Linear Algebra and Differential Equations

UNIT I

System of Linear Equations and its Applications

Basic Matrices

- **Definition**

- A matrix is an arrangement of mn elements in m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Types of Matrices

Row Matrix :- Row matrix (or row vector) is a matrix with one row.

- $\boldsymbol{r} = [r_1 \quad r_2 \quad r_3 \quad \cdots \quad r_n]$

Column Matrix :- Column vector is a matrix with only one column.

- $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$

Square Matrix :-When the row and column dimensions of a matrix are equal (m = n) then the matrix is called a square matrix. In other words A matrix is said to be square, if the number of rows and number of columns are equal.

Types of Matrices

1. **Diagonal Matrix** :- A Scalar matrix in which all non diagonal elements are

- Zero is called diagonal matrix.

2. **Unit Matrix** :- A Diagonal matrix in which all diagonal elements are one (1) is called unit matrix.

3. **Identity Matrix** :- A square matrix in which elements of main diagonal are 1 and other elements are zero is called an identity matrix.

- $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Types of Matrices

- **Upper Triangular Matrix :-**A matrix in which elements below main diagonal are zero is called Upper Triangular Matrix.
- **Lower Triangular Matrix :-**A matrix in which elements above main diagonal are zero is called Lower Triangular Matrix.
- **Singular Matrix** :- A square matrix is said to be singular if$|A| = 0$
- **Non –Singular Matrix** :- A square matrix is said to be non-singular if$|A| \neq 0$
- **Symmetric Matrix :-** A square Matrix A is said to be symmetric matrix if $A=A^T$
- **Skew Symmetric Matrix** :- A square matrix A is said to be skew symmetric if $A=-A^T$
- **Orthogonal Matrix:-** A square matrix A is orthogonal iff $A A^T = A^T A = I$

Algebra of Matrices

- **Matrix Equality**

- Two (m x n) matrices A and B are equal if and only if each of their elements are equal. i.e. A = B if and only if $a_{ij} = b_{ij}$

  for i = 1,...,m; j = 1,...,n.

- **Matrix Addition**
- If A and B are two matrices then
-
- $$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & & a_{2n} + b_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Algebra of Matrices

- **Scalar Multiplication**
- Multiplication of a matrix A by a scalar defined as
- $$\alpha \boldsymbol{A} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & & \alpha a_{2n} \\ \vdots & & \ddots & \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{bmatrix}$$

Algebra of Matrices

- **Matrix Multiplication**
- The product of two matrices A and B is defined only if the number of columns
- of A is equal to the number of rows of B. If A is (m x p) and B is (p x n), the product is an (m x n) matrix C. i.e. $\boldsymbol{C}_{m\times n} = \boldsymbol{A}_{m\times p}\,\boldsymbol{B}_{p\times n}$
- $\boldsymbol{C} = \boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & & a_{1p} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{1n} \\ \vdots & & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pn} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1p}b_{p1} & a_{11}b_{12} + \cdots + a_{1p}b_{p2} & \cdots & a_{11}b_{1n} + \cdots + a_{1p}b_{pn} \\ a_{21}b_{11} + \cdots + a_{2p}b_{p1} & a_{21}b_{12} + \cdots + a_{2p}b_{p2} & & a_{21}b_{1n} + \cdots + a_{2p}b_{pn} \\ \vdots & & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mp}b_{p1} & a_{m1}b_{12} + \cdots + a_{mp}b_{p2} & \dots & a_{m1}b_{1n} + \cdots + a_{mp}b_{pn} \end{bmatrix}$$

Algebra of Matrices

- **Matrix Inverse**
- If A is an (n x n) square matrix and there is a matrix B with the property that $\boldsymbol{A}B = \boldsymbol{I}$. Then B is defined to be the inverse of A and is denoted A^{-1}.
- Inverse of a matrix by adjoint method
- $A^{-1} = \frac{1}{|A|} adjA$
- adj A= transpose of Cofactor matrix

Elementary Transformations

- The following three types of transformations, performed on any non zero matrix A, are called elementary transformations.
- The interchange of i^{th} and j^{th} row denoted by R_{ij}(same for column i.e. C_{ij})
- The multiplication of each element of i^{th} row by non zero scalar k is denoted by kR_i
- Multiplication of each element of j^{th} row by scalar k and adding to the corresponding element of i^{th} row is denoted by (R_i+kR_j).
- **Minor**
- The minor of an element of matrix A is a determinant obtained by omitting the row and the column in which the element is present.

