2.1: Green's Functions

I have spent so much time/space discussing the potential distributions created by a single point charge in various conductor geometries because for any of the geometries, the generalization of these results to the arbitrary distribution $\rho(\mathbf{r})$ of free charges is straightforward. Namely, if a single charge q, located at some point $\mathbf{r}'$, creates the electrostatic potential

$$\phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}qG\left(\mathbf{r},\mathbf{r}'\right), \tag{2.202}$$

then, due to the linear superposition principle, an arbitrary charge distribution (either discrete or continuous) creates the potential

Spatial Green's function

$$\phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\sum_j q_jG\left(\mathbf{r},\mathbf{r}_j\right)=\frac{1}{4\pi\varepsilon_0}\int\rho\left(\mathbf{r}'\right)G\left(\mathbf{r},\mathbf{r}'\right)d^3r'. \tag{2.203}$$

The function $G\left(\mathbf{r},\mathbf{r}'\right)$ is called the (spatial) Green's function – the notion very fruitful and hence popular in all fields of physics.[65] Evidently, as Eq. (1.35) shows, in the unlimited free space

$$G\left(\mathbf{r},\mathbf{r}'\right)=\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}, \tag{2.204}$$

i.e. the Green's function depends only on one scalar argument – the distance between the field-observation point $\mathbf{r}$ and the field-source (charge) point $\mathbf{r}'$. However, as soon as there are conductors around, the situation changes. In this course, I will only discuss the Green's functions that vanish as soon as the radius-vector $\mathbf{r}$ points to the surface (S) of any conductor:[66]

$$\left.G\left(\mathbf{r},\mathbf{r}'\right)\right|_{\mathbf{r}\in S}=0. \tag{2.205}$$

With this definition, it is straightforward to deduce the Green's functions for the solutions of the last section's problems in which conductors were grounded $(\phi=0)$. For example, for a semi-space $z\geq 0$ limited by a conducting plane $z=0$ (Fig. 26), Eq. (185) yields

$$G=\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}-\frac{1}{\left|\mathbf{r}-\mathbf{r}''\right|},\qquad \text{with } \rho''=\rho' \text{ and } z''=-z'. \tag{2.206}$$

We see that in the presence of conductors (and, as we will see later, any other polarizable media), the Green's function may depend not only on the difference $\mathbf{r}-\mathbf{r}'$, but on each of these two arguments in a specific way.

So far, this looks just like re-naming our old results. The really non-trivial result of this formalism for electrostatics is that, somewhat counter-intuitively, the knowledge of the Green's function for a system with grounded conductors (Fig. 30a) enables the calculation of the field created by voltage- biased conductors (Fig. 30b), with the same geometry. To show this, let us use the so-called Green's theorem of the vector calculus.[67] The theorem states that for any two scalar, differentiable functions $f(\mathbf{r})$ and $g(\mathbf{r})$, and any volume V,

$$\int_V\left(f\nabla^2g-g\nabla^2f\right)d^3r=\oint_S(f\nabla g-g\nabla f)_n d^2r, \tag{2.207}$$

where S is the surface limiting the volume. Applying the theorem to the electrostatic potential $\phi(\mathbf{r})$ and the Green's function G (also considered as a function of $\mathbf{r}$), let us use the Poisson equation (1.41) to replace $\nabla^2\phi$ with $(-\rho/\varepsilon_0)$, and notice that G, considered as a function of $\mathbf{r}$, obeys the Poisson equation with the δ-functional source:

$$\nabla^2G\left(\mathbf{r},\mathbf{r}'\right)=-4\pi\delta\left(\mathbf{r}-\mathbf{r}'\right). \tag{2.208}$$

(Indeed, according to its definition (202), this function may be formally considered as the field of a point charge $q=4\pi\varepsilon_0$.) Now swapping the notation of the radius-vectors, $\mathbf{r}\leftrightarrow\mathbf{r}'$, and using the Green's function symmetry, $G\left(\mathbf{r},\mathbf{r}'\right)=G\left(\mathbf{r}',\mathbf{r}\right)$,[68] we get

$$-4\pi\phi(\mathbf{r})-\int_V\left(-\frac{\rho\left(\mathbf{r}'\right)}{\varepsilon_0}\right)G\left(\mathbf{r},\mathbf{r}'\right)d^3r'=\oint_S\left[\phi\left(\mathbf{r}'\right)\frac{\partial G\left(\mathbf{r},\mathbf{r}'\right)}{\partial n'}-G\left(\mathbf{r},\mathbf{r}'\right)\frac{\partial\phi\left(\mathbf{r}'\right)}{\partial n'}\right]d^2r'. \tag{2.209}$$

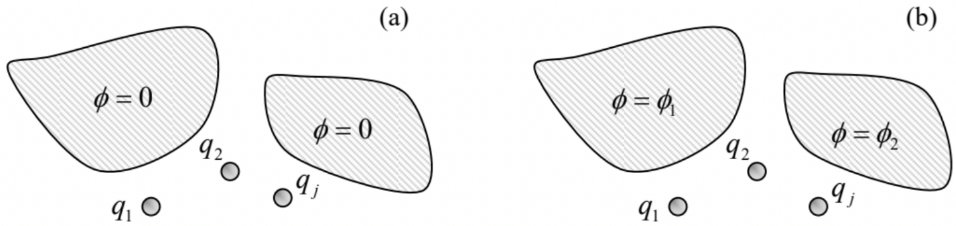

Fig. 2.30. Green's function method allows the solution of a simpler boundary problem (a) to be used to find the solution of a more complex problem (b), for the same conductor geometry.

