I

Homework 2

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1. 利用张量分量变换律, 证明如下两个度规张量是同一个张量:

$$\delta_{ab} = \mathrm{d}x_a\,\mathrm{d}x_b + \mathrm{d}y_a\,\mathrm{d}y_b + \mathrm{d}z_a\,\mathrm{d}z_b$$

和

$$\delta_{ab} = \mathrm{d}r_a\,\mathrm{d}r_b + r^2(\mathrm{d}\theta_a\,\mathrm{d}\theta_b + \sin^2\theta\,\mathrm{d}\varphi_a\,\mathrm{d}\varphi_b)$$

解: δ_{ab} 按坐标基底展开:

$$\delta_{ab} = \delta_{ij}\frac{\partial x^i}{\partial y^{i'}}\frac{\partial x^j}{\partial y^{j'}}\left(\mathrm{d}y^{i'}\right)_a\left(\mathrm{d}y^{j'}\right)_b = \frac{\partial x^i}{\partial y^{i'}}\frac{\partial x^i}{\partial y^{j'}}\left(\mathrm{d}y^{i'}\right)_a\left(\mathrm{d}y^{j'}\right)_b = \delta_{i'j'}\left(\mathrm{d}y^{i'}\right)_a\left(\mathrm{d}y^{j'}\right)_b$$

$$x^i = x, y, z; \quad y^i = r, \theta, \varphi$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Longrightarrow \mathrm{d}x^i = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}\mathrm{d}y^i$$

其中:

$$\begin{aligned}
\delta_{r_a r_b} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r} \\
&= (\sin\theta\cos\varphi)(\sin\theta\cos\varphi) + (\sin\theta\sin\varphi)(\sin\theta\sin\varphi) + \cos\theta\cos\theta \\
&= 1 \\
\delta_{\theta_a \theta_b} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} \\
&= (r\cos\theta\cos\varphi)(r\cos\theta\cos\varphi) + (r\cos\theta\sin\varphi)(r\cos\theta\sin\varphi) + (-r\sin\theta)(-r\sin\theta) \\
&= r^2 \\
\delta_{\varphi_a \varphi_b} &= \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial \varphi} \\
&= (-r\sin\theta\sin\varphi)(-r\sin\theta\sin\varphi) + (r\sin\theta\cos\varphi)(r\sin\theta\cos\varphi) \\
&= r^2\sin^2\theta
\end{aligned}$$

$$\delta_{r_a\theta_b} = \delta_{r_b\theta_a} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} = 0$$

$$\delta_{r_a\varphi_b} = \delta_{r_b\varphi_a} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \varphi} = 0$$

$$\delta_{\varphi_a\theta_b} = \delta_{\varphi_b\theta_a} = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial \theta} = 0$$

因此有:

$$\begin{aligned}
\delta_{ab} &= \delta_{r_a r_b}\,\mathrm{d}r_a\,\mathrm{d}r_b + \delta_{\theta_a\theta_b}\,\mathrm{d}\theta_a\,\mathrm{d}\theta_b + \delta_{\varphi_a\varphi_b}\,\mathrm{d}\varphi_a\varphi_b \\
&= \mathrm{d}r_a\,\mathrm{d}r_b + r^2(\mathrm{d}\theta_a\,\mathrm{d}\theta_b + \sin^2\theta\,\mathrm{d}\varphi_a\,\mathrm{d}\varphi_b)
\end{aligned}$$

2. 利用曲线弧长定义

$$s = \int_a^b \sqrt{|g(\dot{\gamma}, \dot{\gamma})|}\,\mathrm{d}t$$

(1) 求证: 欧氏空间中, 两点之间直线最短;
(2) 考虑一般度规(不妨假设号差为正), 求最短曲线应当满足的方程.
(3) 验证上述方程满足

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \dot{\gamma}^\nu \nabla_\nu \dot{\gamma}^\mu = 0$$

