## Riemann-Silberstein representation of the Maxwell equations set

M. V. Cheremisin A.F.Ioffe Physical-Technical Institute, St.Petersburg, Russia (Dated: October 30, 2018)

The complete set of Maxwell equations is represented by a single equation using Riemann-Silberstein complex vector of electromagnetic field. The consistent derivation of the Lorenz gauge condition is presented. We demonstrate that Fourier form of invariants **EB** and  $E^2 - B^2$  are proportional to dissipated power and equal to zero respectively.

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The relative orientation of the magnetic and electric fields in a certain co-ordinate system is of great importance since  $E^2 - B^2$ , **EB** are invariants upon Lorents[1] relativistic transformation. In this paper we represent the complete set of Maxwell equations by a single equation using Riemann-Silberstein[2, 3] complex vector  $\mathbf{F} =$  $\mathbf{E} + i\mathbf{B}$  whose product contains the both invariants.

We start from the conventional Maxwell equations set:

$$rot\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \text{div}\mathbf{B} = 0, \tag{1}$$

$$\begin{aligned}
\cot \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} & \text{div} \mathbf{B} &= 0, \\
\cot \mathbf{B} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J} & \text{div} \mathbf{E} &= 4\pi\rho, \end{aligned} \tag{1}$$

where the current  $\mathbf{J} = \mathbf{J}^{\mathrm{free}} + c \cdot \mathrm{rot} \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$  and the charge density  $\rho = \rho^{\mathrm{free}} - \mathrm{div} \mathbf{P}$  are generalized with respect to those for free carriers  $\mathbf{J}^{\mathrm{free}}$ ,  $\rho^{\mathrm{free}}$  taking into account the polarization  $\mathbf{P}$  and magnetization  $\mathbf{M}$ . In what follows we will search the electromagnetic fields caused by preassigned charge and current densities. Thus, we disregard the relationships  $\mathbf{J}^{\text{free}}(\mathbf{E}), \mathbf{P}(\mathbf{E})$  and  $\mathbf{M}(\mathbf{B})$  known for electromagnetic fields in continuous media.

The Eqs.(1) leads to conventional definition  $\mathbf{B} = \text{rot} \mathbf{A}$ ,  $\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$  known for vector  $\mathbf{A}$  and scalar  $\phi$  potential respectively. With the help of the above notation Eqs.(2) yield

$$\Box \mathbf{A} = -\frac{4\pi}{c} \mathbf{J} + \nabla \psi, \qquad \Box \phi = -4\pi \rho - \frac{1}{c} \frac{\partial \psi}{\partial t}.$$
 (3)

where  $\square$  is the d'Alembertian operator, then  $\psi = \text{div} \mathbf{A} +$  $\frac{1}{c}\frac{\partial \phi}{\partial t}$  is a certain combination of potentials. Note that  $\psi$  is arbitrary function being, however, invariant upon Lorentz relativistic transformations similar to Maxwell's equations itself. Nevertheless, in order to decouple Eqs.(3) an extra condition

$$\operatorname{div} \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0. \tag{4}$$

is usually implemented. The later is known as the gauge condition put forward first by Lorenz[4]. Historically, the latter was erroneously attributed[5] to H. A. Lorentz, who made, in turn, the enormous contribution to classical electrodynamics. The crucial point of our paper concerns the rigorous derivation of Eq.(4). We insist the later arises from the Lorentz relativistic transformations

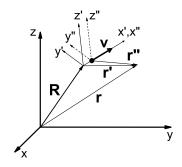


FIG. 1: Moving charge in laboratory L(bold), stationary L'(thin) and charge-centered L''(dashed) coordinate frames.

of the potentials  $\mathbf{A}, \phi$  counted for every charge-centered coordinate system where the respective charge is assumed to be static.