Let us apply this relation to the volume V of free space between the conductors, and the boundary S drawn immediately outside of their surfaces. In this case, by its definition, Green's function $G(\mathbf{r}, \mathbf{r}')$ vanishes at the conductor surface $(\mathbf{r} \in S)$ – see Eq. (205). Now changing the sign of $\partial n'$ (so that it would be the outer normal for conductors, rather than free space volume V), dividing all terms by 4π, and partitioning the total surface S into the parts (numbered by index j) corresponding to different conductors (possibly, kept at different potentials ϕ_k), we finally arrive at the famous result:[69]

Potential via Green's function

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}')\, d^3r' + \frac{1}{4\pi} \sum_k \phi_k \oint_{S_k} \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} d^2r'. \tag{2.210}$$

While the first term on the right-hand side of this relation is a direct and evident expression of the superposition principle, given by Eq. (203), the second term is highly non trivial: it describes the effect of conductors with non-zero potentials ϕ_k (Fig. 30b), using the Green's function calculated for the

similar system with grounded conductors, i.e. with all $\phi_k = 0$ (Fig. 30a). Let me emphasize that since our volume V excludes conductors, the first term on the right-hand side of Eq. (210) includes only the stand-alone charges in the system (in Fig. 30, marked q_1, q_2, etc.), but not the surface charges of the conductors – which are taken into account, implicitly, by the second term.

In order to illustrate what a powerful tool Eq. (210) is, let us use to calculate the electrostatic field in two systems. In the first of them, a plane, circular, conducting disk, separated with a very thin cut from the remaining conducting plane, is biased with potential $\phi = V$, while the rest of the plane is grounded – see Fig. 31.

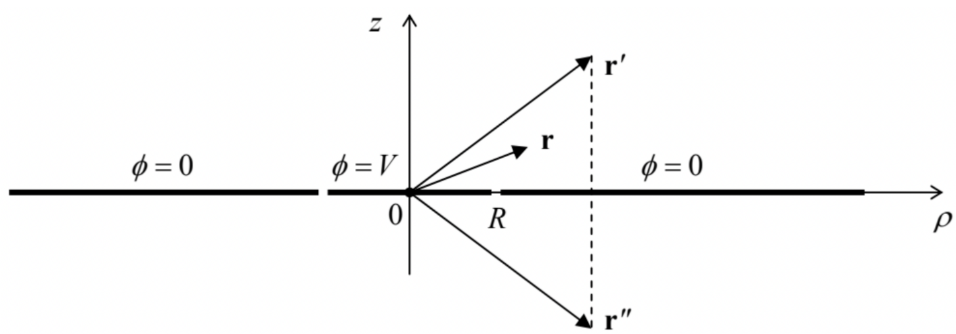

Fig. 2.31. A voltage-biased conducting disk separated from the rest of a conducting plane.

If the width of the gap between the disk and the rest of the plane is negligible, we may apply Eq. (210) without stand-alone charges, $\rho(\mathbf{r}') = 0$, and the Green's function for the uncut plane – see Eq. (206).[70] In the cylindrical coordinates, with the origin at the disk's center (Fig. 31), the function is

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}') = & \frac{1}{\left[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + (z - z')^2\right]^{1/2}} \\ & - \frac{1}{\left[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + (z + z')^2\right]^{1/2}}. \end{aligned} \tag{2.211}$$

(The sum of the first three terms under each square root in Eq. (211) is just the squared distance between the horizontal projections $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ of the vectors $\mathbf{r}$ and $\mathbf{r}'$ (or $\mathbf{r}''$) correspondingly, while the last terms are the squares of their vertical displacements.)

Now we can readily calculate the derivative participating in Eq. (210), for $z \geq 0$:

$$\left.\frac{\partial G}{\partial n'}\right|_s=\left.\frac{\partial G}{\partial z'}\right|_{z'=+0}=\frac{2z}{\left(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2\right)^{3/2}}. \tag{2.212}$$

Due to the axial symmetry of the system, we may take φ for zero. With this, Eqs. (210) and (212) yield

$$\phi=\frac{V}{4\pi}\oint_S\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}d^2r'=\frac{Vz}{2\pi}\int_0^{2\pi}d\varphi'\int_0^R\frac{\rho'd\rho'}{\left(\rho^2+\rho'^2-2\rho\rho'\cos\varphi'+z^2\right)^{3/2}}. \tag{2.213}$$

