Department of Computer Science
Ashoka University

Discrete Mathematics: CS-1104-1 & CS-1104-2

Assignment 7

Collaborators: None **Name: Rushil Gupta**

1 Straightforward

1. (a) $a_n = \begin{cases} a_{n-1} + a_{n-2} + a_{n-5} & \text{if } n > 5 \\ 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$

   (b) $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 9$

   (c) $A(x) = \sum_{i=0}^{\infty} a_i x^i = 1 + x + 2x^2 + 3x^3 + 5x^4 + 9x^5 + \sum_{i=6}^{\infty} a_i x^i$

   Using $a_n = a_{n-1} + a_{n-2} + a_{n-5}$, we get:

$$A(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 9x^5 + \sum_{i=6}^{\infty}(a_{i-1} + a_{i-2} + a_{i-5})x^i$$

$$A(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 9x^5 + x\sum_{i=6}^{\infty} a_{i-1}x^{i-1} + x^2\sum_{i=6}^{\infty} a_{i-2}x^{i-2} + x^5\sum_{i=6}^{\infty} a_{i-5}x^{i-5}$$

$$A(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 9x^5 + x\sum_{i=5}^{\infty} a_i x^i + x^2\sum_{i=4}^{\infty} a_i x^i + x^5\sum_{i=1}^{\infty} a_i x^i$$

$$\begin{aligned} A(x) =& 1 + x + 2x^2 + 3x^3 + 5x^4 + 9x^5 \\ &+ x(A(x) - 1 - x - 2x^2 - 3x^3 - 5x^4) \\ &+ x^2(A(x) - 1 - x - 2x^2 - 3x^3) \\ &+ x^5(A(x) - 1) \end{aligned}$$

$$\begin{aligned} A(x) =& 1 + x + 2x^2 + 3x^3 + 5x^4 + 9x^5 \\ &+ xA(x) - x - x^2 - 2x^3 - 3x^4 - 5x^5 \\ &+ x^2A(x) - x^2 - x^3 - 2x^4 - 3x^5 \\ &+ x^5A(x) - x^5 \end{aligned}$$

$$\begin{aligned} A(x) &= 1 + xA(x) + x^2A(x) + x^5A(x) \\ A(x)(1 - x - x^2 - x^5) &= 1 \\ A(x) &= \frac{1}{1 - x - x^2 - x^5} \end{aligned}$$

   (d) Verifying for $n = 5$. The ways to write 5 as a sum of 1, 2 and 5 are:

   - $1 + 1 + 1 + 1 + 1$
   - $1 + 1 + 1 + 2$
   - $1 + 1 + 2 + 1$
   - $1 + 2 + 1 + 1$
   - $2 + 1 + 1 + 1$
   - $1 + 2 + 2$
   - $2 + 2 + 1$
   - $2 + 1 + 2$
   - 5

   Clearly, there are 9 ways to write 5 as a sum of 1, 2 and 5. So, $a_5 = 9$. Moreover, the coefficient of x^5 in $A(x)$'s taylor expansion is 9, which means $a_5 = 9$.

2. (a) We have a_n as the number of binary strings of length n which contain the substring "10". Let b_n be the number of binary strings of length n which do not contain the substring "10". This means, $a_n + b_n = 2^n$.

Now, given a string of size n, if it ends with 1, then we just have a_{n-1} ways. If it ends with 0, then we have a_{n-1} ways. But, we also include strings of size $n-2$ with do not have a "10" $= b_{n-2}$, since we can add 1 to their end. This means:

$$a_n = a_{n-1} + a_{n-1} + 2^{n-2} - a_{n-2}$$

(b) For $n \leq 1$, $a_n = 0$.
$a_0 = 0$, $a_1 = 0$, $a_2 = 1$, $a_3 = 4$, $a_4 = 11$.