Rank of a Matrix

- **Rank of a Matrix**
- The matrix A is said to be of rank r if
- 1) there exists at least one minor of the order r which is not equal to zero.
- 2) every minor of the order $r + 1$ is equal to zero .
- The rank of matrix A is the maximum order of its non vanishing minor.
- It is denoted as $\rho(A) = r$

Find Rank of a Matrix

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Echelon form or Row Echelon form

- Given matrix is said to be in echelon form if it satisfies the following properties:

a) Zero row of matrix (if any) occurs at bottom of the matrix .

b) The first nonzero number from the left of a nonzero row is 1. This is called **leading 1**.

c) For two successive(consecutive) nonzero rows , the leading 1 of upper appears to the left of the leading 1 of lower row.

- **Note: Rank of the matrix is equal to number of non zero rows in its echelon form.**

pivot elements

$$\begin{bmatrix} 1 & 2 & 3 & 5 & -7 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6 \times 5}$$

Leading entry – First non-zero element in a row called as leading entry or **pivot element**.

Questions

- Q 1] Reduce the following matrix to its Echelon Form and hence find its rank.

- 1. $A = \begin{bmatrix} 4 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ $ans: \rho(A) = 3$

  2. $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ $ans: \rho(A) = 4$

- 3. $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ $ans: \rho(A) = 3$

  4. $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $ans: \rho(A) = 2$

- 5. $A = \begin{bmatrix} 4 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ $ans: \rho(A) = 3$

  6. $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$ $ans: \rho(A) = 2$

- 7. A= $\begin{bmatrix} 1 & 5 & 7 \\ 2 & 10 & 14 \\ 3 & 15 & 21 \end{bmatrix}$ $ans: \rho(A) = 1$

  8. A= $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $ans: \rho(A) = 2$

Questions

9) $A = \begin{bmatrix} -2 & 2 & -1 & 2 \\ 0 & 3 & 3 & -3 \\ 1 & -4 & 2 & 2 \end{bmatrix}$ $Ans: \rho(A) = 3$

10) $A = \begin{bmatrix} 4 & -3 & -4 & -2 \\ -4 & 2 & 1 & -4 \\ -1 & -3 & 1 & -4 \end{bmatrix}$ $Ans: \rho(A) = 3$

11) $\mathrm{A} = \begin{bmatrix} -4 & 1 & 4 \\ 3 & 4 & -3 \end{bmatrix}$ $Ans: \rho(A) = 2$

12) $\mathrm{A} = \begin{bmatrix} -4 & -2 & -1 \\ -2 & -3 & 0 \end{bmatrix}$ $Ans: \rho(A) = 2$

Non Homogeneous system of equation:

- AX = B where B is non zero is called Non Homogeneous system of equation.

- $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 +-- -a_{1n}x_n = b_1$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 +-- -a_{2n}x_n = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 +-- -a_{mn}x_n = b_m$$

- The matrix form of the system is

- $\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{21} \cdots a_{2n} \\ \vdots & \vdots \quad \vdots \quad \vdots \\ a_{m1} & a_{m1} \cdots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ bm \end{bmatrix}$

- i.e., AX=B

where A is called the coefficient matrix, X is called the matrix of variables and B is called the matrix of constants.

