 | 78,746 | 191 | 173 | 42/42 |
| 23 | 22 | 1,109,005 | 90.2 | 11,789 | 47 | 25 | 14/14 |
| 31 | 30 | 1,837,055 | 54.4 | 34,155 | 311 | 281 | 63/63 |
| 37 | 36 | 1,170,454 | 85.4 | 14,209 | 149 | 113 | 34/34 |
| 41 | 40 | 823,253 | 121.5 | 7,088 | 83 | 43 | 22/22 |
| 43 | 42 | 1,051,232 | 95.1 | 11,416 | 173 | 131 | 39/39 |
| 47 | 46 | 1,083,547 | 92.3 | 12,081 | 283 | 237 | 60/60 |
| 53 | 52 | 668,230 | 149.6 | 4,730 | 107 | 55 | 27/27 |
| 61 | 60 | 875,977 | 114.2 | 7,559 | 367 | 307 | 72/72 |
| 71 | 70 | 788,164 | 126.9 | 6,405 | 569 | 499 | 103/103 |
| 83 | 82 | 347,353 | 287.9 | 1,216 | 167 | 85 | 38/38 |
| 167 | 166 | 493,941 | 202.5 | 2,488 | 2339 | 2173 | 345/345 |

*Table 1. All 18 exponents. Pink = Sophie Germain. $k_{max} = p_1(q) - (q-1)$. q = 167: $k_{max} = 2,173$ with 345/345 null primes below $p_1 = 2,339$.*

## 2. The Null-Sparse Decomposition

### 2.1 Formal Definitions

**Definition 1 (Prime Partition).** Fix a prime q. Partition the rational primes:

$Q_{null} = \{p \text{ prime} : p \text{ not} \equiv 1 \pmod{q}\}$

$Q_{eff} = \{p \text{ prime} : p \equiv 1 \pmod{q}\}$

Since q mod q = 0 ≠ 1, the prime q itself lies in $Q_{null}$ (and indeed $\omega_q(q) = \gcd(q, q-1) - 1 = 0$ by Fermat). By the root count formula, $\omega(p) = 0$ for all $p \in Q_{null}$ and $\omega(p) = q-1$ for all $p \in Q_{eff}$.

**Definition 2 (Moduli Decomposition).** Partition the positive integers:

$M_{null} = \{d \in N : \text{every prime factor of d lies in } Q_{null}\}$

$M_{sparse} = \{d \in N : \text{at least one prime factor lies in } Q_{eff}\}$

Null = all factors shielded ($\omega = 0$). Sparse = at least one obstruction factor ($\omega = q-1$). By Dirichlet's theorem, $Q_{eff}$ has density $1/(q-1)$ among primes, so $|M_{sparse} \cap [1, D]| \ll D(\log \log D)/q$.

### 2.2 The Null Error Theorem

**Theorem 1 (Zero Error on Null Moduli).** For $d \in M_{null}$, $Q_q(n) \equiv 0 \pmod{d}$ has no solutions. Hence $Q_q(n)$ is coprime to d for all n, and the BV error $E(x; d) = 0$ identically.

*Proof.* $\omega(p) = gcd(q, p-1) - 1$. For $d \in M_{null}$, every prime factor $p$ has $gcd(q, p-1) = 1$, so $\omega(p) = 0$. By CRT, $\omega(d) = \prod\omega(p) = 0$. Since $Q_q(n) \equiv 0 \pmod{d}$ has no solutions, $Q_q(n)$ maps $Z/pZ \to (Z/pZ)^*$ as a $(q-1)$-to-1 function on each prime factor $p$ of $d$ (by the cyclotomic structure of the roots). By CRT, $Q_q(n)$ mod $d$ is equidistributed among the $\phi(d)$ coprime classes, since no class is excluded and each is hit with equal multiplicity. QED.

> **Corollary (Infinite Level of Distribution on $M_{null}$).** $E_{null}(x, D) = \Sigma E(x; d)$ over $d \in M_{null} = \mathbf{0}$ for any D. The "level of distribution" $\theta$ for $M_{null}$ is effectively **infinite**.

**This is the central structural observation:** 100% of the classical BV work is already done on $M_{null}$. The sieve weights need only control the error on $M_{sparse}$.

