Defying the Degree Barrier

Arithmetic Shielding and Bimodal Effective Degree

in the Titan Polynomial Family $Q_q(n) = n^q - (n-1)^q$

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Abstract

The Bateman-Horn conjecture predicts that prime density of a degree-d polynomial scales as S(f)/d, creating the "degree barrier." We study $Q_q(n) = n^q - (n-1)^q$ for 18 prime q from 3 to 167, with n up to 10^8 (~44 million primes). We prove the unified root count w(p) = gcd(q, p-1) - 1, implying perfect shielding (w = 0) for all p with q not dividing (p-1), including p = q itself (Fermat's Little Theorem). S(f) exhibits a **bimodal distribution**: when 2q+1 is composite, shielding is maximized (q=167: d=166, d_{eff}=11.7); when 2q+1 is prime (Sophie Germain), a penalty factor (q+2)/(2q) approaching 1/2 halves S(f). The Bateman-Horn *heuristic* provides an excellent quantitative fit to all 18 datasets, with relative deviations of 0.3-3.1%.

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1. Introduction

For an irreducible polynomial f(n) of degree d, the Bateman-Horn conjecture [1] provides a heuristic asymptotic formula (unproven, but supported by extensive numerical evidence): $pi_f(x)$ ~ (S(f)/d) Li(x). For a generic polynomial (S ~ 1), density scales as ~1/(d ln N), creating the degree barrier. The Titan family $Q_q(n)$ systematically exhibits $S(Q_q)$ growing with q, with bimodal structure governed by the Sophie Germain property of q.

2. Algebraic Structure

2.1 The Titan Polynomial

For prime q, $Q_q(n)$ has degree d = q-1 and is irreducible (via the cyclotomic connection $Q_q(n+1) = n^{q-1}$ $Phi_q(1+1/n)$, see Washington [5]).

2.2 Unified Root Counting Theorem

Theorem 1. For any prime q and any prime p: $\mathbf{w_f(p) = gcd(q, p-1) - 1}$.

Proof. The congruence reduces to u^q = 1 (mod p) with gcd(q, p-1) solutions. Excluding u = 1 gives gcd(q, p-1) - 1 roots. **Case p = q:** By Fermat, $Q_q(n)$ = 1 (mod q) for all n. Hence w(q) = 0. Formula: gcd(q, q-1) - 1 = 0. QED.

Corollary: w(p) = 0 when q does not divide (p-1) (boost p/(p-1) > 1); w(p) = q-1 when p = 1 (mod q) (obstruction). All p <= q are boost primes.

2.3 Arithmetic Shielding

By *arithmetic shielding* we mean the systematic absence of local obstructions (w(p) = 0) at small primes, resulting in a predominance of boost factors in the singular series. This is not a new conjecture but an observable consequence of Theorem 1.

The boost includes all p <= q. By Mertens' third theorem, the product of p/(p-1) for p <= q grows as e^{gamma} ln(q), where gamma = 0.5772 is the Euler-Mascheroni constant. This logarithmic divergence is the principal source of growth in S(f): as q grows, more primes contribute boost factors before the first obstruction.

2.4 The Sophie Germain Bifurcation

Proposition 2. When q is Sophie Germain (2q+1 prime), the first obstruction at p = 2q+1 contributes penalty: **(q+2)/(2q) = 1/2 + 1/q**, approaching 1/2.

Two tracks: **Shielded** (2q+1 composite, e.g. q=47, 167) with large S(f); **Penalized** (q is SG, e.g. q=23, 83) with S(f) nearly halved.

q	2q+1	SG?	1st p=1(q)	Factor	S(f)	d_eff
3	7	YES	7	0.833	3.362	0.59
5	11	YES	11	0.700	3.678	1.09
7	15	-	29	0.821	5.231	1.15
11	23	YES	23	0.591	4.012	2.49
13	27	-	53	0.788	5.651	2.12
17	35	-	103	0.853	6.803	2.35
19	39	-	191	0.911	8.886	2.03
23	47	YES	47	0.543	4.263	5.16
31	63	-	311	0.906	9.633	3.11
37	75	-	149	0.764	7.360	4.89
41	83	YES	83	0.524	5.742	6.97
43	87	-	173	0.762	7.712	5.45
47	95	-	283	0.840	8.683	5.30
53	107	YES	107	0.519	6.062	8.58
61	123	-	367	0.839	9.162	6.55
71	143	-	569	0.879	9.597	7.29
83	167	YES	167	0.512	4.939	16.60
167	335	-	2339	0.929	14.244	11.65

