Homework 1

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1. 对于电磁场, 我们已知其动量-能量-应力张量可以写作:

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu}{}_{\sigma}F^{\nu\sigma} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

证明其为无迹的, 即:

$$\eta^{\mu\nu}T_{\mu\nu} = 0.$$

解:
能量-动量-应力张量:

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu}{}_{\sigma}F^{\nu\sigma} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

那么

$$\begin{aligned}
\eta_{\mu\nu}T^{\mu\nu} &= \frac{1}{\mu_0}\left(\eta_{\mu\nu}F^{\mu}{}_{\sigma}F^{\nu\sigma} - \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \\
&= \frac{1}{\mu_0}\left(F_{\nu\sigma}F^{\nu\sigma} - \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)
\end{aligned}$$

我们知道:

$$\eta_{\mu\nu}\eta^{\mu\nu} = 4$$

因此:

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{1}{\mu_0}\left(F_{\nu\sigma}F^{\nu\sigma} - \frac{1}{4}\times 4F_{\rho\sigma}F^{\rho\sigma}\right) = 0.$$

2. 利用上题给出的动量-能量-应力张量和麦克斯韦方程组, 证明:

$$\partial_\mu T^{\mu\nu} = -F^{\nu}{}_{\sigma}J^{\sigma}$$

实际上本公式反应了电磁场的能动量守恒, 请简单说明.
解:

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \frac{1}{\mu_0}\partial_\mu\left(F^{\mu}{}_{\sigma}F^{\nu\sigma} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \\
&= \frac{1}{\mu_0}\left[(\partial_\mu F^{\mu}{}_{\sigma})F^{\nu\sigma} + F^{\mu}{}_{\sigma}(\partial_\mu F^{\nu\sigma}) - \frac{1}{2}F^{\rho\sigma}(\partial_\nu F_{\rho\sigma})\right]
\end{aligned}$$

利用 Maxwell 方程 :

$$\partial_\nu F^{\mu\nu} = \mu_0 J^{\mu},$$
$$\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0$$

有:

$$\frac{1}{2}F^{\rho\sigma}(\partial_\nu F_{\rho\sigma}) = \frac{1}{2}F^{\rho\sigma}(-\partial_\rho F_{\sigma\nu} - \partial_\sigma F_{\nu\rho})$$

代入:

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \frac{1}{\mu_0}\left[(\partial_\mu F^{\mu}{}_{\sigma})F^{\nu\sigma} + F^{\mu}{}_{\sigma}(\partial_\mu F^{\nu\sigma}) + \frac{1}{2}F^{\rho\sigma}(\partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho})\right] \\
&= \frac{1}{\mu_0}\left[-\left(\partial_\mu F^{\sigma'\mu}\right)\eta_{\sigma\sigma'}F^{\nu\sigma} - F^{\mu\sigma}(\partial_\mu F_{\nu\sigma}) + \frac{1}{2}F^{\mu\sigma}\partial_\mu F_{\sigma\nu} + \frac{1}{2}F^{\sigma\mu}\partial_\mu F_{\nu\sigma}\right] \\
&= -\frac{1}{\mu_0}\mu_0 J^\sigma F^{\nu}{}_{\sigma} = -F^{\nu}{}_{\sigma}J^\sigma
\end{aligned}$$

解释:
取 $\mu = \nu = 0$, 有:

$$\begin{aligned}
\frac{\partial T^{00}}{\partial t} = \frac{1}{\mu_0}\frac{\partial}{\partial t}\left(F^{0}{}_{\sigma}F^{0\sigma} + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right) = -F^{0}{}_{\sigma}J^\sigma \\
\Longrightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}E^2 + \frac{1}{2}B^2\right) + \mu_0\vec{E}\cdot\vec{J} = 0
\end{aligned}$$

对无源系统, 有:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}E^2 + \frac{1}{2}B^2\right) = 0$$

即系统能动量守恒.

3. 已知，在 2 维 Mincowski 时空中, 当采用闵式坐标时, 两相邻间的时空间隔是:

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2.$$

请给出同样的时空中用如下坐标:

$$u = ct - x, \quad v = ct + x$$

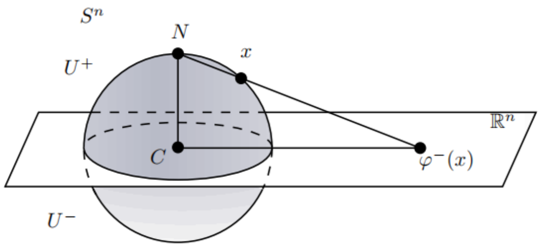
时，两相邻点间的间隔表达式. 实际上本题的坐标称为光锥坐标，在广义相对论，弦论等领域有重要的意义.
解:
由

$$\begin{cases} u = ct - x \\ v = ct + x \end{cases} \Rightarrow \begin{cases} \mathrm{d}u = c\,\mathrm{d}t - \mathrm{d}x \\ \mathrm{d}v = c\,\mathrm{d}t + \mathrm{d}x \end{cases}$$

故有:

$$\begin{aligned}
\mathrm{d}s^2 &= -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 = -(c\,\mathrm{d}t + \mathrm{d}x)(c\,\mathrm{d}t - \mathrm{d}x) \\
&= -\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

4. 写出如图二维球面投影至交于球心的平面的变换关系.



