Department of Computer Science
Ashoka University

Discrete Mathematics: CS-1104-1 & CS-1104-2

Assignment 3

Collaborators: None

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1 Straightforward

1. (a) Let $\mathbb{P} = \{p_n | n \in \mathbb{N}\}$ be the set of all prime numbers, where p_n is the n^{th} prime number.
But since $\mathbb{N}$ is countably infinite, we also need to show that there are an infinite number of primes.

Suppose there are a finite number of primes.
Let $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all primes.
Then, we can construct a new number $N = p_1 \cdot p_2 \cdot \cdots \cdot p_n + 1$.
N is not divisible by any of the primes in P, so it must be divisible by a new prime number p_{n+1}.
This is a contradiction, so there must be an infinite number of primes.

Now, since we know that there are an infinite number of primes, we can prove that P has the same cardinality as $\mathbb{N}$ by constructing a bijection between the two sets.

Let $f : \mathbb{N} \to \mathbb{P}$ be a function defined as follows:

$$f(n) = p_n$$

This function is a bijection, so $\mathbb{N}$ and $\mathbb{P}$ have the same cardinality.

(b) Let P_R be the set of all points on a circle of radius R.
Let $P_{R'}$ be the set of all points on a circle of radius R'.

Using the polar coordinate system, we get:
$P_R = \{(Rcos(\theta), Rsin(\theta)) \mid \theta \in [0, 2\pi)\}$
$P_{R'} = \{(R'cos(\theta), R'sin(\theta)) \mid \theta \in [0, 2\pi)\}$

Let $f : P_R \to P_{R'}$ be a function defined as follows:

$$f((Rcos(\theta), Rsin(\theta))) = (R'cos(\theta), R'sin(\theta))$$

This function is a bijection, so P_R and $P_{R'}$ have the same cardinality.

2. (a) Consider the L.H.S:

$$\bigcup_{i=1}^{n} A_i = (x \in A_1) \vee (x \in A_2) \vee ... \vee (x \in A_n)$$

Now, consider the R.H.S:

$$(\bigcap_{i=1}^{n} A_i^c)^c = \neg((x \in A_1^c) \wedge (x \in A_2^c) \wedge ... \wedge (x \in A_n^c))$$

$$= \neg((x \notin A_1) \wedge (x \notin A_2) \wedge ... \wedge (x \notin A_n))$$

$$= (x \in A_1) \vee (x \in A_2) \vee ... \vee (x \in A_n)$$

Since the L.H.S and R.H.S are the same, we can conclude that the given statement is true.

(b) Consider the L.H.S:

$$\bigcap_{i=1}^{n} A_i = (x \in A_1) \wedge (x \in A_2) \wedge ... \wedge (x \in A_n)$$

Now, consider the R.H.S:

$$(\bigcup_{i=1}^{n} A_i^c)^c = \neg((x \in A_1^c) \vee (x \in A_2^c) \vee ... \vee (x \in A_n^c))$$
$$= \neg((x \notin A_1) \vee (x \notin A_2) \vee ... \vee (x \notin A_n))$$
$$= (x \in A_1) \wedge (x \in A_2) \wedge ... \wedge (x \in A_n)$$

Since the L.H.S and R.H.S are the same, we can conclude that the given statement is true.

3. We will use induction to prove the law for n functions.

Base Case ($n = 2$): By De Morgan's Law for two functions, we know that if we have two invertible functions $f_1 : A_1 \to A_2$ and $f_2 : A_2 \to A_3$, then

$$(f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1}$$

is true.

Inductive Step: Assume the statement holds for $n = k$, that is, given invertible functions $f_1, f_2, \ldots, f_k$, we have:

$$(f_k \circ \cdots \circ f_1)^{-1} = f_1^{-1} \circ \cdots \circ f_k^{-1}$$

Now, consider $n = k + 1$ with invertible functions $f_1, f_2, \ldots, f_k, f_{k+1}$. We need to show that:

$$(f_{k+1} \circ f_k \circ \cdots \circ f_1)^{-1} = f_1^{-1} \circ \cdots \circ f_k^{-1} \circ f_{k+1}^{-1}$$

Proof: Start by considering the composition of the first k functions and then apply f_{k+1}:

$$f_{k+1} \circ (f_k \circ \cdots \circ f_1)$$

By the base case with $n = 2$, we can take the inverse of this composition to be:

$$(f_k \circ \cdots \circ f_1)^{-1} \circ f_{k+1}^{-1}$$

By the induction hypothesis, this is equal to:

$$f_1^{-1} \circ \cdots \circ f_k^{-1} \circ f_{k+1}^{-1}$$

Since function composition is associative, we can rearrange the parentheses without changing the order of the functions:

$$f_1^{-1} \circ \cdots \circ (f_k^{-1} \circ f_{k+1}^{-1})$$

Thus, we have shown that the inverse of the composition of $k + 1$ functions is the composition of their inverses in reverse order. By PMI, the statement is proved for all natural numbers n. $\square$

4. (a) False. Consider the case when $(x \in D) \wedge (x \in A^c)$. Then, $x \notin A$, so $\forall x(x \in A \wedge x \in D) \equiv False$.