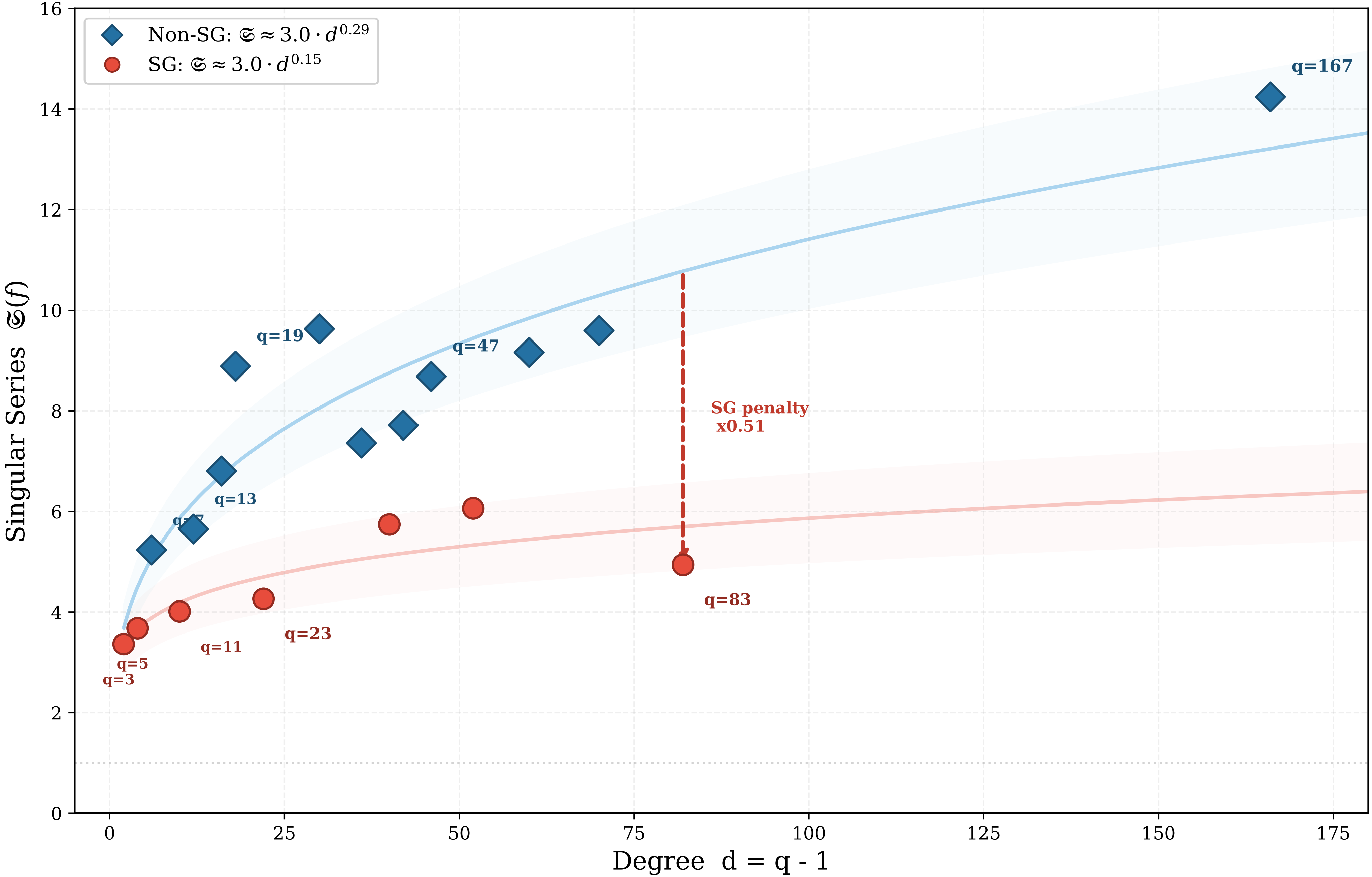
Pink = Sophie Germain (penalized).