(c) Finding $A(x)$:

$$\begin{aligned}
A(x) &= \sum_{i=0}^{\infty} a_i x^i = 0 + 0x + \sum_{i=2}^{\infty} a_i x^i \\
&= \sum_{i=2}^{\infty} (a_{i-1} + a_{i-1} + 2^{i-2} - a_{i-2}) x^i \\
&= \sum_{i=2}^{\infty} (2a_{i-1} + 2^{i-2} - a_{i-2}) x^i \\
&= 2x \sum_{i=2}^{\infty} a_{i-1} x^{i-1} + x^2 \sum_{i=2}^{\infty} 2^{i-2} x^{i-2} - x^2 \sum_{i=2}^{\infty} a_{i-2} x^{i-2} \\
&= 2x \sum_{i=1}^{\infty} a_i x^i + x^2 \sum_{i=0}^{\infty} 2^i x^i - x^2 \sum_{i=0}^{\infty} a_i x^i \\
&= 2xA(x) - x^2 A(x) + x^2 \left(\frac{1}{1-2x}\right) \\
A(x)(1 - 2x + x^2) &= x^2 \left(\frac{1}{1-2x}\right) \\
A(x) &= \frac{x^2}{(1-2x)(1-x)^2}
\end{aligned}$$

Finding closed form:

$$\begin{aligned}
A(x) &= \frac{(1 - 2x + x^2) - (1 - 2x)}{(1-2x)(1-x)^2} \\
&= \frac{1 - 2x + x^2}{(1-2x)(1-x)^2} - \frac{1 - 2x}{(1-2x)(1-x)^2} \\
&= \frac{1}{1-2x} - \frac{1}{(1-x)^2} \\
&= \sum_{i=0}^{\infty} 2^i x^i - \sum_{i=0}^{\infty} (i+1) x^i \\
&= \sum_{i=0}^{\infty} (2^i - i - 1) x^i
\end{aligned}$$

So, $a_n = 2^n - n - 1$.

(d) For $n = 4$, $a_4 = 2^4 - 4 - 1 = 11$.
Computing manually, there are 16 binary strings of length 4. Out of these, the following do not contain the substring "10" (b_4):

i. 0000
ii. 0001
iii. 0011
iv. 0111
v. 1111

Clearly, there are 5 strings which do not contain the substring "10". So, $a_4 = 16 - 5 = 11$.

3. (a) Given a decimal string of length n, we can use the following method to find the number of strings where no two consecutive digits are prime. Let a_n be the number of such strings.

If the last digit is prime, then the second last digit can be any number except a prime. There are 4 primes, so there are 6 non-primes. This means, there are $4 \cdot 6 \cdot a_{n-2}$ ways.

If the last digit is non-prime, any substring of size 2 can be a prime. This means, there are $6 \cdot a_{n-1}$ ways.

Therefore:

$$a_n = 24a_{n-2} + 6a_{n-1}$$

(b) When $n = 0$, there is only 1 string, which is the empty string. So, $a_0 = 1$.
When $n = 1$, there are 10 strings, since any digit can be placed. So, $a_1 = 10$.

Otherwise, if $n \geq 2$, use the recurrence relation to find a_n.

(c)

$$\begin{aligned}
A(x) &= \sum_{i=0}^{\infty} a_i x^i = 1 + 10x + \sum_{i=2}^{\infty} a_i x^i \\
&= 1 + 10x + \sum_{i=2}^{\infty} (24a_{i-2} + 6a_{i-1})x^i \\
&= 1 + 10x + 24x^2 \sum_{i=0}^{\infty} a_i x^i + 6x \sum_{i=1}^{\infty} a_i x^i \\
&= 1 + 10x + 24x^2 A(x) + 6x(A(x) - 1) \\
&= 1 + 10x + 24x^2 A(x) + 6x(A(x)) - 6x \\
A(x)(1 - 24x^2 - 6x) &= 1 + 10x - 6x \\
A(x) &= \frac{1 + 4x}{1 - 24x^2 - 6x}
\end{aligned}$$

(d) Checking for $n = 4$:

$a_4 = 24a_2 + 6a_3 = 24a_2 + 6(24a_1 + 6a_2) = 24 \cdot 10 + 6 \cdot (24 \cdot 1 + 6 \cdot 10) = 240 + 6 \cdot 84 = 240 + 504 = 744.$

Using the generating function's taylor expansion, we get the coefficient of x^4 as 744, which means $a_4 = 744$.

Clearly, the answer is correct.