(b) $x \notin A \to x \in A^c$.
Now, $\exists x(x \in A^c)$ is only true when A^c is non-empty.
So, the given statement is true if $A \neq D$ and false if $A = D$.

(c) $A^c = x \notin A$.
$D \backslash A = x \in D \wedge x \notin A$.
Since D is the domain of discourse, $x \in D$ is always true.
So, $A^c = D \backslash A$ is always true.

(d) By definition, D being the domain of discourse, D contains all elements.
So, $D^c := \{x \notin D\}$ is an empty set.
Since an empty set is a subset of every set, $D^c \subseteq A$ is true.

5. (a) Subsets:
   - $\{(1,1),(1,2),(2,1),(2,2)\}$
   - $\{(2,1),(2,2),(3,1),(3,2)\}$
   - $\{(1,2),(1,3),(2,2),(2,3)\}$
   - $\{(2,2),(2,3),(3,2),(3,3)\}$
   - $\{(1,1),(1,3),(3,1),(3,3)\}$
   - $\{(1,2),(2,1),(2,3),(3,2)\}$

   (b) Over here, the idea is to find any 3 points such that they do not form a straight line.

   So, we can take T, which is the set of all subsets of $S \times S$ which are vertices of a triangle:

$$T := \{\{a,b,c\} | (a,b,c \in S \times S) \wedge \neg(straightLine(a,b,c))\}$$

   Now, all we need to do is define a function $straightLine(a,b,c)$ which returns true if a,b,c are on the straight line.

   We can define this function as follows:

$$straightLine(a,b,c) := (a_1 - c_1) \cdot (b_2 - c_2) = (b_1 - c_1) \cdot (a_2 - c_2)$$

   This function returns true if a,b,c are on the same straight line because it checks if the slopes of the lines formed by a,b and b,c are equal. Using python, here are the 76 possible subsets:

$\{(1,1),(1,2),(2,3)\}$	$\{(1,3),(2,1),(2,3)\}$	$\{(2,1),(2,3),(3,2)\}$	$\{(2,3),(3,1),(3,2)\}$
$\{(2,2),(2,3),(3,1)\}$	$\{(1,2),(1,3),(2,2)\}$	$\{(1,3),(2,2),(2,3)\}$	$\{(1,1),(2,3),(3,1)\}$
$\{(1,1),(1,2),(3,1)\}$	$\{(1,1),(2,1),(2,3)\}$	$\{(2,1),(2,2),(3,2)\}$	$\{(2,1),(2,3),(3,1)\}$
$\{(2,2),(2,3),(3,3)\}$	$\{(1,3),(3,1),(3,2)\}$	$\{(1,2),(2,1),(3,3)\}$	$\{(1,2),(3,2),(3,3)\}$
$\{(2,1),(2,3),(3,3)\}$	$\{(1,2),(2,1),(2,3)\}$	$\{(1,3),(3,1),(3,3)\}$	$\{(1,1),(1,3),(3,2)\}$
$\{(1,3),(2,2),(3,2)\}$	$\{(1,3),(3,2),(3,3)\}$	$\{(1,1),(2,3),(3,2)\}$	$\{(1,1),(3,1),(3,2)\}$
$\{(1,1),(1,3),(3,3)\}$	$\{(1,1),(2,2),(2,3)\}$	$\{(1,1),(2,2),(3,2)\}$	$\{(1,2),(2,3),(3,2)\}$
$\{(2,2),(2,3),(3,2)\}$	$\{(2,2),(3,1),(3,3)\}$	$\{(2,1),(3,2),(3,3)\}$	$\{(2,2),(3,2),(3,3)\}$
$\{(1,2),(2,1),(3,1)\}$	$\{(1,1),(2,2),(3,1)\}$	$\{(1,1),(1,3),(2,3)\}$	$\{(1,2),(3,1),(3,3)\}$
$\{(1,3),(2,1),(3,3)\}$	$\{(1,3),(2,3),(3,2)\}$	$\{(1,1),(3,2),(3,3)\}$	$\{(1,1),(2,1),(3,3)\}$
$\{(2,1),(2,2),(3,3)\}$	$\{(1,3),(2,3),(3,1)\}$	$\{(1,2),(1,3),(3,3)\}$	$\{(1,3),(2,2),(3,3)\}$
$\{(2,1),(3,1),(3,2)\}$	$\{(1,2),(1,3),(3,1)\}$	$\{(1,1),(2,3),(3,3)\}$	$\{(2,2),(3,1),(3,2)\}$
$\{(1,1),(1,3),(2,2)\}$	$\{(1,1),(3,1),(3,3)\}$	$\{(1,2),(2,2),(2,3)\}$	$\{(1,2),(3,1),(3,2)\}$
$\{(1,2),(1,3),(3,2)\}$	$\{(1,2),(2,2),(3,3)\}$	$\{(1,1),(1,3),(2,1)\}$	$\{(1,1),(1,3),(3,1)\}$
$\{(1,2),(2,1),(3,2)\}$	$\{(1,3),(2,1),(3,2)\}$	$\{(1,3),(2,1),(2,2)\}$	$\{(1,3),(2,1),(3,1)\}$
$\{(1,1),(1,2),(3,2)\}$	$\{(1,1),(2,1),(3,2)\}$	$\{(1,2),(2,2),(3,1)\}$	$\{(2,1),(2,2),(3,1)\}$
$\{(1,1),(1,2),(2,1)\}$	$\{(1,1),(2,1),(2,2)\}$	$\{(1,2),(1,3),(2,3)\}$	$\{(1,1),(1,2),(2,2)\}$
$\{(2,3),(3,2),(3,3)\}$	$\{(1,2),(2,3),(3,1)\}$	$\{(1,2),(2,3),(3,3)\}$	$\{(2,1),(3,1),(3,3)\}$
$\{(1,2),(1,3),(2,1)\}$	$\{(2,3),(3,1),(3,3)\}$	$\{(1,2),(2,1),(2,2)\}$	$\{(1,1),(1,2),(3,3)\}$