解: 取球半径为1. 取 NC 与映射点 (记 D) 组成的三角形, 可以得到:

$$\mathrm{CD} = \frac{\sqrt{x^2+y^2}}{1-z}$$

再取 CD 与 D 点对 y 轴垂线 T 组成的三角, 求得:

$$Z_x = \frac{x}{1-z}, \quad Z_y = \frac{y}{1-z}$$

因此 二维球面上点 (x, y, z) 投影到球面的变换为

$$f : (x, y, z) \longrightarrow \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

5. V_p 是一线性空间, v 为 V_p 中的矢量, 线性空间满足:
i. V_p 中有零元等价于存在单点曲线 $\gamma_0(\gamma) : I \to p$;
ii. 矢量间的加法关系: $v_1 + v_2 = v_3$;
iii. 数乘关系: $(\alpha v) \to \gamma(c\lambda)$, c 为一常数.
(1) 考虑流形中一点 p 邻域的曲线加法 $\gamma_1^i(\lambda) + \gamma_2^i(\lambda) = \gamma_3^i(\lambda)$, 由此验证 ii;
(2) 验证 iii.
解:
(1) 由于

$$\gamma_1^i(\lambda) + \gamma_2^i(\lambda) = \gamma_3^i(\lambda)$$

则有:

$$v(\gamma_3^i) = v(\gamma_1^i + \gamma_2^i)$$

利用 v 的线性, 有:

$$\begin{aligned} v(\gamma_3^i) &= v(\gamma_1^i + \gamma_2^i) \\ &= v(\gamma_1^i) + v(\gamma_2^i) \end{aligned}$$

即:

$$v_1 + v_2 = v_3$$

(2)

$$\alpha v(\gamma(\lambda)) = v(\alpha\gamma(\lambda))$$

由于 $\alpha\gamma(\lambda)$ 也在 $\gamma(\lambda)$ 的取值范围中, 取

$$\gamma(\lambda_0) = \alpha\gamma(\lambda)$$

记 $c = \gamma_0/\gamma$, 即有:

$$\alpha v(\gamma(\lambda)) \to \alpha\gamma(\lambda) = \gamma(\lambda_0) = \gamma\left(\frac{\lambda_0}{\lambda}\lambda\right) = \gamma(c\lambda)$$

6. f 为流形上的光滑函数, $\mathrm{d}f$ 表示函数的微分, 定义为 $\mathrm{d}f \coloneqq \dfrac{\partial f}{\partial x^i}\,\mathrm{d}x^i$, 其中 $\mathrm{d}(x^i) = \mathrm{d}x^i$, 验证 $\mathrm{d}f$ 与坐标选择无关.
解:
流形上有两套坐标 $\{x^i\}$, $\{y^j\}$, 有:

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \\ &= \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i}\,\mathrm{d}x^i\end{aligned}$$

由于:

$$\mathrm{d}y^j = \frac{\partial y^j}{\partial x^i}\,\mathrm{d}x^i$$

因此:

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i = \frac{\partial f}{\partial y^j}\,\mathrm{d}y^j$$

即 $\mathrm{d}f$ 与坐标选择无关.

7. 证明: 张量 T 的坐标分量在不同的坐标系之间的变换关系为:

$$T'^{j_1\cdots j_l}_{i_1\cdots i_k} = T^{q_1\cdots q_l}_{p_1\cdots p_k}\frac{\partial x^{p_1}}{\partial y^{i_1}}\cdots\frac{\partial x^{p_k}}{\partial y^{i_k}}\frac{\partial y^{j_1}}{\partial x^{q_1}}\cdots\frac{\partial y^{j_l}}{\partial x^{q_l}}$$

其中, $T'^{j_1\cdots j_l}_{i_1\cdots i_k}$, $T^{q_1\cdots q_l}_{p_1\cdots p_k}$ 分别为张量 T 在坐标系 y^i, x^i 中的分量.
解:

$$\begin{aligned}&T^{q_1\cdots q_l}_{p_1\cdots p_k}\frac{\partial x^{p_1}}{\partial y^{i_1}}\cdots\frac{\partial x^{p_k}}{\partial y^{i_k}}\frac{\partial y^{j_1}}{\partial x^{q_1}}\cdots\frac{\partial y^{j_l}}{\partial x^{q_l}} \\ &= T\left(\partial_{p_1}\ldots\partial_{p_k};\mathrm{d}x^{q_1},\ldots,\mathrm{d}x^{q_l}\right)\frac{\partial x^{p_1}}{\partial y^{i_1}}\cdots\frac{\partial x^{p_k}}{\partial y^{i_k}}\frac{\partial y^{j_1}}{\partial x^{q_1}}\cdots\frac{\partial y^{j_l}}{\partial x^{q_l}} \\ &= T\left(\frac{\partial}{\partial x^{p_k}}\frac{\partial x^{p_m}}{\partial y^{i_k}};\mathrm{d}x^{q_n}\frac{\partial y^{j_l}}{\partial x^{q_l}}\right) \\ &= T\left(\frac{\partial}{\partial y^{i_k}}\delta^m_k;\mathrm{d}y^{j_l}\delta^n_l\right) = T\left(\frac{\partial}{\partial y^{i_k}};\mathrm{d}y^{j_l}\right) = T'^{j_1\cdots j_l}_{i_1\cdots i_k}\end{aligned}$$

8. 证明: 张量缩并的定义与坐标无关.
解:
对于缩并张量, 有:

$$T_i{}^i = T\left(\frac{\partial}{\partial x^i};\mathrm{d}x^i\right) = T\left(\frac{\partial}{\partial y^j}\frac{\partial y^j}{\partial x^i};\mathrm{d}x^i\frac{\partial}{\partial y^j}\,\mathrm{d}y^j\right) = T\left(\frac{\partial}{\partial y^j}\delta^j_i;\mathrm{d}y^j\delta^i_j\right) = T\left(\frac{\partial}{\partial y^i};\mathrm{d}y^i\right) = T'_i{}^i$$

其中 $T_i^{\ i}$ 与 $T'_i{}^{i}$ 分别为张量 T 在坐标系 x^i 与 y^i 中的分量.
因此, 张量缩并的定义与坐标选取无关.