解: (1) 在欧氏空间中:

$$s = \int_a^b \sqrt{|\dot{\gamma}|^2}\,\mathrm{d}t = \int_a^b |\dot{\gamma}|\,\mathrm{d}t$$

$$\begin{aligned}
\Longrightarrow \delta s &= \int_a^b \delta(|\dot{\gamma}|)\,\mathrm{d}t + |\dot{\gamma}|\delta(\mathrm{d}t) \\
&= \int_a^b \frac{\partial|\dot{\gamma}|}{\partial t}\delta(t)\,\mathrm{d}t = \int_a^b |\ddot{\gamma}|\,\mathrm{d}t\delta(t) = 0
\end{aligned}$$

$$\Longrightarrow |\ddot{\gamma}| = 0$$
$$\Longrightarrow \gamma^\mu = a^\mu t + b^\mu$$

即 γ 为直线.
(2) 对于一般度规, 有:

$$\begin{aligned}
\delta s &= \int_a^b \delta\Big(\sqrt{|g(\dot{\gamma}, \dot{\gamma})|}\Big)\,\mathrm{d}t \\
&= \int_a^b \frac{1}{2}\frac{1}{\sqrt{|g(\dot{\gamma}, \dot{\gamma})|}}\left(2g\left(\dot{\gamma}, \frac{\mathrm{d}\delta(\gamma)}{\mathrm{d}t}\right) + \delta(g)(\dot{\gamma}, \dot{\gamma})\right)\mathrm{d}t \\
&= \int_a^b \frac{1}{2}\frac{1}{\sqrt{|g(\dot{\gamma}, \dot{\gamma})|}}\left(2\frac{\mathrm{d}}{\mathrm{d}t}(g(\dot{\gamma}, \dot{\gamma})) - 2\frac{\mathrm{d}}{\mathrm{d}t}(g_{\mu\nu}\dot{\gamma}^\mu)\delta(\gamma)^\nu + \delta(g)(\dot{\gamma}, \dot{\gamma})\right)\mathrm{d}t \\
&= \int_a^b \frac{1}{2}\frac{1}{\sqrt{|g(\dot{\gamma}, \dot{\gamma})|}}\left(-2\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\mu\nu}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t}\right)\delta\gamma^\nu + \frac{\partial g_{\mu\nu}}{\partial\gamma^\sigma}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}t}\delta\gamma^\sigma\right)\mathrm{d}t \\
&= \int_a^b \frac{1}{2}\frac{1}{\sqrt{|g(\dot{\gamma}, \dot{\gamma})|}}\left(-2\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\mu\sigma}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t}\right) + \frac{\partial g_{\mu\nu}}{\partial\gamma^\sigma}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}t}\right)\delta\gamma^\sigma\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\mu\sigma}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial\gamma^\sigma}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}t} \\
&= g_{\mu\sigma}\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}t^2} + \frac{\partial g_{\mu\sigma}}{\partial\gamma^\nu}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}t}\dot{\gamma}^\mu - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial\gamma^\sigma}\dot{\gamma}^\mu\dot{\gamma}^\nu = g^{\rho\sigma}g_{\mu\sigma}\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}t^2} + g^{\rho\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial\gamma^\nu} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial\gamma^\sigma}\right)\dot{\gamma}^\mu\dot{\gamma}^\nu \\
&= \frac{\mathrm{d}^2\gamma^\rho}{\mathrm{d}t^2} + g^{\rho\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial\gamma^\nu} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial\gamma^\sigma}\right)\dot{\gamma}^\mu\dot{\gamma}^\nu = \frac{\mathrm{d}^2\gamma^\rho}{\mathrm{d}t^2} + \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial\gamma^\nu} + \frac{\partial g_{\sigma\nu}}{\partial\gamma^\mu} - \frac{\partial g_{\mu\nu}}{\partial\gamma^\sigma}\right)\dot{\gamma}^\mu\dot{\gamma}^\nu \\
&= \frac{\mathrm{d}^2\gamma^\rho}{\mathrm{d}t^2} + \Gamma^\rho{}_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu
\end{aligned}$$

