Homework 2. Propositional Logic. Tableau Proof.

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1. (6) (Attendance and grading issues). Divya Mannava and Vamsi Krishna Pagadala are our graders.

    - If you have any issues/requests about attendance, please contact Vamsi directly and he will take notes and answer your questions. His email is:

      vpagadal@ttu.edu

    - If you have any issues/requests about grading, please write to either Divya or Vamsi whose emails are *dmannava@ttu.edu* and *vpagadal@ttu.edu* respectively. I do work out a rubric with Divya and Vamsi for grading each homework, and we go through a sample set of submissions together on how to grade. Whom do you need to contact if you cannot attend a class or have an attendance issue? What email do you use for that contact? Whom do you contact if you have doubts on the grading of your homework? What email do you use for that contact?

    Solution:

    Whom do you need to contact if you cannot attend a class or have an attendance issue? : Vamsi Krishna Pagadala.

    What email do you use for that contact? : *vpagadal@ttu.edu*.

    Whom do you contact if you have doubts on the grading of your homework?: Divya or Vamsi.

    What email do you use for that contact? : *dmannava@ttu.edu* or *vpagadal@ttu.edu*

2. (14) Study the proof of $\Sigma \subseteq Cn(\Sigma)$ in Section 2.3 of L04 and the note after the proof to learn how to work backwards step by step. A key in the one-step backwards is the application of the definition of a concept to a use of the concept.

    (a) Based on the working backwards method, write a final proof for the following statement: for any proposition α, α is a consequence of $\{\alpha\}$.
    Solution:

Steps	Proof	Reason
a.	Assume $\forall_x$, $x \in \alpha$	-Given α is any proposition value.
b.	For any valuation V, assume V is model of α.	
c.	For any $\sigma \in \alpha$, $V(\sigma)$=T.	-By (b) and Definition of truth valuation.
d.	$V(x)$ = T.	-By (a), (b) and (c).
e.	For any valuation V, If V is a model of α, then $V(x)$ = T.	-By Definition, $V(\tau)$ = T for all $\tau \in \Sigma \Rightarrow$V($\sigma$) = T.
f.	$\alpha \vDash x$	-By definition x is consequences of α and by (e).
g.	$x \in C_n\alpha$.	-By definition, $C_n\Sigma$ be a set of consequences of Σ and by (f).
h.	$\forall_x, x \in \alpha \Rightarrow x \in C_n\alpha$.	-By (g).
i.	$\alpha \subseteq C_n\alpha$.	-By (h).
j.	α is consequences of α.	

(b) Let Σ_1 and Σ_2 be sets of propositions. Using the working backwards method, prove $\Sigma_1 \subseteq \Sigma_2$ implies $Cn(\Sigma_1) \subseteq Cn(\Sigma_2)$.

Solution:

Steps	Proof	Reason
a.	Assume $\forall_x$, $x \in \alpha$	-Given α is any proposition value.
b.	For any valuation V, assume V is model of α.	
c.	For any $\sigma \in \alpha$, $V(\sigma)$=T.	-By (b) and Definition of truth valuation.
d.	$V(x)$ = T.	-By (a), (b) and (c).
e.	Assume $M(\Sigma_1)$ the set of all model of Σ_1	
f.	As $\Sigma_1 \subseteq \Sigma_2$,V(x)$\in \Sigma_2$	-By definition every model of Σ_2 is also model of Σ_1.
g.	$\Sigma_1 \vDash x$.	-By definition, x is consequences of Σ and by (d).
h.	$x \in C_n(\Sigma_1)$.	-By (g).
i.	$x \in \Sigma_1 \Rightarrow x \in C_n(\Sigma_1)$.	-By (h).
j.	$\forall_x, x \in \Sigma_1 \Rightarrow x \in C_n(\Sigma_1)$.	By (a) and (i).
k.	$\Sigma_1 \subseteq C_n(\Sigma_1)$.	
l.	$x \in \Sigma_2 \Rightarrow x \in C_n(\Sigma_2)$.	-By (f) and (k).
m.	$C_n(\Sigma_1) \subseteq C_n(\Sigma_2)$.	-By (l).
n.	$\Sigma_1 \subseteq \Sigma_2 \Rightarrow C_n(\Sigma_1) \subseteq C_n(\Sigma_2)$.	-By (f), (k) and(m).

3. (10) Find the definition of a *model of a set of propositions* and definition of *a proposition is a consequence of a set of propositions* from the textbook. Rewrite each of the definitions using the concept of *a valuation makes a proposition true* (Definition 3.2) where appropriate. In your new definition, you are NOT allowed to directly apply a valuation $\mathcal{V}$ to a proposition σ in the form $\mathcal{V}(\sigma)$. You are NOT allowed to use T directly in your definitions.

