Unit 5

3.1 Linear Algebraic Equation

Here we solve simultaneous linear equations that can be represented as,

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots\cdots\cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots\cdots\cdots + a_{2n} x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{n1} x_1 + a_{n2} x_2 + \cdots\cdots\cdots + a_{nn} x_n &= b_n
\end{aligned}
$$

where a's are constant coefficients and b's are constants.

3.2 Solving small number of equation

There are several methods that are appropriate for small ($n \leq 3$) sets of simultaneous equations and do not required compiler.

3.2.1 Graphical Method

This method is applicable for two variable simultaneous linear equations. If there are two variables as x_1 and x_2 then we can plot equations on Cartesian co-ordinates with one axis corresponding to x_1 and the other to x_2. Because we are solving linear equations so each equation is a straight line. This can be easily illustrated for the general equations,

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 &= b_1 \\
a_{21} x_1 + a_{22} x_2 &= b_2
\end{aligned}
$$

Both equations can be solved for x_2.

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right) x_1 + \frac{b_1}{a_{12}} \qquad \text{And}$$

$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right) x_1 + \frac{b_2}{a_{22}}$$

Thus the equations are now in the form of straight lines: that is

Slope

$$x_2 = (\text{slop})\, x_1 + \text{intercept}.$$

These lines can be graphed on Cartesian co-ordinates with x_2 as the ordinate and x_1 as the abscissa. The value of x_1 and x_2 at the intersection of the lines represent the solution.

For three simultaneous equations each equation would be represented by a plane in a 3-dimensional co-ordinate system, the point where the three points intersect would represent the solution.

3.3 Gauss Elimination Method:

In this method, the variables from the system of linear equations are eliminated successively and the system of equations is therefore reduced to an upper triangular system from which the variables are determinate by back substitution.

Let us consider, the linear equations

$$a_{11}x_1 + a_{12}x_2 + \text{------------------} + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \text{------------------} + a_{2n}x_n = b_2$$
$$\text{--}$$
$$\text{--}$$
$$a_{n1}x_1 + a_{n2}x_2 + \text{------------------} + a_{nn}x_n = b_n$$

This can be represent in matrix form as,

$$AX = B$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix};\ X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

We actually work on $[A|B]$. Matrix describe as,

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

To eliminate x_1 from second equation, we multiply the first equation by $\left(-\frac{a_{21}}{a_{11}}\right)$ and then add it to the second equation. Similarly, to eliminate x_1 from third equation we multiply the first equation by $\left(-\frac{a_{31}}{a_{11}}\right)$ and add it to the third. By continuing this way we can eliminate x_1 from all the equations. Hence we get $[A:B]$ matrix as

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

Here a_{11} is known as pivot element.

In the second step we eliminate x_2 from third, fourth, ---------, n^{th} equation. By proceeding this way we will get lower triangular zero. We have only upper triangular system.

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} & b_n \end{bmatrix}$$

Now from back substitution we can find the solution of the system of given equations.
For example, if we have three equations then we will get [A:B] matrices.

$$[A \mid B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & a_{33} & b_3 \end{bmatrix}$$

We get equations as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 &= b'_2 \\ a'_{33}x_3 &= b'_3 \end{aligned}$$

Now using back substitution we get the value of variables as

$$x_3 = \frac{b'_3}{a'_{33}}$$

$$x_2 = \frac{b'_2 - a'_{23}x_3}{a_{22}}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

Hence we have the value of x_1, x_2, and x_3.

3.4 Pitfalls (Drawbacks) of Elimination Method:

Gauss elimination is good for large equations, but there are some pitfalls that must be explored before writing a general compute program to implement this method.

(1) Division by Zero:

The Gauss elimination method is known as naïve (basic) gauss elimination; because during both the elimination and the back substitution phase it is possible that a division by zero can occur.
For example,

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 4x_1 + 6x_2 + 7x_3 &= -3 \\ 2x_1 + x_2 + 6x_3 &= 5 \end{aligned}$$

In this example we will find division by zero because $a_{11} = 0$.

(2) Round-off Errors:

On the time when we solve equations having fractional coefficients. We can get long fraction on the time of computation. Which will difficult to evaluate. To reduce this problem we can round-off the decimal part but after rounding-off, next coefficient will be evaluated according to modified value. So it will have small round-off error.