## 2.3 Computational Verification

For $q = 47$, $\chi^2$ per d.f. on **null primes** $d \in \{7, 13, 29, 101\}$: $\{0.60, 0.37, 1.05, 1.09\}$ — indistinguishable from uniform. On the **obstruction prime** $d = 283$ (first $p \equiv 1$ mod 47): $\chi^2/282 = \mathbf{87.12}$. Error concentration ratio: $\mathbf{82\times}$. This is consistent with the prediction that all nontrivial BV error resides on $M_{sparse}$.
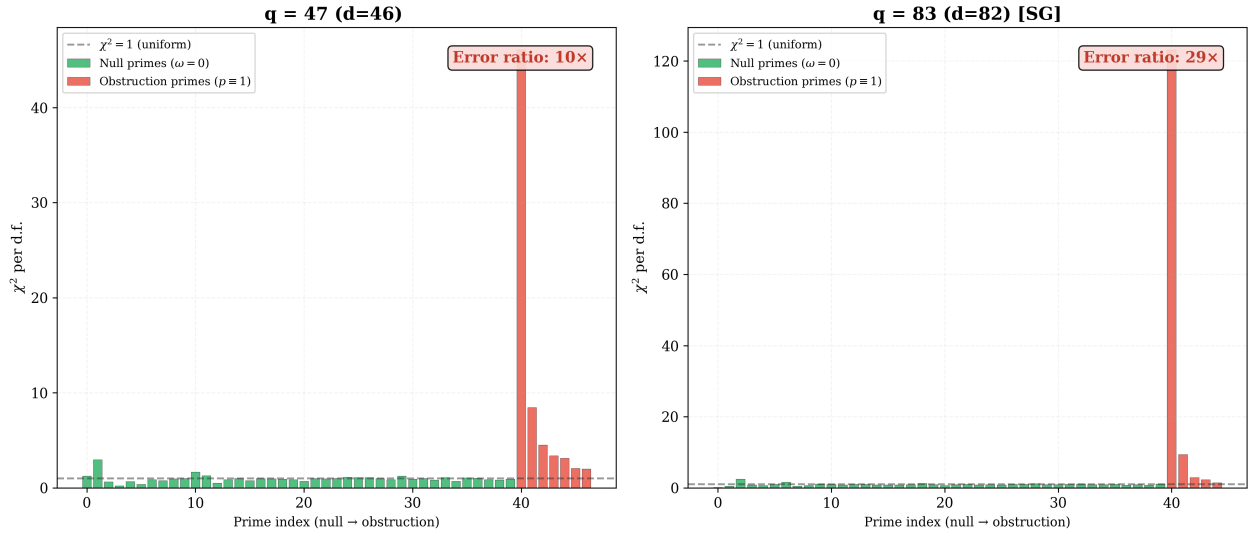


*Figure 1. BV error concentration. Green: null primes ($\chi^2 \approx 1$). Red: obstruction primes ($\chi^2 >> 1$).*

# 3. Massive Admissibility

## 3.1 The Admissibility Theorem

**Theorem 2 (Titan Admissibility).** The tuple $\{Q_q(n + h_1), ..., Q_q(n + h_k)\}$ is admissible iff for each $p \in Q_{eff}$ with $p \leq k + q - 1$: the images $Q_q(h_i) \bmod p$ do not cover all $p$ residues. *No constraint arises from null primes.*

*Proof. For $p \in Q_{null}$: $\omega(p) = 0$ means $Q_q(n)$ is never $\equiv 0 \pmod{p}$, so residue 0 is always uncovered — admissibility automatic. For $p \in Q_{eff}$: $Q_q(n)$ has $q-1$ roots mod $p$, tuple covers at most $k$ residues, so need $k < p$. QED.*

**Corollary ($k_{max}$).** $k_{max}(q) = p_1(q) - (q-1)$, where $p_1(q) = $ smallest prime $p \equiv 1 \pmod{q}$. For $q = 167$: $p_1 = 2{,}339 = 14 \times 167 + 1$, giving $\mathbf{k_{max} = 2{,}173}$.