3. Conjectures

Conjecture 1: $S(Q_q)$ grows as O(ln q), with SG track at ~half the non-SG track.

Conjecture 2 (Effective Degree Collapse): $d_{eff}/(q-1)$ tends to 0. Consistent with all observed data up to q=167.

Figure 1: The Sophie Germain Bifurcation



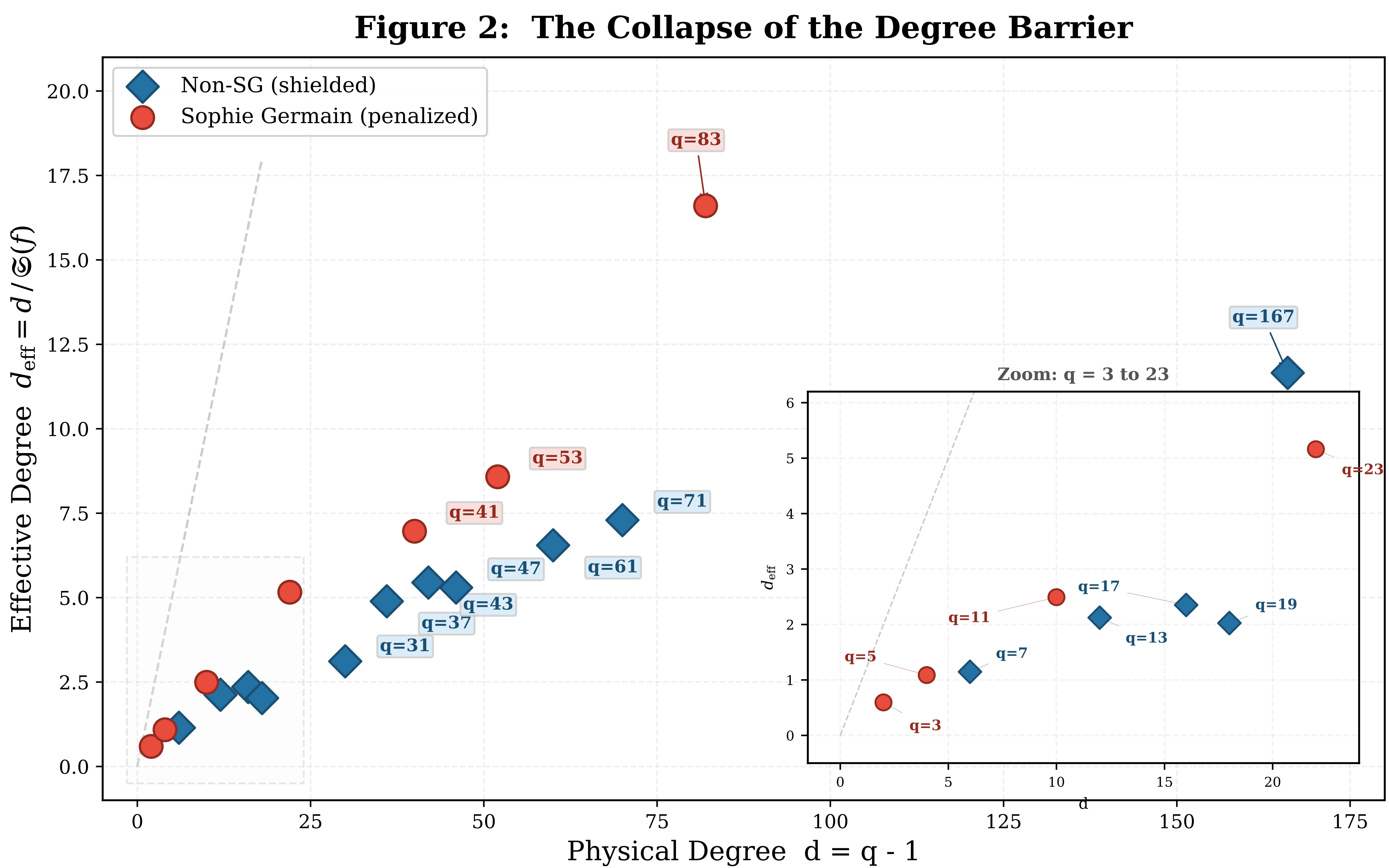


Figure 2: The Collapse of the Degree Barrier
Non-SG (shielded)
Sophie Germain (penalized)
Effective Degree $d_{\mathrm{eff}} = d\,/\,\mathscr{G}(f)$
Physical Degree d = q - 1
q=83
q=167
q=53
q=71
q=41
q=47
q=61
q=43
q=37
q=31
Zoom: q = 3 to 23
d_{eff}
d
q=23
q=17
q=19
q=11
q=13
q=7
q=5
q=3