Non Homogeneous system of equation

- **Augmented Matrix:**
- If AX=B is a system of m equations in n variables (unknowns) then the matrix [A|B]is called augmented matrix and is written as
- $$[A:B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ \vdots & \vdots & \cdots & \vdots & & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$$
- **Consistency:**
- **Consistent system:** System AX = B is said to be consistent iff AX = B has solution
- **Inconsistent System:** System AX = B is said to be inconsistent iff AX = B has no solution

Non Homogeneous system of equation

To solve AX=B

- If $\rho(A) = \rho([A:B]$ = n (Number of unknowns) Then above system is consistent and has unique solution.
- If $\rho(A) = \rho([A:B])$ = r $\prec n(\text{Number of unknowns})$ Then system is consistent and system has infinitely many solutions.{' n-r' free parameter solutions }
- If$\rho(A) \neq \rho([A:B])$; then system has no solution.
- If the given system is consistent then rewrite equations again from Echelon form . By back substitution we get the solution.

Non Homogeneous system of equation

- Test for consistency of the following system of equations and if consistent then solve the following equations;

1) 3x + y + 2z =3; 2x - 3y - z = -3; x + 2y + z = 4 — Ans: x=1, y =2, z =-1

2) 5x + 3y +7z =4; 3x + 26y +2z =9; 7x +2y +10z = 5 — Ans: $x = \frac{7-16t}{11}, y = \frac{3+t}{11}, z = t$

3) 2x+6y+11=0; 6x+20y-6z+3=0; 6y-18z+1=0 — Ans: system is inconsistent

4) x+2y+2z=1;2x+2y+3z=3;x-y+3z=5 — Ans: x= 1 y = -1 z =1

5) 2x + z = 4; x - 2y +2z = 7; 3x + 2y =1 — Ans: $x = 2 - \frac{t}{2}, y = \frac{-5}{2} + \frac{3t}{4}, z = t$

6) 4x-2y+6z=8; x+y-3z=-1; 15x-3y+9z=21 — Ans: x=1, y =3t-2, z = t

7) x + y + z = 4; 2x +5y -2z = 3 — Ans: $x = \frac{17-7t}{3}, y = \frac{-5+4t}{t}, z = t$

8) 2x+3y+4z=11;x+5y+7z =15;3x+11y+13z=25 — Ans: x=2, y =-3, z=4

Problems

Q.1) Investigate the values of λ and μ so that the equations:

$$2x + 3y + 5z = 9;$$

$$7x + 3y - 2z = 8;$$

$$2x + 3y + \lambda z = \mu$$

have

(i) no solution

(ii) a unique solution

(iii) an infinite number of solutions.

Problems

Q. 2) Show that the system of equations,

$$x_1 + 2x_2 + 3x_3 = \lambda x_1;$$

$$3x_1 + x_2 + 2x_3 = \lambda x_2;$$

$$2x_1 + 3x_2 + x_3 = \lambda x_3$$

can possess a non trivial solution only if $\lambda = 6$. Obtain general solution for real values of λ.

Q.3) Show that the equations,

$$ax + by + cz = 0;$$

$$bx + cy + az = 0;$$

$$cx + ay + bz = 0$$

has a non trivial solutions only if $a + b + c = 0$ or $a = b = c$.

Problems

Q.4) Test for consistency and if consistent, solve the following system of equations:

$$2x - y -= 2$$
$$x + 2y + z = 2$$
$$4x - 7y - 5z = 2$$

Q.5) For what values of k the equations:

$$x + y + z = 1;$$
$$2x + y + 4z = k;$$
$$4x + y + 10z = k^2$$

have infinite number of solutions? Hence find solutions.

Q. 6) Show that the system

$$3x + 4y + 5z = \alpha;$$
$$4x + 5y + 6z = \beta;$$
$$5x + 6y + 7z = \gamma$$

is consistent only when α, β, γ are in arithmetic progression.

Problems

1. Investigate the values of λ and μ so that the equations:

$$2x + 3y + 5z = 9; 7x + 3y - 2z = 8; 2x + 3y + \lambda z = \mu \text{ have}$$

(i) no solution (ii) a unique solution (iii) an infinite number of solutions.