This integral is not overly pleasing, but may be readily worked out at least for points on the symmetry axis $(\rho=0, z\geq 0)$:[71]

$$\phi=Vz\int_0^R\frac{\rho'd\rho'}{\left(\rho'^2+z^2\right)^{3/2}}=\frac{V}{2}\int_0^{R^2/z^2}\frac{d\xi}{(\xi+1)^{3/2}}=V\left[1-\frac{z}{\left(R^2+z^2\right)^{1/2}}\right] \tag{2.214}$$

This result shows that if $z\rightarrow 0$, the potential tends to V (as it should), while at $z\gg R$,

$$\phi\rightarrow V\frac{R^2}{2z^2}. \tag{2.215}$$

Now let us use the same Eq. (210) to solve the (in :-)famous problem of the cut sphere (Fig. 32). Again, if the gap between the two conducting semi-spheres is very thin $(t<<R)$, we may use Green's function for the grounded (and uncut) sphere. For a particular case $\mathbf{r}'=d\mathbf{n}_z$, this function is given by
Eqs. (197)-(198); generalizing the former relation for an arbitrary direction of vector $\mathbf{r}'$, we get

$$G=\frac{1}{\left[r^2+r'^2-2rr'\cos\gamma\right]^{1/2}}-\frac{R/r'}{\left[r^2+\left(R^2/r'\right)^2-2r\left(R^2/r'\right)\cos\gamma\right]^{1/2}},\qquad \text{for } r,r'\geq R, \tag{2.216}$$

where γ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$, and hence $\mathbf{r}''$ – see Fig. 32.

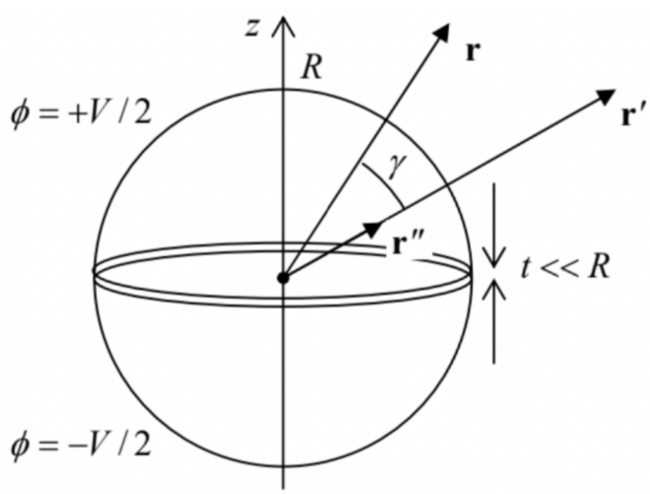

Fig. 2.32. A system of two separated, oppositely biased semi-spheres.

Now, calculating the Green's function's derivative,

$$\left.\frac{\partial G}{\partial r'}\right|_{r'=R+0}=-\frac{\left(r^2-R^2\right)}{R\left[r^2+R^2-2Rr\cos\gamma\right]^{3/2}}, \tag{2.217}$$

and plugging it into Eq. (210), we see that the integration is again easy only for the field on the symmetry axis ($\mathbf{r}=z\mathbf{n}_z, \gamma=\theta$), giving:

$$\phi=\frac{V}{2}\left[1-\frac{z^2-R^2}{z\left(z^2+R^2\right)^{1/2}}\right]. \tag{2.218}$$

For $z\rightarrow R$, this relation yields $\phi\rightarrow V/2$ (just checking :-), while for $z/R\rightarrow\infty$,

$$\phi \rightarrow \frac{3R^2}{4z^2}V. \tag{2.219}$$

As will be discussed in the next chapter, such a field is typical for an electric dipole.

Reference

[65] See, e.g., CM Sec. 5.1, QM Secs. 2.2 and 7.4, and SM Sec. 5.5.

[66] G so defined is sometimes called the Dirichlet function.

[67] See, e.g., MA Eq. (12.3). Actually, this theorem is a ready corollary of the better-known divergence ("Gauss") theorem, MA Eq. (12.2).

[68] This symmetry, evident for the particular cases (204) and (206), may be readily proved for the general case by applying Eq. (207) to functions $f(\mathbf{r}) \equiv G(\mathbf{r}, \mathbf{r}')$ and $g(\mathbf{r}) \equiv G(\mathbf{r}, \mathbf{r}'')$. With this substitution, the left-hand side of that equality becomes equal to $-4\pi\left[G(\mathbf{r}'', \mathbf{r}') - G(\mathbf{r}', \mathbf{r}'')\right]$, while the right-hand side is zero, due to Eq. (205).

[69] In some textbooks, the sign before the surface integral is negative, because their authors use the outer normal to the free-space region V rather than that occupied by conductors – as I do.

[70] Indeed, if all parts of the cut plane are grounded, a narrow cut does not change the field distribution, and hence the Green's function, significantly.

[71] There is no need to repeat the calculation for $z \leq 0$: from the symmetry of the problem, $\phi(-z) = \phi(z)$.