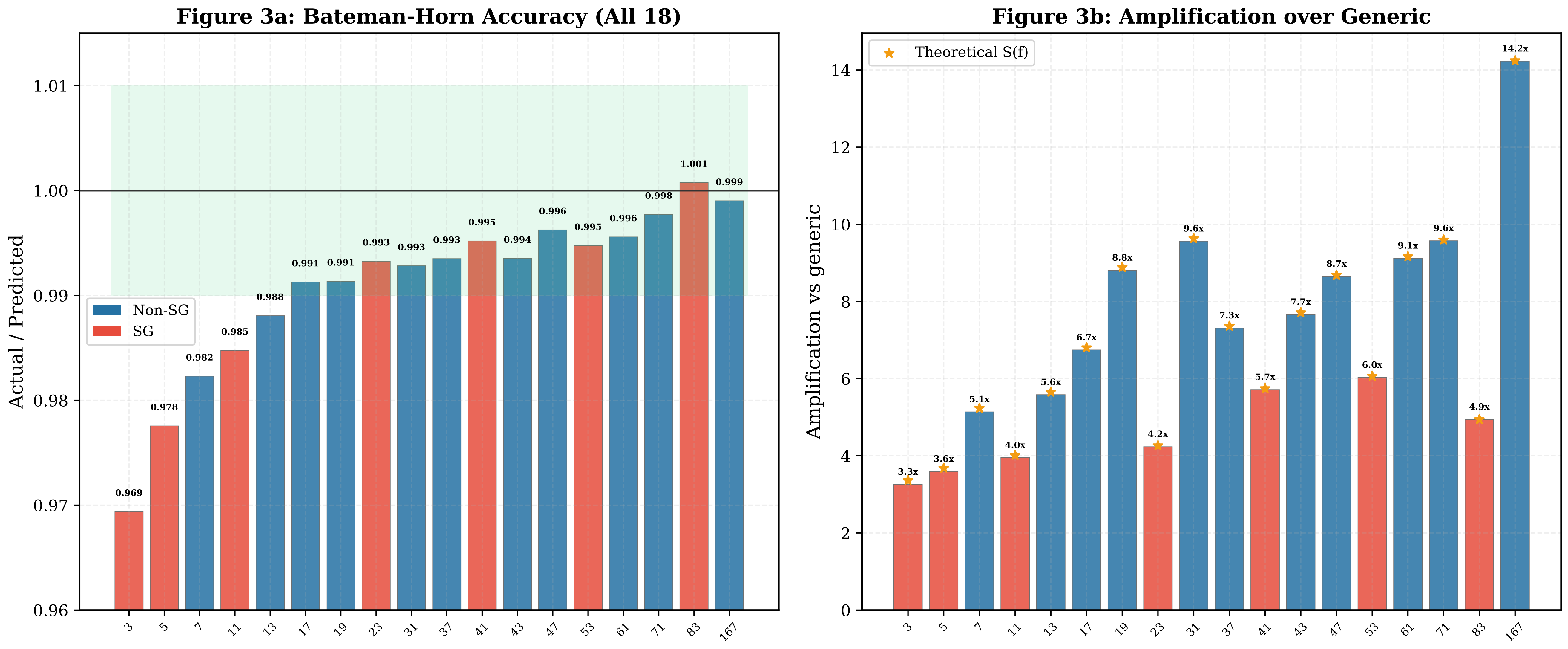