6. First, let's see if p is a function of t.
For p to be a function of t, for every t, there should be exactly one p. Since p is the position (a single point) based on the time, it is evident that p is a function of t.

Now, let's find the domain of $p(t)$.
We know the recorded motion is from $t = 0$ to $t = n > 0$. So, the domain of $p(t)$ is $[0, n]$.

The image of $p(t)$ is the set of all positions of the particle at any given time.

For $p(t)$ to be injective, for every $t_1 \neq t_2$, $p(t_1) \neq p(t_2)$. This in simple words means that the particle must travel in the same direction (since $p \in (R)$). Otherwise, if the particle changes direction or is stationary, it will have the same position at two different times.

7. (a) For a bijection to exist, we need each element from A to map to exactly one unique element in B.

   We know $|A| = |B| = n$. When we start mapping the elements, we see the first element from A can map to any of the n elements from B. The second element from A can map to any of the remaining $n - 1$ elements from B. This goes on till the last element from A can map to the last element from B.

   So, the total number of bijections is $n \times (n - 1) \times \cdots \times 1 = n!$

   (b) First, we show that if f is an injection, then f is a surjection.

   - For f to be an injection, for every $b \in B, \exists a \in A$ such that $f(a) = b$.
   - Since $|A| = |B| = n$, and f is an injection, n elements from A are mapped to n unique elements in B, which implies f is a surjection.

   Now, we show that if f is a surjection, then f is an injection.

   - For f to be a surjection, for every $b \in B, \exists a \in A$ such that $f(a) = b$.
   - Since $|A| = |B| = n$, and f is a surjection, using the fact that all functions must be well defined, we can say that n elements from A are mapped to n unique elements in B, which implies f is an injection.

   Therefore, f is an injection $\leftrightarrow$ f is a surjection.

   (c) For f to be an injection, $|A| \leq |B|$. So, if $m > n$, then there are no injections from A to B.

   Now, if $m \leq n$, we need to choose m elements from n elements in B. This can be done in $\binom{n}{m}$ ways.
   Now, for each of these m elements, we can map them to any of the m elements in A. This can be done in $m!$ ways.

   So, the total number of injections is $\binom{n}{m} \times m! = \frac{n!}{(n-m)!m!} \times m! = \frac{n!}{(n-m)!}$

8. The image of f, $Img(f)$, is defined as $\{f(x)|x \in D\}$, where $D = Dom(f)$ is the domain of f.

   Since f is a function from $D \to D$, for every $x \in Dom(f)$, $f(x) \in Dom(f)$.
   Now, since $Dom(f) = D$ and $Img(f) = \{f(x)|x \in D\}$, we can say that $Img(f) \subseteq D = Dom(f)$.

   Therefore, $Img(f) \subseteq Dom(f)$.

9. (a) Since n is even, f is not injective, since $\forall x(f(-x) = f(x))$.

   A left inverse does not exist because negative numbers do not have a corresponding output in $\mathbb{R}^+ \cap \{0\}$ under f. But, a right inverse does exist.