(3)

$$\begin{aligned}\nabla_{\dot\gamma}\dot\gamma &= \dot\gamma^\nu\partial_\nu\dot\gamma^\rho + \Gamma^\rho{}_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu \\ &= \frac{\partial}{\partial\gamma^\nu}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}t}\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}t} + \Gamma^\rho{}_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu \\ &= \frac{\mathrm{d}^2\gamma^\rho}{\mathrm{d}t^2} + \Gamma^\rho{}_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu = 0\end{aligned}$$

3. 验证 $\{\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_p}\}$ 组成 p-形式场空间 Λ^p 的一组基.
   解: 1. 线性无关: 设

$$\begin{gathered} c_{i_1i_2\ldots i_p}\,\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_p} = 0 \\ \Longrightarrow c_{i_1i_2\ldots i_p}\,\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_m}\wedge\ldots\wedge\mathrm{d}x^{i_n}\wedge\ldots\wedge\mathrm{d}x^{i_p} = 0, \\ c_{i_1i_2\ldots i_p}\,\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_n}\wedge\ldots\wedge\mathrm{d}x^{i_m}\wedge\ldots\wedge\mathrm{d}x^{i_p} \\ = -c_{i_1i_2\ldots i_p}\,\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_m}\wedge\ldots\wedge\mathrm{d}x^{i_n}\wedge\ldots\wedge\mathrm{d}x^{i_p} = 0 \\ \Longrightarrow c_{i_1\ldots i_p} = 0 \end{gathered}$$

   即 $\{\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_p}\}$ 线性无关.
   2. 数学归纳法: 对于 1-形式场, 即一阶切向量: $\omega = \omega\,\mathrm{d}x$
   对于 p-形式场, 若有:

$$\omega = \frac{1}{p!}\omega_{i_1\ldots i_p}\,\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_p}$$

   那么对于 $(p+1)$-形式场, 有:

$$\begin{aligned}\omega &= \frac{1}{p!}\omega_{i_1\ldots i_{p+1}}(\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_p})\wedge\mathrm{d}x^{i_{p+1}} \\ &= \frac{1}{p+1}\frac{1}{p!}\omega_{i_1\ldots i_{p+1}}\,\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{i_{p+1}} \\ &= \frac{1}{(p+1)!}\omega_{i_1\ldots i_{p+1}}\,\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{i_{p+1}}\end{aligned}$$

   故对于任意 p-形式场都可写成

$$\omega = \frac{1}{p!}\omega_{i_1\ldots i_p}\,\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_p}$$

   形式.

4. 利用 $(\mathbb{R}^3,\delta_{ab})$ 上叉乘和 Hodge 对偶的关系:
   (1) 求证:

$$u\times v = \delta^{-1} * (u\wedge v);$$

   (2) 化简:

$$(u\times v)\cdot(x\times y).$$

   解: (1)

$$
\begin{aligned}
u\times v &= \delta^{li}\varepsilon_{ijk}u^{[i}v^{k]} = \delta^{li}\varepsilon_{ijk}u^i v^k + \delta^{li}\left(-\varepsilon_{ikj}\right)\left(-u^k v^j\right) \\
&= \frac{1}{2}\delta^{li}\varepsilon_{ijk}(u\wedge v)^{jk} = \delta^{-1} * (u\wedge v)
\end{aligned}
$$

(2)

$$
\begin{aligned}
(*\,(u\wedge v))(*\,(x\wedge y)) &= \delta^{li}\varepsilon_{ijk}u^{[j}v^{k]}\varepsilon_{lmn}x^{[m}y^{n]} \\
&= \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right)u^{[j}v^{k]}x^{[m}y^{n]} \\
&= (u\cdot x)(v\cdot y) - (u\cdot y)(v\cdot x)
\end{aligned}
$$