Following [6] let us consider(see Fig.1) the elementary charge moving at a certain velocity v counted for laboratory coordinate system L. In addition, we introduce a mobile coordinate system L" moving synchronously with the charge. The charge velocity is assumed to be directed along x"-axis of the mobile frame. Then, let us make use of the stationary co-ordinate system L' whose axes coincide with those of the mobile frame L'' at a moment t = 0.

We argue that the charge is motionless within the mobile frame L". The respective electric field is described by scalar potential  $\phi'' = \frac{e}{|r''|}$ , where  $\mathbf{r}''$  is the radius vector of the observation point within the mobile frame L''. For stationary coordinate system L' both the scalar and vector potentials are present[6]

$$\mathbf{A}' = \phi' \frac{\mathbf{v}}{c}, \qquad \phi' = \frac{\phi''}{\sqrt{1 - v^2/c^2}} = \frac{e}{R^*}$$
 (5)

because of the Lorentz relativistic transformations[1]. Here,  $R^* = [(x'-vt)^2 + (1-v^2/c^2)(y'^2+z'^2)]^{1/2}$ , where x', y', z' denote the coordinates of the observation point counted for stationary frame L', then  $v = |\mathbf{v}|$ . We emphasize that the potentials specified by Eq.(5) depend on the difference x' - vt. Neglecting the spatial dependence of the charge velocity we obtain  $\operatorname{div} \mathbf{A}' = (\mathbf{v}, \nabla \phi')/c$  and, furthermore, recover the Lorenz gauge condition  $\psi' = 0$ 

for sole charge within stationary frame L'. This result is valid overall the stationary frame for arbitrary moment and charge velocity. Remarkably, the same condition remains valid for a sole charge within laboratory frame L. Indeed, the stationary frame L' can be superposed on the laboratory frame L under appropriate axes rotation and subsequent translation(see radius vector  $\mathbf{R}$  in Fig.1) of L'-frame center. In general, the axis rotation and(or) center translation of an arbitrary frame is known to keep the single Lame's indices[7]. The divergence of vector potential remains constant, i.e.  $\operatorname{div} \mathbf{A} = \operatorname{div} \mathbf{A}'$ . Moreover, the scalar potential remains unchanged as well, therefore  $\phi = \phi'$ . As a result, the Lorentz gauge is justified for sole charge within the laboratory frame L as well.

Let us now evaluate the summation over the all elementary charges  $\sum_j [\psi_j = 0]$ , where the subscript denotes j-th elementary charge. Obviously, for laboratory frame L we find both the vector  $\mathbf{A} = \sum_j \mathbf{A}_j$  and the scalar  $\phi = \sum_j \phi_j$  potentials being find exactly those embedded into Eqs.(3,4). With the help of Lorentz gauge specified by Eq.(4), the Eqs.(3) obey a familiar form

$$\mathbf{\Delta} \mathbf{A} = -\frac{4\pi}{c} \mathbf{J}, \qquad \mathbf{\Phi} = -4\pi\rho \tag{6}$$

and, moreover, result in conventional charge conservation law.

Let us discuss the gauge invariance conditions[6]. In general, the electromagnetic fields  $\mathbf{E}, \mathbf{B}$  remain unchanged under the substitution  $\mathbf{A} \to \mathbf{A} + \nabla f$ ,  $\phi \to \phi - \frac{1}{c} \frac{\partial f}{\partial t}$ , where f is an arbitrary scalar function. Searching the respective solution of Eqs.(6) one immediately find an extra equation  $\Box f = 0$  for presumably arbitrary function f. We attribute this finding to potentials  $\mathbf{A}, \phi$  and electromagnetic fields  $\mathbf{E}, B$  are known to obey d'Alembert-like homogeneous equation for charge-free media. We emphasize that the gauge invariance conditions yield the replacement  $\psi \to \psi + \Box f$  and, hence provide the Lorenz gauge  $\psi = 0$  identity.

We now use the Riemann-Silberstein complex vector  $\mathbf{F} = \mathbf{E} + i\mathbf{B}$  and, then make an attempt to re-write the Maxwell equations set. At first, we note the identity

$$\dot{\mathbf{F}} = \dot{\mathbf{E}} - ic \cdot \text{rot}\mathbf{E},\tag{7}$$

which will be helpful in further investigation.

