Homework 5 . Predicate Logic: Syntax and Semantics

Rushikesh Khamkar

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1. **Consider the following sentence.**

   Every number greater than or equal to 4 can be written as the sum of two prime numbers.

   (a) **Write *a language* (as defined in Def 2.1) such that some formulas of the language can be used to represent the sentence above.**

   Solution:

Distinctive primitive symbols	**Answer**
Variables	a, b, c
Constants	4
Connectives	$\vee, \wedge, \neg, \rightarrow, \leftrightarrow$
Quantifiers	$\forall, \exists$
Punctuation	$,\ (\ )$
Predicate Symbols	P, Q, R
Function Symbols	f

   (b) **Write a formula of your language that should reflect the meaning of the sentence above.**

   Solution:

   Formula - $((\forall a P(a,4))((\exists c Q(c))R(a,f(b,c))$

Formula of language	Meaning of the sentence
P(a,4)	The condition where all integers in the predicate are greater than or equal to 4.
Q(b), Q(c)	The condition where all-prime-numbers are predicate.
R(a,f(b,c))	The clause that states that a = b+c.
f(b,c)	The equation that symbolizes (b + c).

(c) **In terms of your language, write three example terms, three example atomic formulas, and three example formulas.**

Solution:

Definition	Example
Terms	$a, b, c,$ $f(b, c)$
Atomic Formulas	$P(a, 4),$ $Q(b),$ $Q(c),$ $R(a, f(b, c))$
Formulas	$P(a, 4),$ $Q(b),$ $Q(c),$ $R(a, f(b, c)),$ $((\forall a P(a, 4))((\exists b Q(b))(\exists c Q(c))R(a, f(b, c))))$

2. **Write a formula to represent the following information. Your formula should be as close as possible to the intended meaning of these sentences.**

(a) ***There is a mother to all children.***

Solution:

$\exists x\ \forall y$ Mother(x, y) where, Mother(x, y) = x is mother of children y.

(b) **ALL ITEMS NOT AVAILABLE AT ALL STORES.**
Note, the sentence above is a disclaimer in the weekly flyer of specials of a grocery store chain.

Solution:

$\forall x\ \forall y\ \neg$ available(x, y) where, available(x, y) $=$ x items are available at y stores.

3. **Given the language defined in Def 2.1, which of the following are formulas defined by Def 2.5?**

   (a) $f(x, c)$

   (b) $R(c, f(d, z))$

   (c) $\forall x(P(x))$

   (d) $((\exists x)(((\forall y)P(z)) \rightarrow R(x, y)))$

   Solution:

Formulas	**Formulas Defined or Not?**
$f(x, c)$	Formula is not defined by Def 2.5
$R(c, f(d, z))$	Formula is defined by Def 2.5
$\forall x(P(x))$	Formula is defined by Def 2.5
$((\exists x)(((\forall y)P(z)) \rightarrow R(x, y)))$	Formula is defined by Def 2.5

4. **Given**

$$((\exists x)(((\forall y)P(z)) \rightarrow R(x, y))),$$

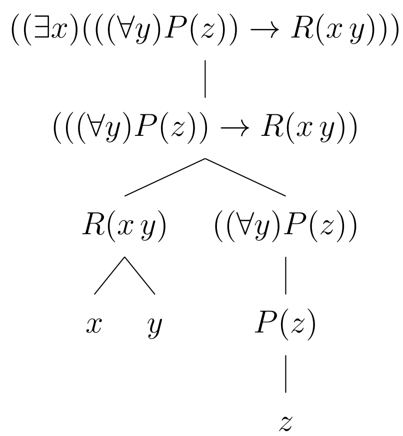
   (a) **List all its subformulas.**

   Solution:

List of Sub formulas from above given Formula
$P(z)$
$((\forall y)P(z))$
$R(x, y)$
$(((\forall y)P(z)) \rightarrow R(x, y))$
$((\exists x)(((\forall y)P(z)) \rightarrow R(x, y)))$

   (b) **Draw its formation tree.**

   Solution:



5. **Which of the following terms are substitutable for x in the corresponding formulas?**

   (a) $f(z, y)$ in $((\exists y)(P(y) \wedge R(x, z)))$.

   Solution:
   It is not substitutable, Since y is already constrained by $\exists y$ the substitution $f(z, y)$ for x results in the wrong result because x is not bound in this situation.

   (b) $g(f(z, y), a)$ in $((\exists x)(P(x) \wedge R(x, y)))$.

   Solution:
   It is not substitutable, Since x is constrained by $\exists x$.

6. **Connect predicate calculus to the study of this course. To focus on the substance, we need to extend (informally) predicate calculus (syntax) as follows**

   - **You can use sets or proposition as a parameter of a predicate.**
   - **You can use $=$ as a predicate symbol in the normal way. For example, that two terms t_1 and t_2 are equal can be represented by $t_1 = t_2$.**
   - **You can use a variable to refer to a function or a set or a proposition. (e.g., $((\exists \mathcal{V})\mathcal{V}(x) = T)$ where T is constant.)**

   The form of formula will be expanded accordingly to the extension above. Consider the definition of *consequence* (Def 3.7 of Part I).