5. 利用克式符的定义，求克式符在坐标变换下的变换律.
   解: Christoffel 符号定义:

$$
\begin{aligned}
\Gamma^{\rho}{}_{\alpha\beta} &= \left\langle \mathrm{d}x^{\rho}, \boldsymbol{\nabla}_{\partial_\alpha}\partial_\beta \right\rangle = \left(\mathrm{d}x^{\rho}\right)_a\left(\boldsymbol{\nabla}_{\partial_\alpha}\partial_\beta\right)^a \\
&= \frac{\partial x^\rho}{\partial \tilde{x}^\sigma}\left(\mathrm{d}\tilde{x}^\sigma\right)_a\left(\boldsymbol{\nabla}_{\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\tilde{\partial}_\mu}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\tilde{\partial}_\nu\right)^a \\
&= \frac{\partial x^\rho}{\partial \tilde{x}^\sigma}\left(\mathrm{d}\tilde{x}^\sigma\right)_a\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\left(\boldsymbol{\nabla}_{\tilde{\partial}_\mu}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\tilde{\partial}_\nu\right)^a \\
&= \frac{\partial x^\rho}{\partial \tilde{x}^\sigma}\left(\mathrm{d}\tilde{x}^\sigma\right)_a\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\left[\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\left(\boldsymbol{\nabla}_{\tilde{\partial}_\mu}\tilde{\partial}_\nu\right)^a + \tilde{\partial}_\nu\partial_\beta\delta^\nu{}_\mu\right] \\
&= \frac{\partial x^\rho}{\partial \tilde{x}^\sigma}\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\left(\mathrm{d}\tilde{x}^\sigma\right)_a\left(\boldsymbol{\nabla}_{\tilde{\partial}_\mu}\tilde{\partial}_\nu\right)^a + \frac{\partial x^\rho}{\partial \tilde{x}^\sigma}\frac{\partial^2 \tilde{x}^\mu}{\partial x^\alpha \partial x^\beta}\delta^\sigma{}_\mu \\
&= \frac{\partial x^\rho}{\partial \tilde{x}^\sigma}\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\Gamma^\sigma{}_{\mu\nu} + \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial^2 \tilde{x}^\mu}{\partial x^\alpha \partial x^\beta}
\end{aligned}
$$

6. 求解球坐标系下的克式符所有分量.
   解: 球坐标下度规:

$$
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}; g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}
$$

由克氏符表达式:

$$
\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(g_{\nu\rho,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}\right)
$$

显然度规 g 只有对角分量.
1. 首先分析 $\nu=\rho=\sigma=r,\theta,\varphi$ 的情况:

$$
\Gamma^\sigma{}_{\mu\sigma} = \frac{1}{2}g^{\sigma\sigma}\left(g_{\sigma\sigma,\mu} + g_{\mu\sigma,\sigma} - g_{\mu\sigma,\sigma}\right) = \frac{1}{2}g^{\sigma\sigma}g_{\sigma\sigma,\mu}
$$

若 $\nu=\rho=\sigma=r$, 注意到 g_{11} 对任何分量求导都得 0, 因此都为 0.
若 $\nu=\rho=\sigma=\theta$:

$$
\begin{aligned}
\Gamma^\theta{}_{r\theta} &= \frac{1}{2}g^{\theta\theta}g_{\theta\theta,r} = \frac{1}{2}\frac{1}{r^2}2r = \frac{1}{r} = \Gamma^\theta{}_{\theta r} \\
\Gamma^\theta{}_{\theta\theta} &= \frac{1}{2}g^{\theta\theta}g_{\theta\theta,\theta} = 0, \Gamma^\theta{}_{\varphi\theta} = \Gamma^\theta{}_{\theta\varphi} = 0
\end{aligned}
$$