   1) *model of a set of propositions* Σ

   Solution:

Model of a set of propositions
Let, Σ be a (possibly infinite) set of propositions.
σ is the consequence of Σ V be a truth valuation function that assigns to each proposition α a unique truth value $V(\alpha)$.
we say that a valuation V is a model of Σ.
a valuation V of α makes the proposition α true, i.e. $V(\alpha)$ = T for every σ ϵ Σ, denoted by $M(\Sigma)$.

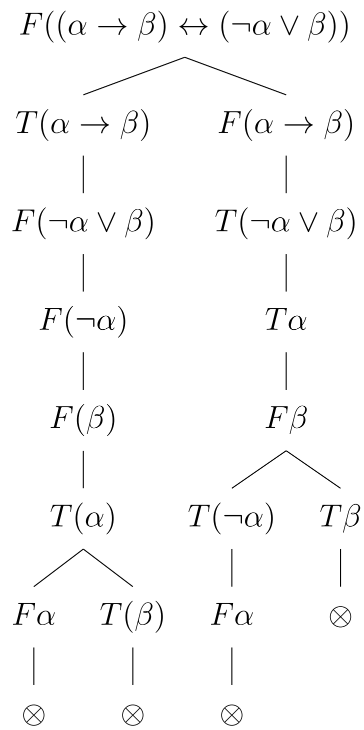
   2) *a proposition is a consequence of a set of propositions*

   Solution:

Proposition is a consequence of a set of propositions
Let, Σ be a (possibly infinite) set of propositions.
$\mathcal{V}$ be a truth valuation function that assigns to each proposition.
α a unique truth value $\mathcal{V}(\alpha)$, then, we say that, a proposition (σ) is a consequence of a set of propositions (Σ).
It written as $\Sigma \vDash \sigma$, if for any valuation V of τ makes the proposition τ true, i.e. , ($V(\tau)$ = T for all τ ϵ Σ) $\Rightarrow$ $V(\sigma)$ = T.

4. (15) Give a tableau proof of $((\alpha \to \beta) \leftrightarrow (\neg\alpha \vee \beta))$.

   Solution:



Here, $F(\neg\alpha \vee \beta)$ is chosen first as it does not have branching later. After that I have chosen $T(\alpha \to \beta)$ as it has branching.
Similarly, $F(\alpha \to \beta)$ is chosen first as it does not have branching later. After that $T(\neg\alpha \vee \beta)$ is chosen as it has branching.

5. (10) Which entries of the tableaux in Figure 1 are *reduced*? Which are not?

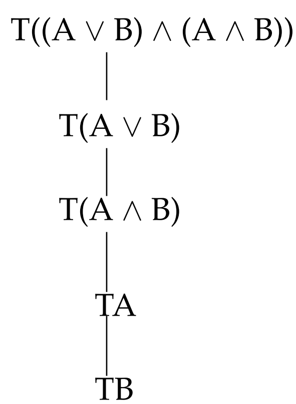


Figure 1: A tableau

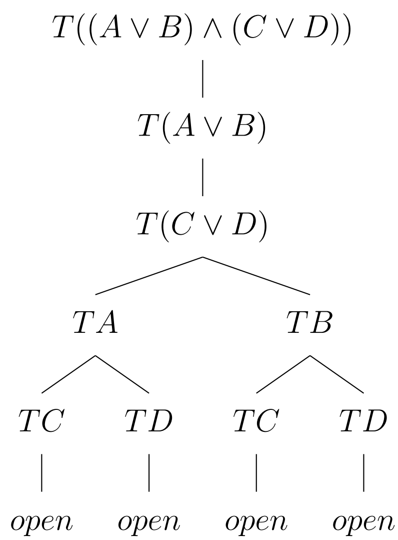
Solution:

Reduced entries of the tableaux in Figure are as below:
$T((A \vee B) \wedge (A \wedge B))$, $T(A \wedge B)$, TA, TB

Entries that are NOT reduced of the tableaux in Figure are as below:
$T(A \vee B)$

6. (15) Draw the CST of $T((A \vee B) \wedge (C \vee D))$.