2. Show that the system of equations,

$$x_1 + 2x_2 + 3x_3 = \lambda x_1; 3x_1 + x_2 + 2x_3 = \lambda x_2;\ 2x_1 + 3x_2 + x_3 = \lambda x_3$$

can possess a non trivial solution only if $\lambda = 6$. Obtain general solution for real values of λ.

3. Show that the equations, $ax + by + cz = 0; bx + cy + az = 0; cx + ay + bz = 0$ has a nontrivial solutions only if $a + b + c = 0$ or $a = b = c$.

Homogeneous System of Equations

- The system AX=B is called homogeneous system if matrix B is a null matrix.
- It is denoted by AX=Z (AX=O)

  Here A is coefficient matrix, X is variable matrix and Z or O is null matrix.

- Note:- Homogeneous system is always consistent.

The system of equation is given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + -- -a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + -- -a_{2n}x_n = 0$$

……………………………………………………………..

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + -- -a_{mn}x_n = 0$$

Homogeneous System of Equations

- If $\rho(A) = \rho(A,Z)$ = Number of variables, then the system has trivial solution ($x = 0, y = 0, z = 0$)

- If $\rho(A) = \rho(A,Z) <$ Number of variables, then the system has non-trivial

- Consider the system of equations $AX=0$ where A is the square matrix then

  i. If $|A|=0$, then system possesses a nontrivial solution. The solution can be obtained by rank method.

  ii. If $|A| \neq 0$, then the system possesses only trivial solution.

Q. 1) $x+2y+2z=0$; $2x+2y+3z=0$; $x-y+3z=0$

Problems

Q. 2) Show that the system of equations,

$$x_1 + 2x_2 + 3x_3 = \lambda x_1;$$

$$3x_1 + x_2 + 2x_3 = \lambda x_2;$$

$$2x_1 + 3x_2 + x_3 = \lambda x_3$$

can possess a non trivial solution only if $\lambda = 6$. Obtain general solution for real values of λ.