The problem of round-off error can become particularly important when large numbers of equations are to be solved. This is due to the fact that every result is dependent on previous results. Consequently an error in the early step will tend to propagate, that is, it will cause errors in subsequent steps.

(3) ~~Ill-conditioned System~~: Ill- Conditioned System ⇒

Ill-conditioned systems are those where small changes in coefficients results in large changes in the solution. An alternative interpretation of ill-conditioning is that a wide range of answers can approximately satisfy the equations. Because round-off error can reduce small changes in the coefficients, there artificial changes can lead to large solution errors for ill-conditioned system, as illustrated in the following example

$$x_1 + 2x_2 = 10$$
$$1.1x_1 + 2x_2 = 10.4$$

Answer $x_1 = 4$, $x_2 = 3$

However, with a slight change of the coefficient a_{21} from 1.1 to 1.05 the result is changes to $x_1 = 8$, $x_2 = 1$

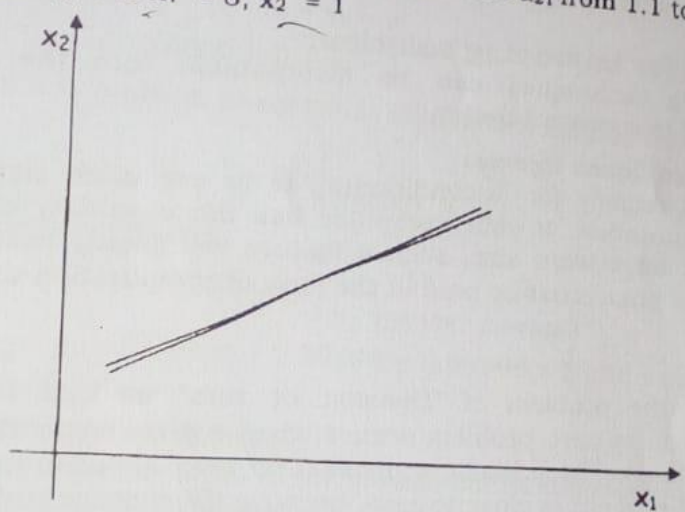

Figure 3.4 (Ill condition System)

Figure 3.4 demonstrate ill condition system, here we can not find the exact cross point of two lines.

Mathematically we can characterize this situation as when the slopes of two equations are nearly to equal then ill-condition occurs.

i.e.

$$a_{11} x_1 + a_{12} x_2 = b_1 \text{ ------------------(1)}$$
$$a_{21} x_1 + a_{22} x_2 = b_2 \text{ ------------------ (2)}$$

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_2}{a_{12}}$$

$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$

$$\frac{a_{11}}{a_{12}} \approx \frac{a_{21}}{a_{22}} \qquad \text{or}$$

$$a_{11}\,a_{22} - a_{12}\,a_{21} \approx 0$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \approx 0$$

In other words we can say that an ill-conditioned system is one with a determinant close to zero or zero.

(4) Singular System:

The case when any two equations represent parallel lines or the case where the two lines are co-incident known as singular system. Singular system gives zero or infinite solution. Use figure 3.1 and 3.2 if required.

Problem 1: Solve the following system by Gauss elimination method

$x_1+2x_2+3x_3=8$

$2x_1+4x_2+9x_3=8$

$4x_1+3x_2+2x_3=2$

Solution: We can write given equation in matrix form as [A|B] as follows

$$[A|B]= \begin{bmatrix} 1 & 2 & 3 & 8 \\ 2 & 4 & 9 & 8 \\ 4 & 3 & 2 & 2 \end{bmatrix}$$

Use the following row transformations to make a_{21} and a_{31} zero

$R_2 = R_1 * \left(-a_{21}/a_{11}\right) + R_2$	$R_3 = R_1 * \left(-a_{31}/a_{11}\right) + R_3$
$R_2 = R_1 * (-2) + R_2$	$R_3 = R_1 * (-4) + R_3$
$R_2 \Rightarrow 1*(-2)+2=0$	$R_3 \Rightarrow 1*(-4)+4= 0$
$\Rightarrow 2*(-2)+4=0$	$\Rightarrow 2*(-4)+3= -5$
$\Rightarrow 3*(-2)+9=3$	$\Rightarrow 3*(-4)+2=-10$
$\Rightarrow 8*(-2)+8=-8$	$\Rightarrow 8*(-4)+2=-30$

$$[A|B]= \begin{bmatrix} 1 & 2 & 3 & 8 \\ 0 & 0 & 3 & -8 \\ 0 & -5 & -10 & -30 \end{bmatrix}$$

