INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
THIRUVANANTHAPURAM 695 547

End Semester Examination - July 2022

B.Tech - II Semester

MA121 - Vector Calculus and Ordinary Differential Equations

Date: 01/07/2022 **Time: 09.30 am - 11.30 am** **Max. Marks: 40**

Answer all questions

1. (a) Let $E = \left\{ \dfrac{i}{2^{10,000}} \; ; \; i = 0, 1, 2, \cdots, 2^{10,000} \right\}$
$f : E \to \mathbb{R}$ be defined by
$$f_n(x) = x^n.$$
Does f_n converges uniformly to some f? [2]

   (b) Does the sequence
$$f_n(x) = \frac{nx}{1 + n^3x^2} + x^n(x-1)(1 - \cos x), \; x \in [0, 1]$$
converges uniformly. [2]

2. State whether the following are true/false. If true, prove it. If not give a counter example (with proof).

   (a) $f_n \xrightarrow{u} f, \; g_n \xrightarrow{u} g \;\Rightarrow\; f_n g_n \xrightarrow{u} fg.$ [1.5]

   (b) $f_n \xrightarrow{u} f$ in $[a, b] \;\Rightarrow\; \displaystyle\int_a^b f_n(x)dx \longrightarrow \int_a^b f(x)dx.$ [1.25]

   (c) $f_n \xrightarrow{u} f$ in $[a, b] \;\Rightarrow\; \displaystyle\lim_{n \to \infty} f_n{}'(x) = f(x) \; \forall \; x \in [a, b].$ [1.25]

3. A function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called homogeneous of degree $n \geq 1$ if $g(tx, ty) = t^n g(x, y)$ for all $t \in \mathbb{R}$ and for all $(x, y) \in \mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a homogeneous function of degree $n \geq 1$. Show that $D_{\vec{v}} f|_{(0,0)}$ exists for all direction vector $\vec{v}$ and find its value. Show that following function f is a homogeneous function and find $D_{\vec{v}} f|_{(0,0)}$ along all directions $\vec{v}$.
$$f(x, y) = \begin{cases} \dfrac{xy(x+y)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$
Is the above function f differentiable at $(0, 0)$? [2+1+1]

4. (a) Find the arc length function of the path $\gamma(t) = (\frac{2\sqrt{2}}{3} t^{3/2}, \, t \sin t, \, t \cos t)$ between $(0, 0, 0)$ and $(\cancel{\pi^2}, 0, \cancel{-\pi^2})$ and hence find the length of the curve. Find the Cartesian equation corresponding the given parametric equation of the curve. Is the given parametrization arc-length parametrization of the curve? [1+0.5+0.5]
$\left(\frac{2\sqrt{2}}{3}\pi^{3/2}, 0, -\pi\right)$

   (b) Find curvature of a circle. [2]

5. (a) Let $\vec{F} = (xy^2e^z, \; x^2ye^z + y, \; \frac{1}{2}x^2y^2e^z + z)$ be a vector field and C be a curve joining the points in the given order $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 0)$ by straight line segments. Evaluate $\displaystyle\int_C \vec{F}$. Is $\vec{F}$ a conservative vector field? Justify your answer. [2]

(b) Using Green's theorem find the double integral $\iint_G (y-1)\, dx\, dy$ in terms of line integral where G is given by $G: x^2+y^2 \leq 4$. [2]

6. (a) Let $F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ and C a simple positively oriented smooth loop in $\mathbb{R}\backslash\{(0,0)\}$. With proper arguments find the value of $\int_C F$. **[If a integral is well known, you do not need to calculate; may put the value directly]** [2]

   (b) Using Green's theorem find the area of the region G bounded by the ellipse $4x^2+9y^2=1$ and the straight line $y=0$ in the upper half plane of $\mathbb{R}^2$. [2]

7. Consider the IVP:

$$\frac{dy}{dx} = 1+y^2, \quad y(0)=0.$$

   (a) Using the *Picard's Theorem*, determine the "best possible" value of h $(h>0)$ such that the IVP has a unique solution in the interval $|x| \leq h$.

   (b) Is it possible to determine existence of the unique solution of the IVP in the same interval using the *non-local existence Theorem*? Justify your answer. [2+1]

8. (a) Let ϕ_1, ϕ_2 be two real valued differentiable functions on the interval $[a,b]$. Prove that if ϕ_1 and ϕ_2 are linearly dependent on $[a,b]$, then the Wronskian $W(\phi_1,\phi_2)(x) = 0$, for all $x \in [a,b]$. [2]

   (b) Write particular solution of the equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = xe^{2x}$$

   in terms of linear combination of suitable functions. [2]
   (Determining the coefficients is not required.)

9. Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{-3x}/x, \quad x>0.$$

[5]

10. Let ϕ_1 and ϕ_2 be two linearly independent series solutions of the equation

$$(x-x^2)\frac{d^2y}{dx^2} + (1-5x)\frac{dy}{dx} - 4y = 0$$

about the point $x=0$ for the region $x>0$.

   (a) Classify the point $x=0$ with justification and determine the indicial equation.

   (b) Write the form of the solutions ϕ_1 and ϕ_2.
   (Determining the coefficients is not required.) [2+2]

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