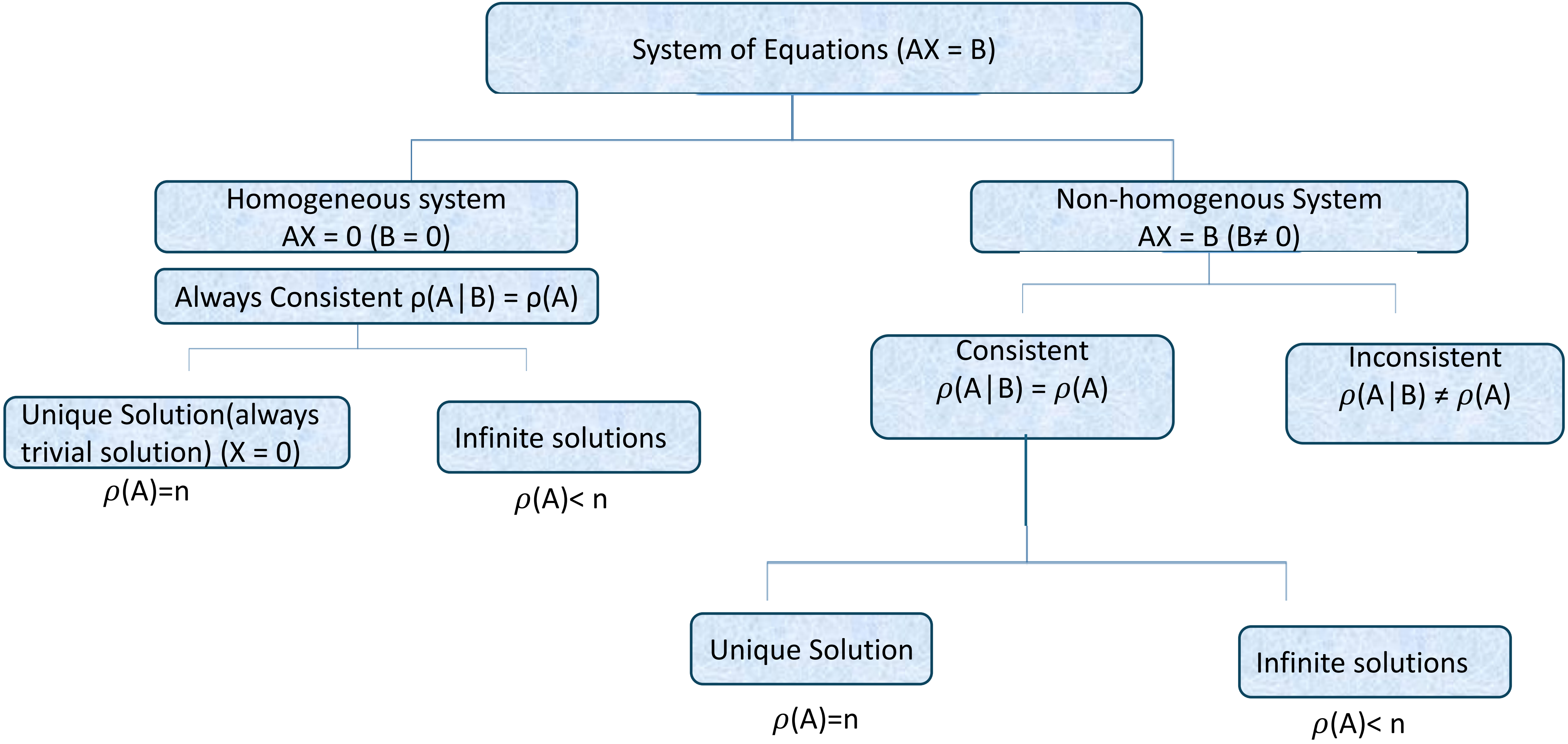
Q.3) Show that the equations,

$$ax + by + cz = 0;$$

$$bx + cy + az = 0;$$

$$cx + ay + bz = 0$$

has a non trivial solutions only if $a + b + c = 0$ or $a = b = c$.


System of Equations (AX = B)
Homogeneous system
AX = 0 (B = 0)
Always Consistent ρ(A|B) = ρ(A)
Unique Solution(always trivial solution) (X = 0)
ρ(A)=n
Infinite solutions
ρ(A)< n
Non-homogenous System
AX = B (B≠ 0)
Consistent
ρ(A|B) = ρ(A)
Inconsistent
ρ(A|B) ≠ ρ(A)
Unique Solution
ρ(A)=n
Infinite solutions
ρ(A)< n

Gauss Jordan Elimination Method

- Gauss –Jordan elimination method is a numerical method which is used to solve system of linear equations.
- It involves performing elementary row operations on an augmented matrix to transform it to reduced row echelon form.
- Example of reduced row echelon form

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & 5 & 3 \end{bmatrix}$

Steps to solve the system of equations by Gauss-Jordan elimination

Step 1 : Write the augmented matrix of the linear system.

Step 2 : Use row operations to reduce the augmented matrix to reduced row echelon Form.

Step 3 : Interpret the final matrix as linear system

Step 4 : Use back substitution and write the solution.

Examples on Gauss Jordan Method

Solve following system of equations using Gauss Jordan Elimination method.

Q.1 $x + y + z = 5$

$2x + 3y + 5z = 8$

$4x + 5z = 2$

Solution:

Step 1: Write the system in matrix for

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$$

AX=B

Step 2: Write the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{bmatrix}$$

Step 3: Reduce above matrix to row echelon form by elementary row transformation

- $R_2 - 2R_1$ & $R_3 - 4R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{bmatrix}$$

➢ $R_3 + 4R_2$

$$[\text{A:B}] \sim \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{bmatrix}$$

➢ $(\frac{1}{13})\ R_3$

$$[\text{A:B}] \sim \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

➢ $R_2 - 3R_3$ & $R_1 - R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

➢ $R_1 - R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Now matrix is transformed to reduced row echelon form, by using back substitution,

- we can write the solution of system as

$$x = 3$$
$$y = 4$$
$$z = -2$$

Example 1. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

Example 2. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

Example 3. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} 4y + z = 2 \\ 2x + 6y - 2z = 3 \\ 4x + 8y - 5z = 4 \end{cases}$$

Solve following system of equations using Gauss Jorden Elimination method.