By using the first two row transformation,the term a_{22} became zero, so in next row transformation pivot element(i.e. a_{22}) is zero .we can not use direct row transformation on above matrix,so we use row echange to make a_{22} as non zero.

$$[A|B]= \begin{bmatrix} 1 & 2 & 3 & 8 \\ 0 & -5 & -10 & -30 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

Now, there is no need to use third row transformation. Since, lower triangular matrix is zero, as required in gauss elimination method

We can write expression for this matrix as

$x_1+2x_2+3x_3= 8$

$-5x_2 -10x_3= -30$

$3\, x_3= -8$

Now by using back substitution,

$3\, x_3=-8 \quad => x_3=-8/3$

put $x_3=-8/3$ in $5x_2 -10x_3= -30$

put x_3 and x_2 in $x_1+2x_2+3x_3= 8$

we get $x_1 = -20/3$; $x_2= 34/3$; $x_3=-8/3$

Problem 2: Solve the following system by Gauss elimination method

x+2y+3z=10

x+3y-2z = 7

2x-y+ z = 5

Solution:

We can write given equation in matrix form as [A|B] as follows

$$[A|B]=\begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 3 & -2 & 7 \\ 2 & -1 & 1 & 5 \end{bmatrix}$$

Use the following row transformations to make a_{21} and a_{31} zero

$R_2 = R_1 x \left(-a_{21}/a_{11}\right) + R_2$

$R_2 = R_1 x\, (-1) + R_2$

$R_3 = R_1\, x \left(-a_{31}/a_{11}\right) + R_3$

$R_3 = R_1 x\, (-2) + R_3$

$$[A|B]=\begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & 1 & -5 & -3 \\ 0 & -5 & -5 & 15 \end{bmatrix}$$

$R_{3}=R_2 \times \left(-a_{32}/a_{22}\right)+R_3$

$R_3= R_2 x (5)+ R_3$

$$[A|B]=\begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & -30 & -30 \end{bmatrix}$$

Since, lower triangular matrix is zero, as required in gauss elimination method

We can write expression for this matrix as

x+2y+3z= 10

y-5z= -3

-30z= -30

Now by using back substitution,

-30z=-30=Z=>1

Put z=1 in y-5z=-3=>y=2

put z and y in x+2y+3z=10=>x=3

we get x=3; y=2; z=1

Numerical Integration :- Numerical integration is the approximate computation of an integral using numerical techniques.

The numerical computation of an integral is sometimes called quadrature. Some methods of numerical integration are as Trapezoidal methods, Simpson's $1/3^{rd}$ method and Simpson's $3/8^{th}$ method.

The process of finding the value of the integral of a function of a single variable is known a Quadrature.

The general formula for quadrature is-

$$\int_{x_0}^{x_n} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots + \text{up to}(n+1)\text{terms}\right]$$

Trapezoidal Rule:- The general formula for quadrature is-

$$\int_{x_0}^{x_n} f(x)dx = nh[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0$$

$$+ \cdots + \text{up to}(n+1)\text{terms}]$$

Put n=1 into above equation ,then we get

$$\int_{x_0}^{x_1} f(x)dx = h[y_0 + \frac{1}{2}\Delta y_0]$$

$$=h[y_0 + \frac{1}{2}(y_1 - y_0)]$$

$$= \frac{h}{2}[y_0 + y_1]$$

Similarly,

$$\int_{x_1}^{x_2} f(x)dx = \frac{h}{2}[y_1 + y_2]$$

$$\int_{x_2}^{x_3} f(x)dx = \frac{h}{2}[y_2 + y_3]$$

....

$$\int_{x_{n-2}}^{x_{n-1}} f(x)dx = \frac{h}{2}[y_{n-2} + y_{n-1}]$$

and

$$\int_{x_{n-1}}^{x_n} f(x)dx = \frac{h}{2}[y_{n-1} + y_n]$$

Now adding these n integrals, we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

This is known as the Trapezoidal rule.