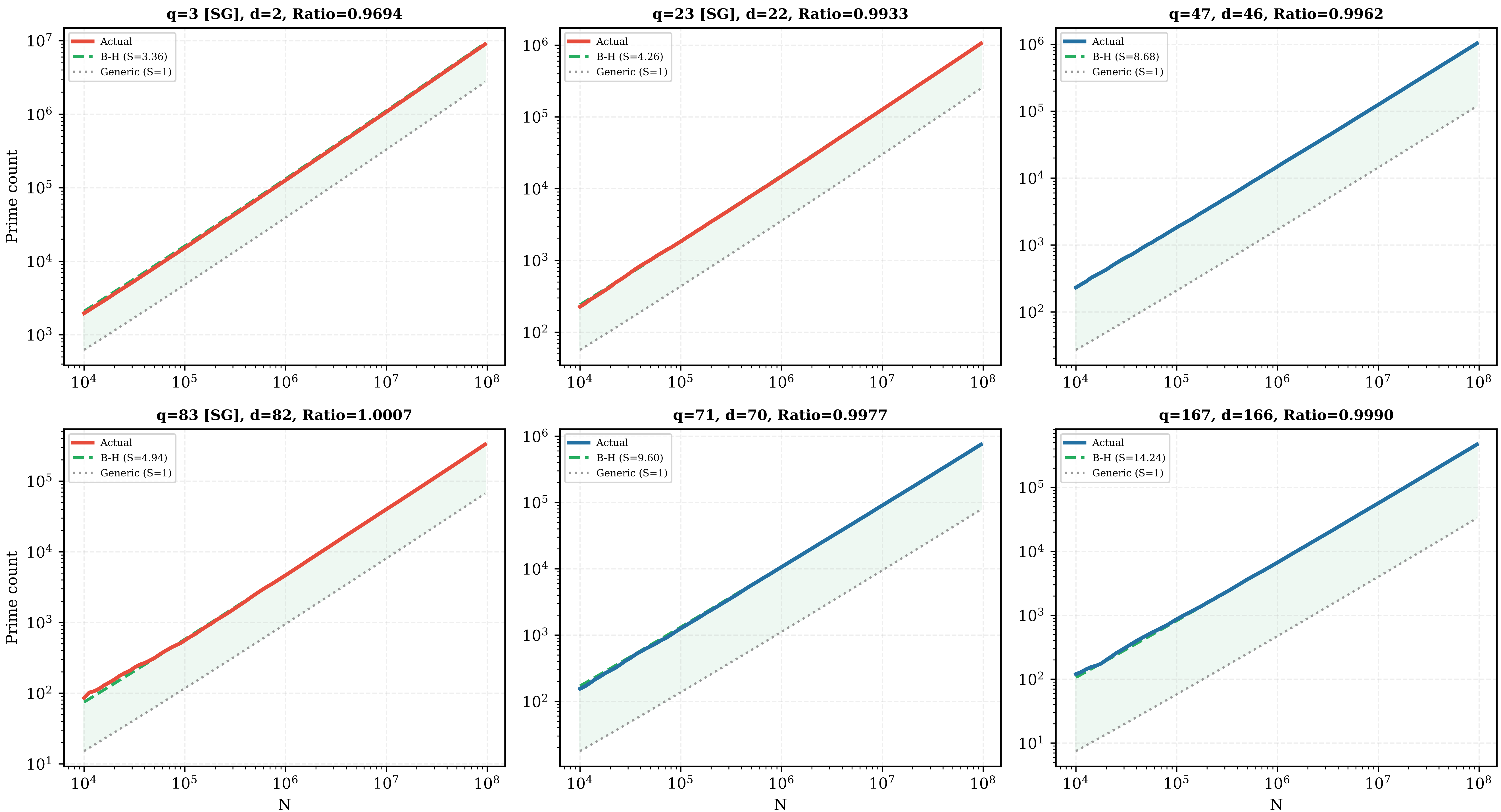