Q.4 $x + y + z = 9$

$2x - 3y + 4z = 13$

$3x + 4y + 5z = 40$ {Answer: $x = 1; y = 3; z = 5$}

Q.5 $x + 3y + z = 10$

$x - 2y - z = -6$

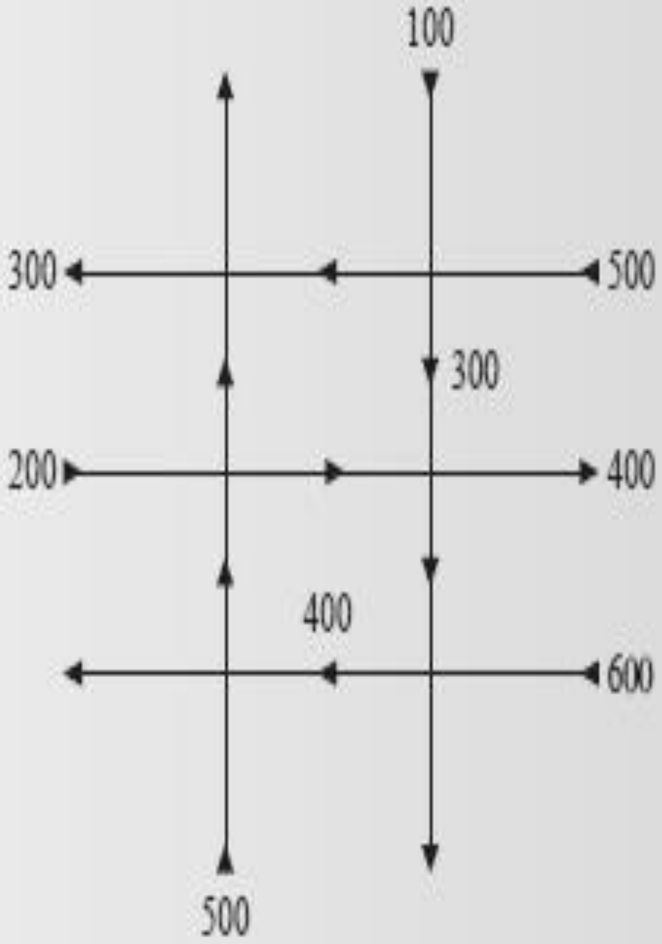
$2x + y + 2z = 10$ {Answer: $x = 1; y = 2; z = 3$}

Application of linear system of equations

Network Flow :

To study the flow of traffic through city streets, urban planners use mathematical models called directed graphs or digraphs. In these models, edges and points are used to represent streets and intersections, respectively. Arrows are used to indicate the direction of traffic. To balance a traffic network, we assume that the outflow of each intersection is equal to the inflow, and that the total flow into the network is equal to the total flow out.

Q.1 Partial traffic flow information, given by average hourly volume, is known about a network of five streets, as shown in Fig. 1. Complete the flow pattern for the network.



Solution To complete the traffic model, we need to find values for the eight unknown flows, as shown in Fig. 2.