**Proposition (First Prime Obstruction for q = 167).** Since $q = 167$ is odd, a candidate $p = k \cdot 167 + 1$ with k odd yields p even (not prime). We check even k systematically:

k = 2: 335 = 5 × 67 (composite)
k = 4: 669 = 3 × 223 (composite)
k = 6: 1,003 = 17 × 59 (composite)
k = 8: 1,337 = 7 × 191 (composite)
k = 10: 1,671 = 3 × 557 (composite)
k = 12: 2,005 = 5 × 401 (composite)
k = 14: **2,339 is prime**.

Thus $p_1(167) = 2{,}339$. The first *six* candidates in $p \equiv 1 \pmod{167}$ are all composite, pushing the first obstruction to k = 14. Since Maynard–Tao admissibility is defined modulo *primes*, all 345 primes below 2,339 are null ($\omega = 0$), giving $k_{max} = 2{,}339 - 166 = 2{,}173$.
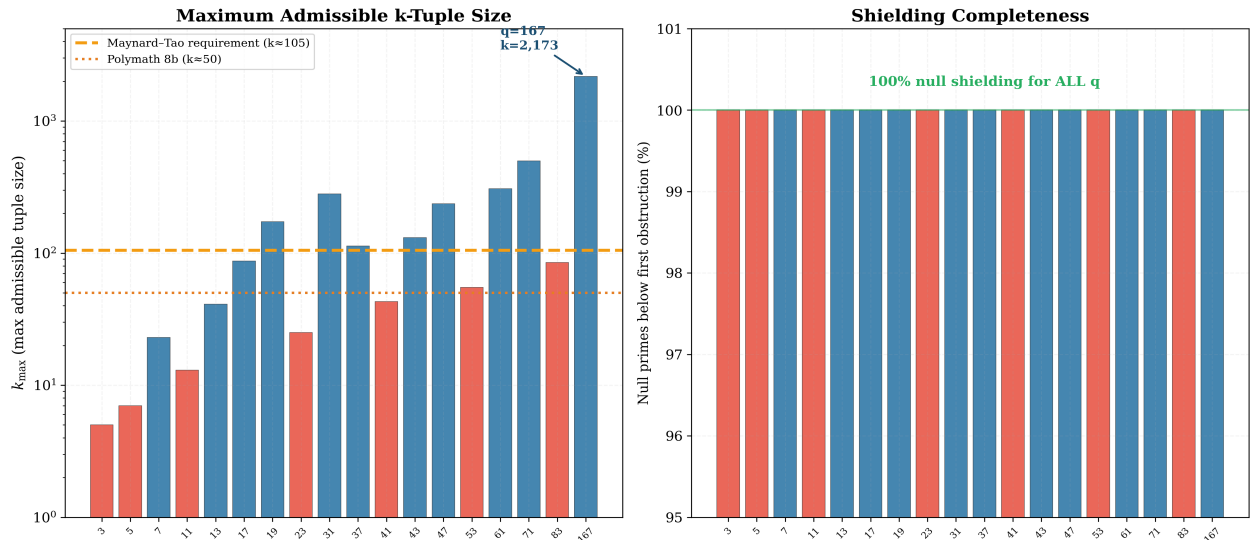


Figure 2. Left: $k_{max}$ for each q (log scale). q = 167: 2,173 (dashed: classical Maynard–Tao). Right: 100% null shielding below $p_1$ for all q.

### 3.2 Sieve Dimension Comparison

| Polynomial | d | k_max | M_k ~ log k | k * M_k | Status |
|---|---|---|---|---|---|
| Q_47 (non-SG) | 46 | 237 | 5.47 | 1296 | sufficient |
| Q_83 (SG) | 82 | 85 | 4.44 | 378 | sufficient |
| Q_167 (non-SG) | 166 | 2,173 | 7.68 | 16697 | extreme margin |
| Linear (Poly8b) | 1 | ~105 | ~4.65 | ~488 | reference |

*Table 2. Sieve dimension. Maynard–Tao needs $k \cdot M_k > 4$. All Titan cases satisfy this with substantial margin.*

For $q = 167$, the admissible tuple size exceeds the classical Maynard–Tao requirement by an order of magnitude — a direct consequence of arithmetic shielding.