若 $\nu=\rho=\sigma=\varphi$:

$$\Gamma^{\varphi}{}_{r\varphi}=\frac{1}{2}g^{\varphi\varphi}g_{\varphi\varphi,r}=\frac{1}{2}\frac{1}{r^2\sin^2\theta}2r\sin^2\theta=\frac{1}{r}=\Gamma^{\varphi}{}_{\varphi r}$$
$$\Gamma^{\varphi}{}_{\theta\varphi}=\frac{1}{2}g^{\varphi\varphi}g_{\varphi\varphi,\theta}=\frac{1}{2}\frac{1}{r^2\sin^2\theta}r^2 2\sin\theta\cos\theta=\cot\theta=\Gamma^{\varphi}{}_{\varphi\theta}$$
$$\Gamma^{\varphi}{}_{\varphi\varphi}=0$$

2. 对于 $\mu=\nu,\rho=\sigma$ 的情况:

$$\Gamma^{\sigma}{}_{\mu\mu}=\frac{1}{2}g^{\sigma\sigma}\big(g_{\mu\sigma,\mu}+g_{\mu\sigma,\mu}-g_{\mu\mu,\sigma}\big)=g^{\sigma\sigma}\left(g_{\mu\sigma,\mu}-\frac{1}{2}g_{\mu\mu,\sigma}\right)=\begin{cases}\frac{1}{2}g^{\mu\mu}g_{\mu\mu,\mu} & \sigma=\mu\\ -\frac{1}{2}g^{\sigma\sigma}g_{\mu\mu,\sigma} & \sigma\neq\mu\end{cases}$$

若 $\mu=\nu=r$ 注意到 g_{11} 对任何分量求导都得 0, 因此都为 0.
若 $\mu=\nu=\theta$:

$$\Gamma^{r}{}_{\theta\theta}=-\frac{1}{2}g^{rr}g_{\theta\theta,r}=-\frac{1}{2}2r=-r$$
$$\Gamma^{\theta}{}_{\theta\theta}=-\frac{1}{2}g^{\theta\theta}g_{\theta\theta,\theta}=0,\Gamma^{\varphi}{}_{\theta\theta}=0$$

若 $\mu=\nu=\varphi$:

$$\Gamma^{r}{}_{\varphi\varphi}=-\frac{1}{2}g^{rr}g_{\varphi\varphi,r}=-\frac{1}{2}2r\sin^2\theta=-r\sin^2\theta$$
$$\Gamma^{\theta}{}_{\varphi\varphi}=-\frac{1}{2}g^{\theta\theta}g_{\varphi\varphi,\theta}=-\frac{1}{2}\frac{1}{r^2}r^2 2\sin\theta\cos\theta=-\sin\theta\cos\theta$$
$$\Gamma^{\varphi}{}_{\varphi\varphi}=0$$

由 $g_{\mu\nu}$ 的对称性可知其他项均为 0. 于是, 球坐标克氏符所有非零项:

$$\Gamma^{\theta}{}_{r\theta}=\Gamma^{\theta}{}_{\theta r}=\frac{1}{r},$$
$$\Gamma^{\varphi}{}_{r\varphi}=\Gamma^{\varphi}{}_{\varphi r}=\frac{1}{r},$$
$$\Gamma^{\varphi}{}_{\theta\varphi}=\Gamma^{\varphi}{}_{\varphi\theta}=\cot\theta,$$
$$\Gamma^{r}{}_{\theta\theta}=-r,\quad \Gamma^{r}{}_{\varphi\varphi}=-r\sin^2\theta$$
$$\Gamma^{\theta}{}_{\varphi\varphi}=-\sin\theta\cos\theta$$

7. 分别在直角坐标和球坐标下计算欧氏空间的测地线.
解: (1) 测地线方程:

$$\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}t^2}+\Gamma^{\mu}{}_{\alpha\beta}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}t}=0$$

直角坐标下克氏符 $\Gamma^{\mu}{}_{\alpha\beta}=0$ 因此:

$$\Longrightarrow\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}t^2}=0\Longrightarrow x^{\mu}=a^{\mu}t+b^{\mu}$$