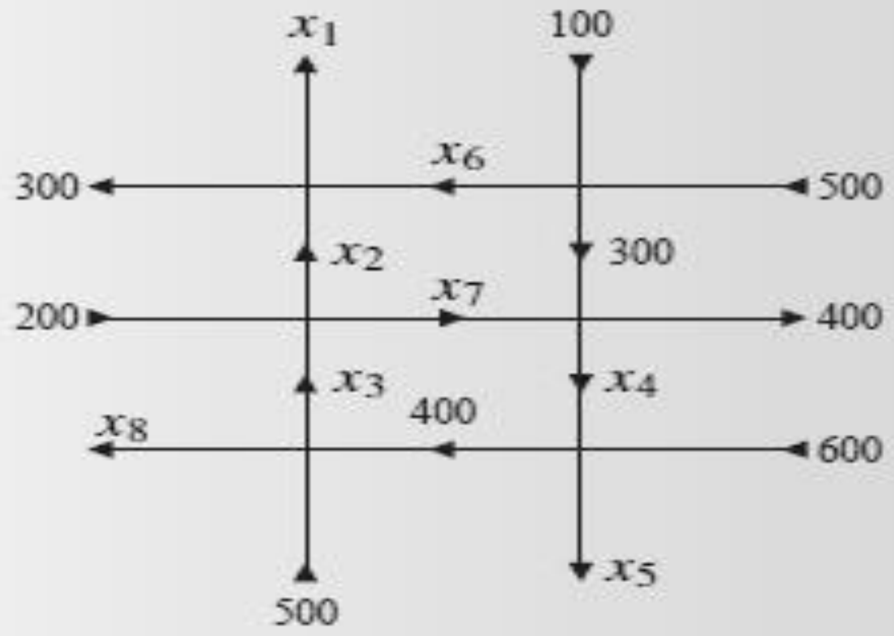


Figure 2

Our assumptions about the intersections give us the set of linear equations

$$\begin{cases} x_2 + x_6 & = 300 + x_1 \\ 100 + 500 & = x_6 + 300 \\ 200 + x_3 & = x_2 + x_7 \\ 300 + x_7 & = 400 + x_4 \\ 400 + 500 & = x_3 + x_8 \\ x_4 + 600 & = 400 + x_5 \end{cases}$$

Moreover, balancing the total flow into the network with the total flow out gives us the additional equation

$$500 + 600 + 500 + 200 + 100 = 400 + x_5 + x_8 + 300 + x_1$$

The final linear system is

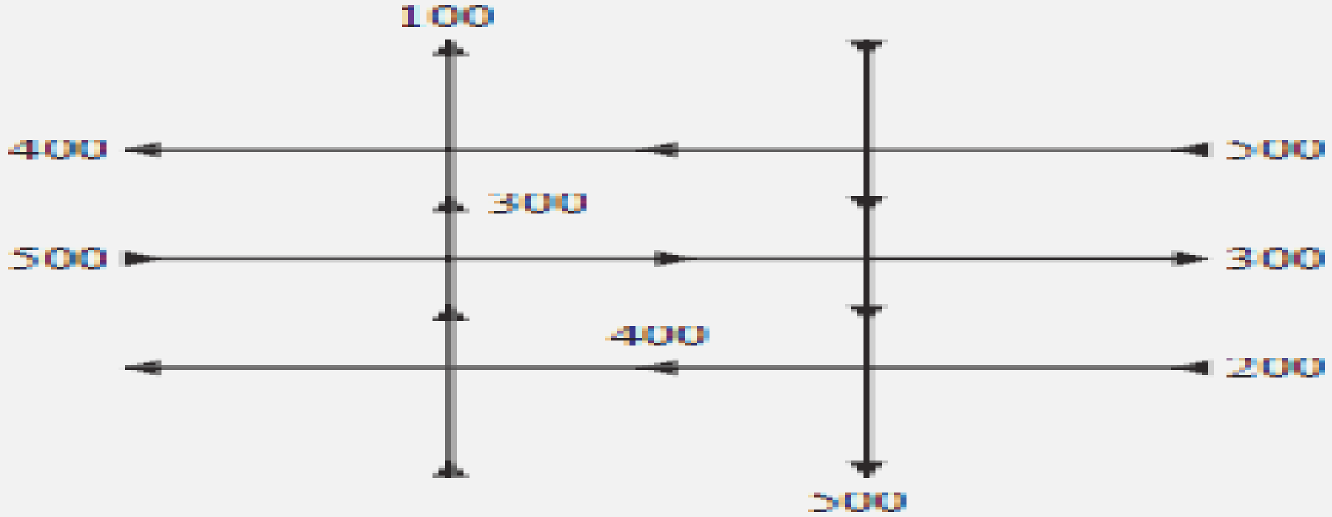
$$\left\{\begin{array}{rcrcrcrcrcrcrcrcl}
-x_1 & + & x_2 & & & & & & & + & x_6 & & & & & = & 300 \\
& & & & & & & & & & x_6 & & & & & = & 300 \\
& & x_2 & - & x_3 & & & & & & & + & x_7 & & & = & 200 \\
& & & & & - & x_4 & & & & & + & x_7 & & & = & 100 \\
& & & & x_3 & & & & & & & & & + & x_8 & = & 900 \\
& & & & & - & x_4 & + & x_5 & & & & & & & = & 200 \\
x_1 & & & & & & & + & x_5 & & & & & + & x_8 & = & 1200
\end{array}\right.$$