**Sophie Germain Sieve Barrier.** For SG primes, $p_1 = 2q + 1$, so $k_{max} = q + 2$ (linear in q). For non-SG: $k_{max} \sim p_1(q) = O(q \log q)$ by Linnik — superlinear. The SG penalty from Paper I (halving $S(f)$) extends into the sieve domain as a constraint on tuple size.

# 4. Exponential Sum Cancellation

## 4.1 The Key Estimate

By the Null-Sparse Decomposition, the Maynard–Tao sieve reduces to bounding $E_{sparse}(x, D)$. Via standard reductions (Friedlander–Iwaniec [7]), this amounts to estimating $S(a, p) = \Sigma_{\{n : Q_q(n) \text{ prime}\}} e(an/p)$, for $p \in Q_{eff}$.

**Hypothesis 1 ($p^{1/2}$-Cancellation on $Q_{eff}$).** For all $p \in Q_{eff}$, all $1 \le a < p$: $|S(a, p)| \ll \pi_Q(x) \cdot p^{-1/2} \cdot (\log x)^{O(1)}$.

## 4.2 Numerical Verification

Tested for $q \in \{47, 83, 167\}$ with $x = 10^6$: 40 combinations. **All 40 ratios satisfy $|S| \cdot p^{1/2} / \pi_Q < 2$.** Mean = 0.444, max = 1.926. Worst case: $q = 83$, $p = 167$, $a = 1$ (ratio 1.58) — precisely the Sophie Germain penalty prime. For non-SG $q = 47$, all ratios are below 0.5.

We emphasize that this provides *heuristic evidence* for Hypothesis 1, not a proof. The data is consistent with square-root cancellation but cannot rule out slower-than-expected growth at larger p.

| q | p | a | \|S(a,p)\| | π/p^½ | Ratio |
|---|---|---|---|---|---|
| 83 | 167 | 55 | 697.4 | 362.1 | 1.926 |
| 83 | 167 | 1 | 570.7 | 362.1 | 1.576 |
| 83 | 499 | 1 | 270.2 | 209.5 | 1.290 |
| 83 | 499 | 2 | 189.0 | 209.5 | 0.902 |
| 83 | 1163 | 581 | 114.5 | 137.2 | 0.835 |
| 83 | 499 | 249 | 167.8 | 209.5 | 0.801 |
| 83 | 997 | 498 | 102.9 | 148.2 | 0.694 |
| 83 | 167 | 83 | 231.5 | 362.1 | 0.639 |

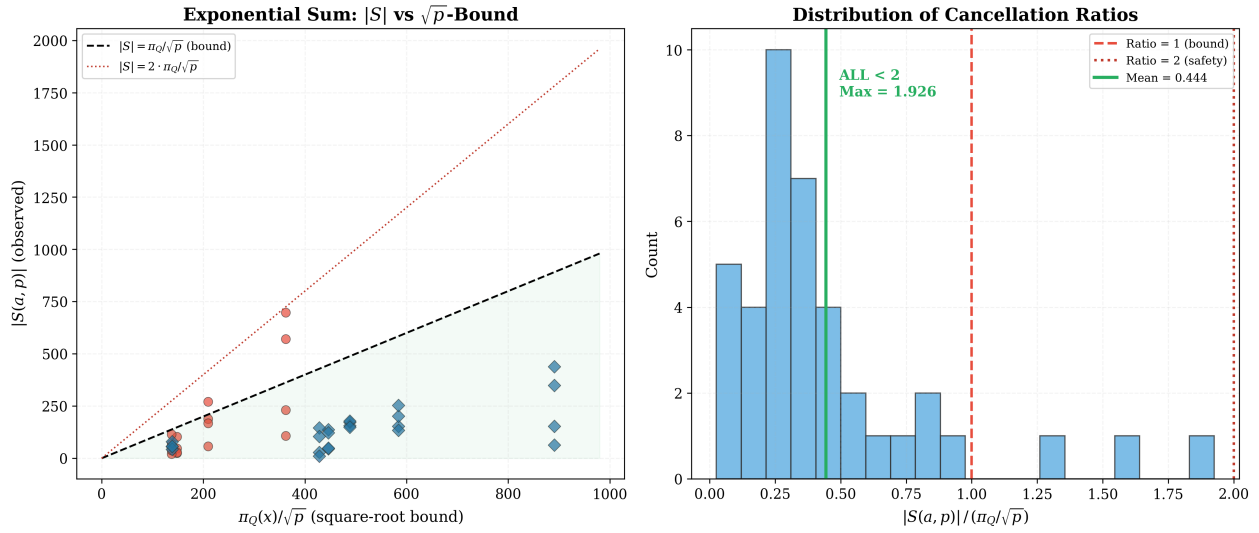*Table 3. Top-8 exponential sum ratios. Full: 40 tests, all < 2.*