Figure 3a: Bateman-Horn Accuracy (All 18)
Actual / Predicted
Non-SG
SG
1.01
1.00
0.99
0.98
0.97
0.96
0.969
0.978
0.982
0.985
0.988
0.991
0.991
0.993
0.993
0.993
0.995
0.994
0.996
0.995
0.996
0.998
1.001
0.999
3
5
7
11
13
17
19
23
31
37
41
43
47
53
61
71
83
167
Figure 3b: Amplification over Generic
Theoretical S(f)
Amplification vs generic
14
12
10
8
6
4
2
0
3.3x
3.6x
5.1x
4.0x
5.6x
6.7x
8.8x
4.2x
9.6x
7.3x
5.7x
7.7x
8.7x
6.0x
9.1x
9.6x
4.9x
14.2x
3
5
7
11
13
17
19
23
31
37
41
43
47
53
61
71
83
167

Figure 4: Bateman-Horn Verification (6 selected exponents)



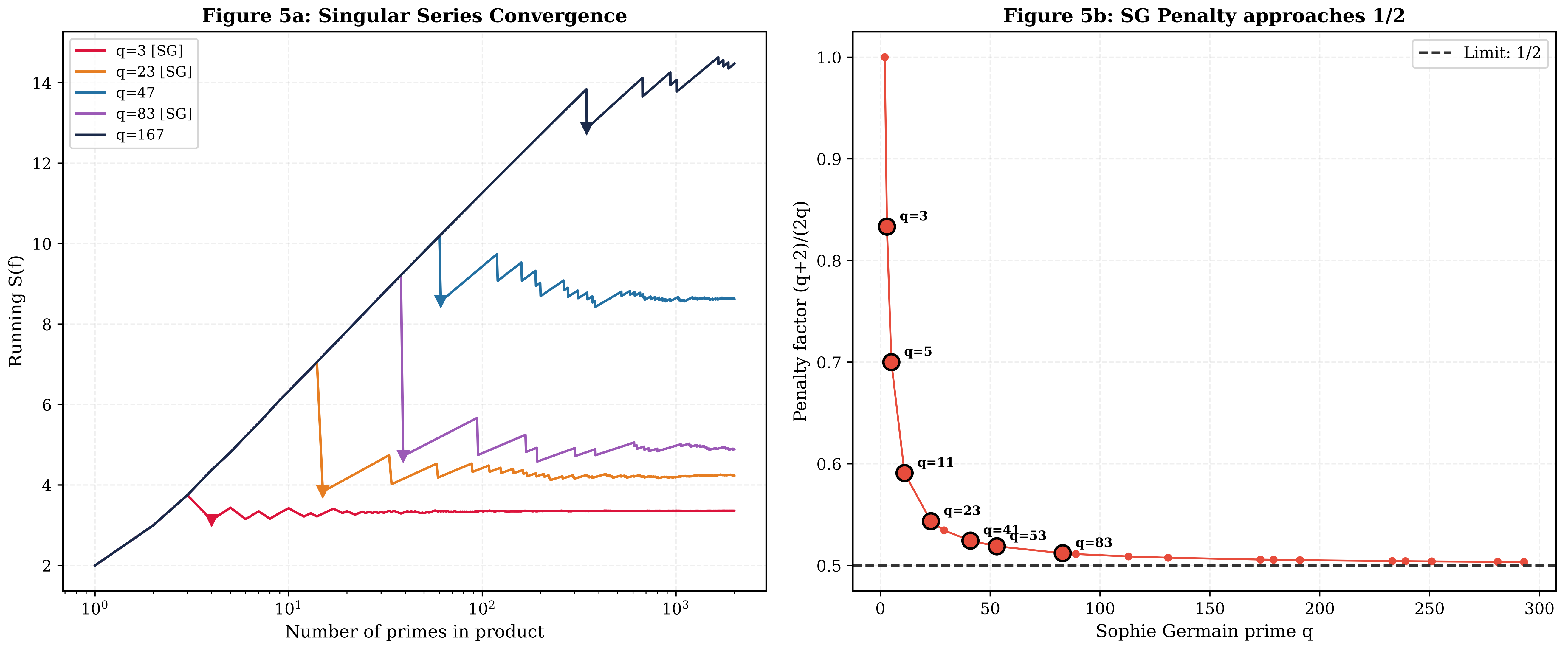


Figure 5a: Singular Series Convergence
q=3 [SG]
q=23 [SG]
q=47
q=83 [SG]
q=167
Running S(f)
Number of primes in product
Figure 5b: SG Penalty approaches 1/2
Limit: 1/2
q=3
q=5
q=11
q=23
q=41
q=53
q=83
Penalty factor (q+2)/(2q)
Sophie Germain prime q

4. Computational Results

Exhaustive primality testing for n in [1, 10^8] across 18 prime exponents q. Primality verified using deterministic Miller-Rabin tests; all computations performed with Python/GMP arithmetic. ~44 million primes total. Source code, raw data, and all figures are publicly available at:

`https://github.com/Ruqing1963/titan-polynomial-prime-shielding`

Bateman-Horn fit: All 18 actual/predicted ratios in [0.969, 1.001], mean 0.991. The heuristic works equally well for both SG and non-SG tracks.

Effective degree collapse: Q_{47} (d=46, d_{eff}=5.3, 8.7x), Q_{167} (d=166, d_{eff}=11.7, 14.2x).

The q=83 case: Highest d_{eff}=16.6 -- an apparent anomaly fully explained by the Sophie Germain mechanism: 167=2*83+1 is prime, penalty 0.512. Compare q=71 (143=11*13 composite): S=9.60, d_{eff}=7.3 (3x better despite lower degree).