The solution is given by

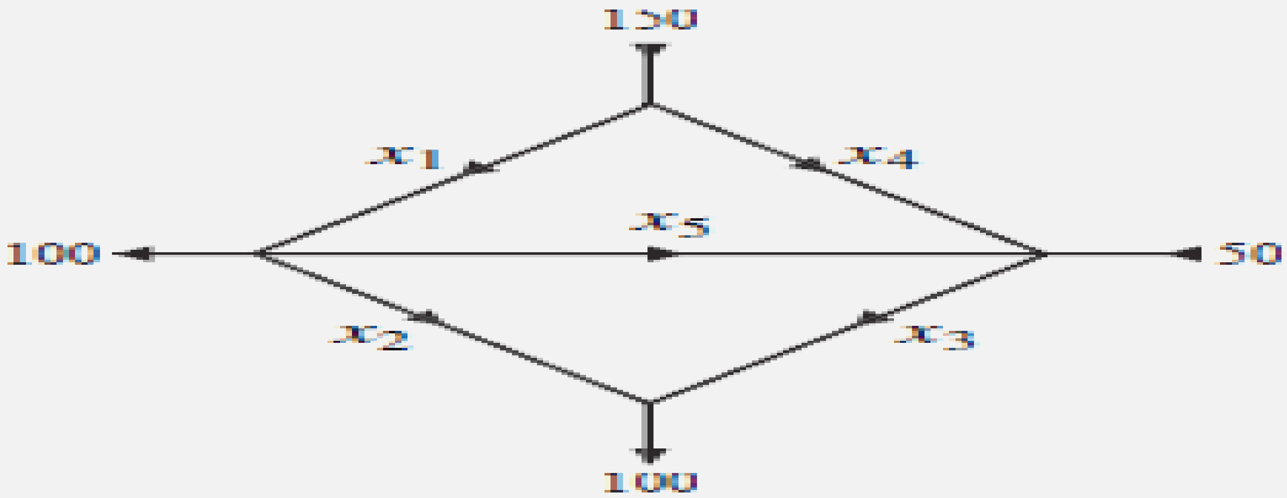
$x_1 = 1100 - s - t \qquad x_2 = 1100 - s - t \qquad x_3 = 900 - t \qquad x_4 = -100 + s$
$x_5 = 100 + s \qquad x_6 = 300 \qquad x_7 = s \qquad x_8 = t$

Notice that x_7 and x_8 are free variables. However, to obtain particular solutions, we must choose numbers for s and t that produce positive values for each x_i in the system (otherwise we will have traffic going in the wrong direction!) For example, $s = 400$ and $t = 300$ give a viable solution.

Q.2 Find the traffic flow pattern for the network in the figure. Flow rates are in cars per hour. Give one specific solution.



Q.3 Find the traffic flow pattern for the network in the figure. Flow rates are in cars per half-hour. What is the current status of the road labeled x_5?



Q.4 Find the traffic flow pattern for the network in the figure. Flow rates are in cars per half-hour. What is the smallest possible value for x_8?