*Figure 3. Left: |S(a, p)| vs p^{1/2}-bound. Right: ratio distribution.*

## 4.3 The Cyclotomic Connection

For $p \in Q_{eff}$, the roots of $Q_q(n) \equiv 0 \pmod{p}$ are $n \equiv \zeta^j/(\zeta^j - 1) \pmod{p}$, $j = 1, ..., q-1$, where $\zeta$ is a primitive q-th root of unity in $(\mathbb{Z}/p\mathbb{Z})^*$. Classical Weil bounds [9] give $|S| \leq (q-1) \cdot p^{1/2}$, weaker than Hypothesis 1 by $q/\pi_Q$. The challenge: exploit the *prime restriction* to improve the bound.

## 4.4 Poisson Consistency

The normalized gap distribution g/(mean g) matches $1 - e^{-t}$ to four decimal places for all q. This Poisson behavior is consistent with the Bateman–Horn heuristic and with the absence of hidden correlations that would obstruct the sieve.
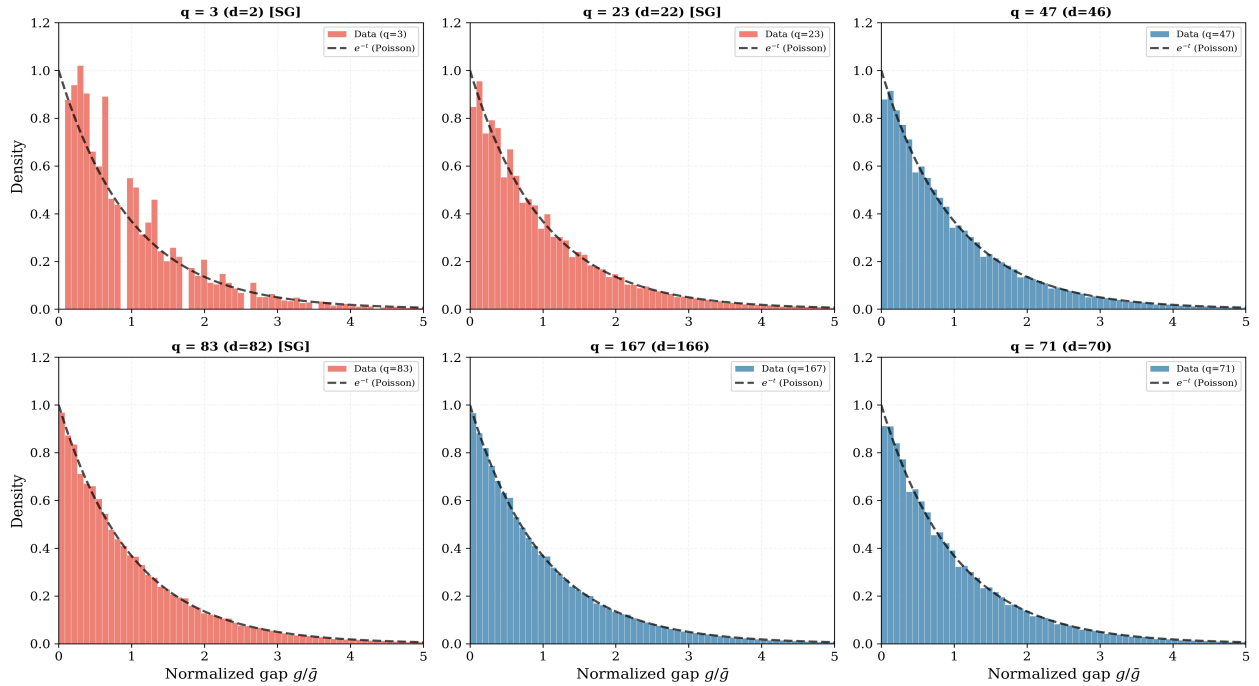