Control: Q_{167} produces 493,941 primes vs ~34,713 for generic d=166: 14.2x amplification.

5. Discussion

Root formula: w(p) = gcd(q,p-1)-1 is notable for its uniformity across all p. No piecewise cases; p=q is automatic.

Prediction test: Power-law on q <= 83 predicts d_{eff} ~ 21-28 for q=167; actual is 11.65. Shielding accelerates at high degree.

Computational strategy: (i) Prefer non-SG exponents. (ii) Sieve effort concentrates on p = 1 (mod q). (iii) For n = 10^{5000}, Q_{167}(n) yields ~830,000-digit candidates at effective degree ~12.

6. Conclusion

1. **Unified mechanism:** w(p) = gcd(q,p-1)-1. p=q is boost (Fermat).
2. **Bimodal:** S splits by SG property. Penalty = (q+2)/(2q) ~ 1/2.
3. **Effective degree collapse:** d_{eff} from 0.59 to 16.60 (mean 6.9x).
4. **Bateman-Horn fit:** All 18 ratios in [0.969, 1.001], conditional on the B-H heuristic.
5. **Testable prediction:** S is low iff 2q+1 is prime. Confirmed all 18 points.
6. **Acceleration:** Shielding grows faster than power-law at high d.

The degree barrier is revealed to be permeable for algebraically structured families. Informally: for the Titan family, the wall is made of paper; for non-Sophie-Germain exponents, it is tissue paper.

References

[1] Bateman & Horn (1962). Math. Comp. 16(79), 363-367.

[2] Hardy & Littlewood (1923). Acta Math. 44, 1-70.

[3] Schinzel & Sierpinski (1958). Acta Arith. 4(3), 185-208.

[4] Crandall & Pomerance (2005). Prime Numbers. Springer.

[5] Washington (1997). Intro. to Cyclotomic Fields. GTM 83, Springer.

[6] Granville (2008). Princeton Companion to Mathematics.

[7] Ribenboim (1996). New Book of Prime Number Records. Springer.