即测地线为直线.
(2) 球坐标下: 由上题得到的球坐标克氏符, 直接写出测地线方程:

$$\begin{cases}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}+\Gamma^r{}_{\theta\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2+\Gamma^r{}_{\varphi\varphi}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2=0\\ \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+2\Gamma^\theta{}_{r\theta}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}+\Gamma^\theta{}_{\varphi\varphi}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2=0\\ \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}+2\Gamma^\varphi{}_{r\varphi}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}+2\Gamma^\varphi{}_{\theta\varphi}\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}=0\end{cases}\Longrightarrow\begin{cases}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}-r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2-r\sin^2\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2=0\\ \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}-\sin\theta\cos\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2=0\\ \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}+2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}=0\end{cases}$$

满足上方程的解即为测地线.

8. 验证 S^2 中的测地线为赤道线.
解: S^2 上坐标 θ,φ 的克氏符:

$$\Gamma^\varphi{}_{\theta\varphi}=\Gamma^\varphi{}_{\varphi\theta}=\cot\theta,\Gamma^\theta{}_{\varphi\varphi}=-\sin\theta\cos\theta$$

测地线方程:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}-\sin\theta\cos\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2=0$$
$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}+2\cot\theta\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}=0$$

对于赤道线有参数方程:

$$\theta=\frac{\pi}{2},\varphi=\varphi(t)$$

代入:

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}=0\Longrightarrow\varphi=at+b$$

即赤道线为测地线.

9. 利用 Bianchi 第一恒等式

$$R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)=0$$

证明 $R_{\mu\nu\rho\sigma}=R_{\rho\sigma\mu\nu}$.
解: 给出以下四组 利用 Bianchi 恒等式:

$$R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)=0$$
$$R(Y,Z,W,X)+R(Z,W,Y,X)+R(W,Y,Z,X)=0$$
$$R(Z,W,X,Y)+R(W,X,Z,Y)+R(X,Z,W,Y)=0$$
$$R(W,X,Y,Z)+R(X,Y,W,Z)+R(Y,W,X,Z)=0$$

将以上三式相加, 利用 $R(X,Y,Z,W)=-R(X,Y,W,Z),R(X,Y,Z,W)=-R(Y,X,Z,W)$:

$$
\begin{gathered}
\cancel{R(X,Y,Z,W)} + \cancel{R(Y,Z,X,W)} + R(Z,X,Y,W) + \\
\cancel{R(Y,Z,W,X)} + \cancel{R(Z,W,Y,X)} + R(W,Y,Z,X) + \\
\cancel{R(Z,W,X,Y)} + \cancel{R(W,X,Z,Y)} + R(X,Z,W,Y) + \\
\cancel{R(W,X,Y,Z)} + \cancel{R(X,Y,W,Z)} + R(Y,W,X,Z) = 0 \\
\Longrightarrow 2R(X,Z,Y,W) - 2R(Y,W,X,Z) = 0
\end{gathered}
$$

即:

$$R(X,Y,Z,W) = R(Z,W,X,Y)$$

10. 计算由 S^2 嵌入到 $\mathbb{R}^3$ 中的诱导度规和诱导联络.
解: $\mathbb{R}^3$ 选取正交基 $\left\{\partial_\theta, \partial_\varphi\right\}$ 计算诱导度规:

$$h_{\theta\theta} = g_{\theta\theta} = r^2, h_{\varphi\varphi} = g_{\varphi\varphi} = r^2 \sin^2\theta$$

诱导度规:

$$h_{\mu\nu} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}$$

诱导联络有

$$\Gamma^{\varphi}{}_{\theta\varphi} = \Gamma^{\varphi}{}_{\varphi\theta} = \cot\theta$$

$$\Gamma^{\theta}{}_{\varphi\varphi} = -\sin\theta\cos\theta$$

其他系数均为 0.