The Maxwell's equations set specified by Eqs.(1,2) yields

$$rot\mathbf{F} = \frac{i}{c}(\dot{\mathbf{F}} + 4\pi\mathbf{J}), \qquad \text{div}\mathbf{F} = 4\pi\rho$$
 (8)

or, in a shortened form

$$\mathbf{r}\dot{\mathbf{F}} = \ddot{\mathbf{I}} - ic \cdot \text{rot}\dot{\mathbf{I}},\tag{9}$$

where  $\dot{\mathbf{I}} = 4\pi(\dot{\mathbf{J}}/c^2 + \nabla \rho)$  is an auxiliary vector. It is rather fascinating that the all Maxwell's equations may be represented by means of a single equation. One can easily find the relationships

$$\mathbf{D}\mathbf{E} = \dot{\mathbf{I}}, \qquad \mathbf{D}\dot{\mathbf{B}} = -ic \cdot \text{rot}\dot{\mathbf{I}}. \tag{10}$$

comparing Eqs.(7),(9).

We now able to discuss in greater details the problem of electromagnetic field invariants. Within Fourier formalism  $\mathbf{F} = \int\limits_{-\infty}^{+\infty} \mathbf{F}_{k,\omega} e^{i\omega t - i\mathbf{k}\mathbf{r}} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d\omega}{2\pi}$  the Eq.(7) yields the algebraic equation

$$\mathbf{F}_{k,\omega} = \mathbf{E}_{k,\omega} + \frac{ic}{\omega} [\mathbf{k} \mathbf{E}_{k,\omega}]. \tag{11}$$

which immediately gives the zero imaginary part of the product  $\operatorname{Im} \mathbf{F}_{k,\omega}^2 = 2(\mathbf{E}_{k,\omega}, \mathbf{B}_{k,\omega}) \equiv 0$ . In contrast, the real part of the product  $\operatorname{Re} \mathbf{F}_{k,\omega}^2 = \mathbf{E}_{k,\omega}^2 - \mathbf{B}_{k,\omega}^2$  remains finite. Indeed, with the help of Fourier form of Eq.(6),(10) we obtain

$$\operatorname{Re}\mathbf{F}_{k\,\omega}^{2} = \kappa^{2}(\mathbf{A}_{k\,\omega}^{2} - \phi_{k\,\omega}^{2}). \tag{12}$$

where  $\kappa^2 = k^2 - \omega^2/c^2$ . According to Eq.(12) the field invariant in terms of Fourier components is proportional to certain combination complied from Fourier components of 4-vector potential. This result is highly expected since the scalar  $\mathbf{A}^2 - \phi^2$  is known[8] to be invariant upon Lorentz relativistic transformations. Apart from Eq.(12), a similar proportionality is occurred for Fourier replicas of 4-vector current  $\mathbf{J}^2 - c^2 \rho^2$  and scalar product  $(\mathbf{A}, \mathbf{J}) - c\phi\rho$  invariants[8]. We emphasize that Eq.(12) can be, in addition, re-written in terms of dissipated power, namely  $\mathrm{Re}\mathbf{F}_{k,\omega}^2 = \frac{4\pi i}{\omega}(\mathbf{J}_{k,\omega},\mathbf{E}_{k,\omega})$ . In absence of dissipation Eqs.(12) gives a familiar result for electromagnetic wave propagating in vacuum when  $\mathbf{E}_{k,\omega}^2 - \mathbf{B}_{k,\omega}^2 \equiv 0$ . The present study points to substantial reduction of invariant multiplicity, at least within Fourier formalism.

In conclusion, we represent set of Maxwell equations by single equation using complex electromagnetic field. The Fourier form of invariants  $E^2-B^2$  and **EB** is proportional to dissipated power and equal to zero respectively.

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