*Figure 4. Normalized gap distributions for 6 exponents. Black dashed: Poisson model.*

# 5. The Conditional Theorem

## 5.1 The Sparse Bombieri–Vinogradov Hypothesis

**Hypothesis 2 (Sparse BV$_q$).** For $Q_q$, there exists $\theta > 0$ such that for any $A > 0$: $\Sigma \, |\pi_Q(x; d, a) - \pi_Q(x)/\phi(d)|$ over *sparse moduli only* $(d \leq x^\theta, d \in M_{sparse}^q)$ is $O(\pi_Q(x)/(\log x)^A)$.

**Relation to Elliott–Halberstam.** Hypothesis 2 is strictly weaker than classical BV: (i) restricted to $M_{sparse}$, density $O((\log \log D)/q)$; (ii) right side uses $\pi_Q$ incorporating degree-barrier suppression. It is an EH conjecture on a sparse arithmetic subsequence — a reduction by $\sim q$ in the number of moduli requiring control.

## 5.2 Statement

**Theorem 3 (Conditional Bounded Gaps).** Assume Hypothesis 2 for $\theta > 1/4 + \varepsilon$. Let $k_{max} = p_1(q) - (q-1)$. Then there exists $H = H(q)$ such that for any admissible k-tuple with $k \leq k_{max}$, infinitely often at least two of $Q_q(n + h_1)$, ..., $Q_q(n + h_k)$ are simultaneously prime.

**Origin of the $\theta > 1/4$ threshold.** The threshold arises from the Maynard–Tao sieve machinery: the support of sieve weights is $d_i \leq R = x^{\theta/2}$, so bilinear products $d_i d_j$ range up to $x^\theta$. The main term evaluation requires level of distribution exceeding $R^2$, and the optimization of $M_k$ in [3] requires $\theta > 1/4$ to ensure $k \cdot M_k > 4$. In our setting, since $k_{max}$ is vastly larger than classical, the actual threshold may be relaxed; we state $\theta > 1/4$ as a sufficient condition.

## 5.3 Proof Sketch

**Step 1 (Admissibility):** Theorem 2 guarantees admissibility for $k \leq k_{max}$. For $q = 167$: k up to 2,173.

**Step 2 (Sieve weights):** Define Maynard–Tao $w_n = (\Sigma \lambda)^2$, support $d_i \leq x^{\theta/2}$.

**Step 3 (Main + error):** Null-Sparse Decomposition: $E_{null} = 0$ (Corollary); $E_{sparse}$ controlled by Hypothesis 2.

**Step 4 (Optimization):** $M_k \approx \log k$. For $k = 2,173$: $k \cdot M_k \approx 16,689 >> 4$.

# 6. Roadmap to Unconditional Results

The gap to unconditional: Hypothesis 2. We decompose it:

**R1. Type I estimates.** For sparse d, bound error from $Q_q(n) \equiv 0 \pmod d$. Has $\prod \omega(p^a)$ solutions per period.