   (a) **Represent it as a formula. You need to introduce all predicates you need in the formula, in the way we did in L11.1 for the subset example. Intuitively, a concept name and its parameters/arguments you identify in the definition is a good candidate of a predicate. You also need to introduce any constant**

or function symbol you need. Note the main "if" in a definition should be understood as "if and only if." Note statement "$\forall x \in A, x \in B$" in the subset definition. We translate it to "$((\forall x)(x \in A \to x \in B))$."

Answer
Predicates: consequence (σ, Σ): σ is a consequence of Σ
equals($\upsilon(\tau), T$): $\upsilon(\tau)$ equals to true.
equals($\upsilon(\sigma), T$): $\upsilon(\sigma)$ equals to true.
Constant: T (True)
Resulting Formula: consequence$(\sigma, \Sigma) \leftarrow$ (($\forall$ ($\tau \in \Sigma$), equals(($\upsilon(\tau), T$) $\to$ equals($\upsilon(\sigma), T$)))

(b) **Let your formula be α. What can you logically derive (some intuition/experience from your earlier study is needed here) from the formula ($\alpha\wedge$ "B is a consequence of $\{A, A \to B\}$")? Note you should translate the English in the formula using predicate(s) you introduced. Your answer to this question has to be in the form of a formula. You need to do a variable substitution (Σ in English definition would be replaced by $\{A, A \to B\}$, and σ by B).**

Answer:
consequence(B, A, $A \to B$) $\leftarrow$ (($\forall$ ($\tau \in A$, $A \to B$), equals(($\upsilon(\tau)$,T) $\to$ equals(υ (B), T)))

7. 1) **Let $A = \{1, 2\}$.**
a) **List all functions from A to A in the form of sets of pair. For example, one function is $\{(1, 1), (2, 2)\}$. If we let the function be named g. Then in the example function, $g(1)$ is 1 and $g(2)$ is 2.**

Solution:
According to the question $A = \{1, 2\}$. All functions from A to A are : $g(A) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

b) **List all unary relations on A. Your relations must be represented as sets.**

Solution:
Assume $U(A)$ represents all unary relations on A. $U(A) = \{1\}, \{2\}, \{1, 2\}$.

2) **Show that $\forall x(p(x) \to q(f(x))) \wedge \forall xp(x) \wedge \exists x\neg q(x)$ is satisfiable. The format of your structure should follow the one in L12.1. In your structure, you must use the domain $A = \{1, 2\}$. You must represent the function assigned to f in the set of pairs. You must represent the relations assigned to p and q in the set forms. Note you have to figure out the constants in the language on which the formula is defined.**

Solution:

Steps	Proof
a.	Assume, Σ = $p(x), q(f(x))$, Domain: A = {1,2}, Constant Assignments: {(1, one), (2, two)}
b.	Function Assignment: f to {(1,2), (2,2)}, p to $p1$: $p1\{1\}$, q to $q1$: $q1\{2\}$
c.	The sentences listed above are a set, as we all know. According to the definition of induction, every $a\exists A$ in a sentence must be designated by a ground word L in order for it to be true.
d.	Constants and constants created by applying function symbols to other constants are considered ground terms.
e.	In this instance, $f(x)$ is grounded, making q true.
f.	Considering that $(p(x) \to q(f(x))$, $p(x)$ is also true.
g.	All members of $\exists$ are real based on the information above, thus $\forall x(p(x) \to q(f(x))) \land \forall x p(x) \land \exists x \neg q(x)$ is satisfiable.

QED

8. **Prove that $\mathcal{A} \models \neg\exists x\varphi(x)$ if and only if $\mathcal{A} \models \forall x\neg\varphi(x)$. You have to follow the proof format we used earlier. Working backward again is a good idea. You have to be able to apply the definitions.**

   Solution:

Steps	Proof	Reason
a.	If each element of the structure is named by a ground term of L that is defined by induction, then the phrase φ satisfies the structure $\mathcal{A}$ i.e., $\mathcal{A} \models \varphi$.	By definition 4.3.
b.	$\mathcal{A} \models \neg\exists$ x $\varphi(x)$ for some ground term t, $\mathcal{A} \models \neg\varphi$ (t) .	By definition 4.3, (ii) and (viii)).
c.	$\mathcal{A} \models \forall x\neg\phi(x)$ for some ground term t, $\mathcal{A} \models \neg\phi(t)$.	By definition 4.3, (ii) and (viii)).
d.	Both $\mathcal{A} \models \neg\exists$ x $\varphi(x)$ and $\mathcal{A} \models \forall x\neg\phi(x)$ are equivalent to the $\mathcal{A} \models \neg\varphi(t)$, hence $\mathcal{A} \models \neg\exists$ x $\varphi(x) \leftrightarrow \mathcal{A} \models \forall x\neg\phi(x)$.	By (b) and (c).
e.	$\mathcal{A} \models \neg\exists$ x $\varphi(x) \leftrightarrow \mathcal{A} \models \forall x\neg\phi(x)$	By (d).

QED