Simpson's 1/3rd Rule => This rule is obtained by putting n=2 in general quadrature formula ,we get

$$\int_{x_0}^{x_2} y\,dx = 2h[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0]$$

$$=2h[y_0 + y_1 - y_0 + \frac{1}{6}(y_2 - 2y_1 + y_0)]$$

$$=\frac{2h}{6}[y_2 - 2y_1 + y_0 + 6y_1]$$

$$=\frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly,

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$\int_{x_4}^{x_6} y\,dx = \frac{h}{3}[y_4 + 4y_5 + y_6]$$

....

$$\int_{x_{n-4}}^{x_{n-2}} y\,dx = \frac{h}{3}[y_{n-4} + 4y_{n-3} + y_{n-2}]$$

and $$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

Adding all the integrals ,we get

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n} ydx = \frac{h}{3}[y_0 + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})$$
$$+ 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + y_n]$$

This is known as Simpson's 1/3rd Rule

Simpson's 3/8 Rule - This rule is obtained by putting n=3 in general quadrature formula ,we get

$$\int_{x_0}^{x_3} y\,dx = 3h[y_0 + \frac{3}{2}\Delta y_0 + \frac{3(3)}{12}\Delta^2 y_0 + \frac{3(1)}{24}\Delta^3 y_0]$$

$$= 3h[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly,In the interval [x3,x6], [x6,x9]……..[x_{n-3},x_n] we get

$$\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

$$\int_{x_6}^{x_9} y\,dx = \frac{3h}{8}[y_6 + 3y_7 + 3y_8 + y_9]$$

…………… ……….. ……….. ……….

$$\int_{x_{n-6}}^{x_{n-3}} y\,dx = \frac{3h}{8}[y_{n-6} + 3y_{n-5} + 3y_{n-4} + y_{n-3}]$$

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \int_{x_{n-3}}^{x_n} y\,dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding all the integrals, we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}) + y_n]$$

This is known as Simpson's 3/8 Rule

Problem 1: Evaluate $\int_1^3 x^3$dx taking h=0.5 by trapezoidal rule

Solution: The table is as follows

Here h=0.5 that's why the value of x is in difference 0.5

x	y = x^3
1	y_0 =1
1.5	y_1 = 3.375
2	y_2 = 8
2.5	y_3 = 15.625
3	y_4 = 27

By trapezoidal formula,

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

(here 4 value of x between 1 to 3 so equation is)

$$\int_1^3 x^3 \, dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3) + y_4)]$$

$$= \frac{0.5}{2}[1+2(3.375+8+15.625)+27]$$

$$= \frac{0.5}{2}[1 + 2(27) + 27]$$

$$= \frac{0.5}{2}[1 + 54 + 27]$$

$$= \frac{0.5}{2}[82]$$

$$\int_1^3 x^3 \, dx = 20.5$$

Problem 2 : Evaluate $\int_1^3 \frac{1}{x}$ dx taking h=0.5 by Simpson's 1/3 and Simpson's 3/8 Rule.

Solution: The table is as follows

Here h=0.5 that's why the value of x is in difference 0.5

x	y = 1/x
1	y_0 =1
1.5	y_1 = 2/3
2	y_2 = 1/2
2.5	y_3 = 2/5
3	y_4 = 1/3

Using Simpson's 1/3 formula

$$\int_1^3 \frac{1}{x} dx = \frac{h}{3}[y_0 + 2(y_2) + 4(y_1 + y_3) + y_4]$$

$$= \frac{0.5}{3} [1 + 2(\frac{1}{2}) + 4(\frac{2}{3} + \frac{2}{5}) + \frac{1}{3}]$$

=1.1

Using Simpson's 3/8 formula

$$\int_1^3 \frac{1}{x} dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2) + 2(y_3) + y_4]$$

$$= \frac{3*0.5}{8}[1 + 3(\frac{2}{3} + \frac{1}{2}) + 2*\frac{2}{5} + \frac{1}{3}]$$

=1.05625