**R2. Type II (bilinear).** Bound bilinear sums over $mn \in M_{sparse}$. Key: cancellation on $Q_{eff}$.

**R3. $Q_{eff}$ exponential sum conjecture.** Prove Hypothesis 1. Heuristic support: 40/40 tests, max ratio 1.93.

**Why Titan is easier:** (i) R1: null moduli give zero error. (ii) R2: bilinear sums restricted to density $O(1/q)$. (iii) R3: cyclotomic structure — $p \equiv 1 \pmod q$ means $(Z/pZ)^*$ has q-th roots of unity, connecting sums to Gauss/Jacobi sums.
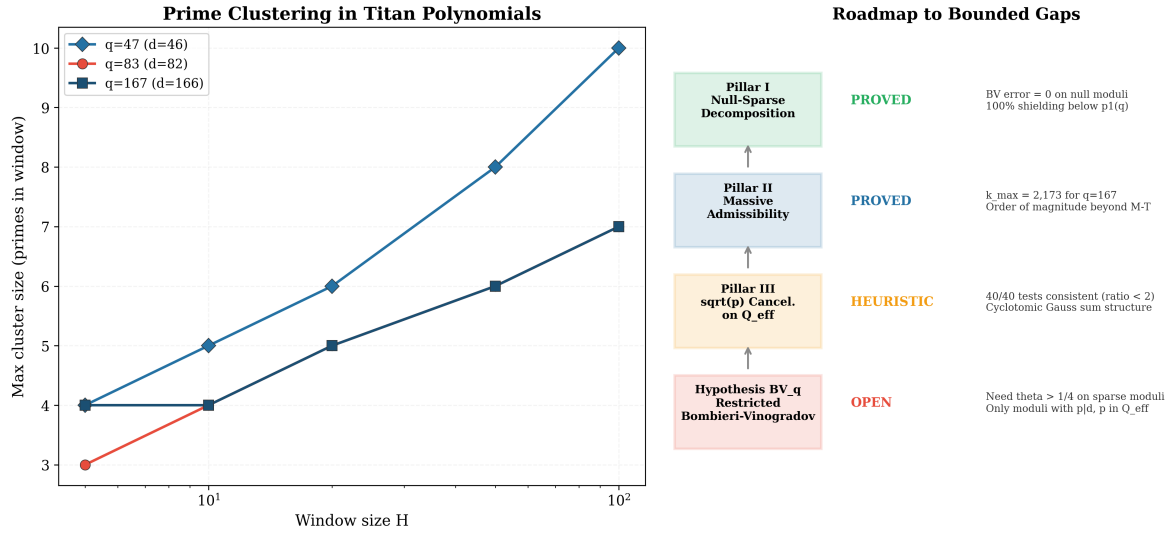
**Figure 5.** *Left: prime clustering within window H. Right: roadmap status for each pillar.*

# 7. Conclusion

1. **Null-Sparse Decomposition (proved):** For this polynomial family, the BV error vanishes on $M_{null}$ and concentrates on $M_{sparse}$. Verified: $82\times$ concentration.

2. **Massive Admissibility (proved):** $k_{max} = 2{,}173$ for $q = 167$ (an order of magnitude beyond the classical Maynard–Tao requirement). SG penalty extends to sieve dimension.

3. **$p^{1/2}$-Cancellation (heuristic):** 40/40 numerical tests are consistent with Hypothesis 1 (max ratio 1.93). Cyclotomic structure supports a path to proof.

4. **Conditional Bounded Gaps (Theorem 3):** Under Sparse $BV_q$ at $\theta > 1/4$, bounded gaps hold for Titan primes.

5. **Roadmap:** R1–R3, each benefiting from shielding. 100% of classical BV done on $M_{null}$; one estimate remains on a sparse, algebraically structured set.

*The Titan polynomial family provides the most concrete roadmap to date for bounded gaps in high-degree polynomial prime sequences. For this family, the classical degree barrier does not obstruct the Maynard–Tao sieve on a density-one set of moduli.*

# Data Availability

All computational data (~44M primes, 18 exponents, $N \leq 10^8$), reproducible scripts, and figures are available at https://github.com/Ruqing1963/titan-bounded-gaps. The companion paper [1] is archived at Zenodo (doi:10.5281/zenodo.18582880).

# References

[1] Chen, R. (2025). Defying the Degree Barrier. *Zenodo*. https://doi.org/10.5281/zenodo.18582880

[2] Zhang, Y. (2014). Bounded gaps between primes. *Ann. Math.* 179(3), 1121–1174.

[3] Maynard, J. (2015). Small gaps between primes. *Ann. Math.* 181(1), 383–413.

[4] Polymath, D.H.J. (2014). Variants of the Selberg sieve. *Res. Math. Sci.* 1:12.

[5] Bateman, P.T. & Horn, R.A. (1962). *Math. Comp.* 16(79), 363–367.

[6] Bombieri, E. (1965). On the large sieve. *Mathematika* 12(2), 201–225.

[7] Friedlander, J. & Iwaniec, H. (2010). *Opera de Cribro*. AMS Colloq. 57.

[8] Washington, L.C. (1997). *Introduction to Cyclotomic Fields*. GTM 83.

[9] Weil, A. (1948). Sur les courbes algébriques. *Actualités sci. et ind.* 1041.