   Solution:



7. (10) (Write definition) Recall the *language* of propositional logic in L02. We now expand it with a new connective *majority*. While most of the original connectives in the language such as $\wedge$ are used in an *infix form* to form a proposition. For example, if α and β are propositions, then $(\alpha \wedge \beta)$ is a proposition. With the expanded language, we can write new propositions. For the new connective *majority*, it allows exactly three parameters, and a prefix form has to be used for it to form a new proposition. For example, for propositional letters A_1, A_2, A_3, *majority(*A_1, A_2, A_3*)* is a proposition. In fact, we can nest these connectives. For example, *majority(majority(*A_1, A_2, A_3*),* $(A_1 \wedge A_2), (A_3 \vee B)$*)* is a new *propositions*, and so is *(majority(*$A_1, A_2, A_3) \wedge A_1)$

   Write a definition of the new *proposition*. You can refer to the definition of original proposition from the book/L02. Clearly, an inductive (recursive) definition is needed here.

   Solution:
   Definition of the propositional logic defines connectives ($\vee$, $\wedge$,$\neg$, $\rightarrow$, $\leftrightarrow$), Parentheses: ((,)) and Propositional letters: A_1, A_2, $\ldots$, B_1, B_2)

   Assume we have considered a new connective *majority*, Now, our propositional language contains symbols like Connectives ($\vee$, $\wedge$, $\neg$, $\rightarrow$, $\leftrightarrow$, *majority*, Parentheses: ((,)) and Propositional letters: A_1, A_2, $\ldots$, B_1, B_2)

   Definition for new proposition is as below:
   (a) Propositional letters are propositions.

(b) If α, β are propositions, then $(\alpha \vee \beta)$, $\neg(\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$, $(majority(\alpha_1, \alpha_2) \wedge \beta)$, $(majority(\alpha_1, \alpha_2) \vee \beta)$, $(\neg(majority(\alpha_1, \alpha_2)))$, $(majority(\alpha_1, \alpha_2,) \rightarrow \beta)$, $(majority(\alpha_1, \alpha_2) \leftrightarrow \beta)$, are propositions.
(c) If and only if it can be obtained by starting with propositional letters, a group of symbols is a proposition (a) and (b).

8. (10) Study carefully the proofs in L06. Prove the completeness result of the tableaux proof, i.e., Theorem 5.3. Follow the proof of soundness result in L06. Do not skip steps in your proof. Your proof should be in the final form (e.g., all labels for statements will be without prefix b or F). You may use lemma 5.4 directly.

   Solution:

Steps	**Proof**	**Reason**
a.	Assume $\forall_\alpha$, $\alpha \in \Sigma$	-Given Σ is any proposition value.
b.	For any valuation V, assume V is model of Σ.	
c.	For any $\sigma \in \Sigma$, $V(\sigma)$=T.	-By (b) and Definition of truth valuation.
d.	$V(\alpha)$ = T.	-By (a), (b) and (c).
e.	If V is model of Σ, then $V(\alpha)$ = T.	-By (d).
f.	Assume finished tableau τ with value of F(α)	By theorem 4.8.
g.	Assume τ had a non-contradictory path P.	
h.	$V(\alpha) \in \Sigma$.	-By definition, Valuation of V that agrees with all its entries and by (g).
i.	F(α) exists in the tableau. So, $V(\alpha)$ = F which contradicting the validity of α.	
j.	Valuation of α can be either T or F. It can't have two different value for same proposition on the same path.	
k.	Every path on τ is contradictory	By (i) and (j).
l.	τ is a tableau proof of α.	
m.	If α is valid then α is tableau provable i.e. $\models\alpha \Rightarrow \models \alpha$.	-By (e) and (l).

9. (10) Study carefully the proofs in L06. Prove lemma 5.2. You have to follow the methods we studied in L06.

   Solution:

Proposition is a consequence of a set of propositions
1. Assume that $\alpha \in \Sigma$ is tableau provable. and $\Sigma(\alpha)$ is contrary.
2. Since α is not a result of Σ, there is a valuation V that gives α the letter F.
3. There is a conflicting tableau as per statement (2) and Valuation V agrees with F(α)
Path P is present on the tableau.
Since the tableau contains contradiction in accordance with claim (3), P must also include contradiction.
T(α) and F(α) exist on P.
V(α) = F and V(α) = T, and V agrees with P.
Denying the assertion that a valuation does not give two values to a single proposition.
There is a valuation V that assigns T β to each instance of $\beta(\Sigma)$.
However, this valuation V already assigns F to α, i.e., V(α) = F, refuting the claim that a consequence always gives a proposition its actual value.
There is a valuation V that assigns T β to each instance of $\beta(\Sigma)$.
It follows that α is a consequence of Σ if there is a tableau proof of α from a set of premises.