   Let $g : R \to R + \cup\{0\}$ be defined as $g(x) = \sqrt[n]{x}$.
   Then, $f(g(x)) = (g(x))^n = x$.

   (b) In this case, f is injective and surjection since a unique output is defined for each input. Therefore, f is a bijection and a full inverse exists.

   Let $g : R \to R$ be defined as $g(x) = \sqrt[n]{x}$.
   Then, $f(g(x)) = (g(x))^n = x$.

   (c) In this case, f is injective and surjection since a unique output is defined for each input. Therefore, f is a bijection and a full inverse exists.

   Let $g : R \to R + \cup\{0\}$ be defined as $g(x) = e^x$.
   Then, $f(g(x)) = ln(e^x) = x$.

   (d) In this case, f is a constant function, so it is not injective and not surjective. Therefore, a left, right or full inverse does not exist.

10. Let f be an invertible function.
    Let g and h be two unique inverses of f.

    Since g and h are inverses of f, we have:

$$f(g(x)) = x$$

$$f(h(x)) = x$$

Now, we need to show that $g = h$.

- Let's assume that $g \neq h$.
- Then, $\exists x$ such that $g(x) \neq h(x)$.
- Now, since f is a function, $f(g(x)) = f(h(x))$.
- But, $f(g(x)) = x$ and $f(h(x)) = x$.
- So, $x = x$, which is true for all x.
- This is a contradiction, so our assumption that $g \neq h$ is false.

Therefore, $g = h$.

2 ¬Straightforward

1. (a) Let $(a_1, b_1) := \{\{a_1\}, \{a_1, b_1\}\}$, and $(a_2, b_2) := \{\{a_2\}, \{a_2, b_2\}\}$.

    Now, we need to show that $(a_1, b_1) = (a_2, b_2) \leftrightarrow (a_1 = a_2 \wedge b_1 = b_2)$.

    Showing $(a_1, b_1) = (a_2, b_2) \rightarrow (a_1 = a_2 \wedge b_1 = b_2)$:

    - Let's assume that $(a_1, b_1) = (a_2, b_2)$.
    - Then, $\{\{a_1\}, \{a_1, b_1\}\} = \{\{a_2\}, \{a_2, b_2\}\}$.
    - This implies that $\{a_1\} = \{a_2\}$ and $\{a_1, b_1\} = \{a_2, b_2\}$.
    - Since $\{a_1\} = \{a_2\}$, $a_1 = a_2$.
    - Since $\{a_1, b_1\} = \{a_2, b_2\}$ and $a_1 = a_2$, $b_1 = b_2$.
    - Therefore, $(a_1, b_1) = (a_2, b_2) \rightarrow (a_1 = a_2 \wedge b_1 = b_2)$.

    Showing $(a_1 = a_2 \wedge b_1 = b_2) \rightarrow (a_1, b_1) = (a_2, b_2)$:

    - Let's assume that $a_1 = a_2$ and $b_1 = b_2$.
    - Then, $\{a_1\} = \{a_2\}$ and $\{a_1, b_1\} = \{a_2, b_2\}$.
    - This implies that $\{\{a_1\}, \{a_1, b_1\}\} = \{\{a_2\}, \{a_2, b_2\}\}$.
    - Therefore, $(a_1 = a_2 \wedge b_1 = b_2) \rightarrow (a_1, b_1) = (a_2, b_2)$.

    Therefore, $(a_1, b_1) = (a_2, b_2) \leftrightarrow (a_1 = a_2 \wedge b_1 = b_2)$.

   (b) $(a, b) = \{\{a\}, \{a, b\}\}$

$$\therefore (a, b, c) = ((a, b), c) = \{\{\{a\}, \{a, b\}\}, \{\{\{a\}, \{a, b\}\}, c\}\}$$

   (c) Let $(x_1, x_2, \ldots, x_n) := ((x_1, x_2, \ldots, x_{n-1}), x_n)$.
    From part (b), we can say $((x_1, x_2, \ldots, x_{n-1}), x_n) = \{\{(x_1, x_2, \ldots, x_{n-1})\}, \{(x_1, x_2, \ldots, x_{n-1}), x_n\}\}$

    We also know, $(x_1, x_2, \ldots, x_{n-1}) := ((x_1, x_2, \ldots, x_{n-2}), x_{n-1})$
    This goes on till $(x_1, x_2) := \{\{x_1\}, \{x_1, x_2\}\}$.

    Therefore, $(x_1, x_2, \ldots, x_n) = \{\{\{x_1\}, \{x_1, x_2\}\}, \{\{\{x_1\}, \{x_1, x_2\}\}, \ldots, x_n\}\}$

3 Bonus

1.