4. (a) We have the following ways to tile:

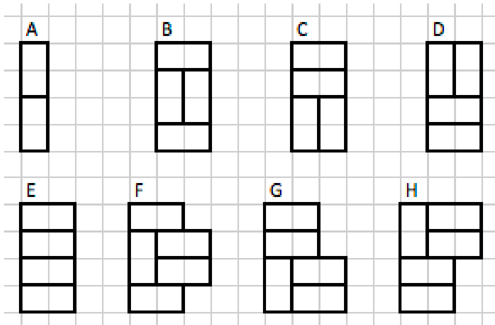

Figure 1: https://math.stackexchange.com/questions/664113/count-the-ways-to-fill-a-4-times-n-board-with-dominoes

Clearly, if we do the following types of tilling, we get the following relations:

i. Type A: $T(4, n-1)$
ii. Type B: $T(4, n-2)$
iii. Type C: $T(4, n-2)$
iv. Type D: $T(4, n-2)$
v. Type E: $T(4, n-2)$

Let the ways in which we count for patterns F, G, H be denoted by $f_{fgh} = f_f + f_g + f_h$.

So, $T(4, n) = T(4, n-1) + 4T(4, n-2) + f_{fgh}(n)$.

Now, consider the "F" tile. This tile can only be extended by a "B" or "F" pattern, with the center let domino removed, which means $f_f(n) = f_b(n-2) + f_f(n-2)$.

Also, the "G" tile can only be extended by a "A", "C" or "G" pattern, with the lower left domino removed, which means $f_g(n) = f_a(n-2) + f_c(n-2) + f_g(n-2)$.

Lastly, the "H" tile can only be extended by a "A", "D" or "H" pattern, with the upper left domino removed, which means $f_h(n) = f_a(n-2) + f_d(n-2) + f_h(n-2)$.

Substituting these values in the original equation, we get:

$$\begin{aligned}
T(4, n) =& T(4, n-1) + 4T(4, n-2) + f_{fgh}(n) \\
=& T(4, n-1) + 4T(4, n-2) + f_b(n-2) + f_f(n-2) + f_a(n-2) \\
&+ f_c(n-2) + f_g(n-2) + f_a(n-2) + f_d(n-2) + f_h(n-2) \\
=& T(4, n-1) + 4T(4, n-2) + f_{abcdfgh}(n-2) + f_a(n-2) \\
=& T(4, n-1) + 4T(4, n-2) + T(4, n-2) - f_e(n-2) + f_a(n-2)
\end{aligned}$$

Since we already know $f_a(n) = T(4, n-1)$ and $f_e(n) = T(4, n-2)$, we get:

$$\begin{aligned}
T(4, n) =& T(4, n-1) + 4T(4, n-2) + T(4, n-2) + T(4, n-3) - T(4, n-4) \\
=& T(4, n-1) + 5T(4, n-2) + T(4, n-3) - T(4, n-4)
\end{aligned}$$

(b) For $T(4, 0) = 1$, $T(4, 1) = 1$, $T(4, 2) = 5$, $T(4, 3) = 11$.
Also, $T(4, n) = 0 \quad \forall n < 0$.

(c) Firstly, for simplicity, let's denote $T(4, n)$ as t_n.

Let $T(x) = \sum_{n=0}^{\infty} t_n x^n$.

Using the initial conditions, we get:

$$\begin{aligned}
T(x) &= 1 + x + 5x^2 + 11x^3 + \sum_{n=4}^{\infty} t_n x^n \\
&= 1 + x + 5x^2 + 11x^3 + \sum_{n=4}^{\infty} (t_{n-1} + 5t_{n-2} + t_{n-3} - t_{n-4}) x^n \\
&= 1 + x + 5x^2 + 11x^3 + x \sum_{n=4}^{\infty} t_{n-1} x^{n-1} + 5x^2 \sum_{n=4}^{\infty} t_{n-2} x^{n-2} \\
&\quad + x^3 \sum_{n=4}^{\infty} t_{n-3} x^{n-3} - x^4 \sum_{n=4}^{\infty} t_{n-4} x^{n-4} \\
&= 1 + x + 5x^2 + 11x^3 + x \sum_{n=3}^{\infty} t_n x^n + 5x^2 \sum_{n=2}^{\infty} t_n x^n \\
&\quad + x^3 \sum_{n=1}^{\infty} t_n x^n - x^4 \sum_{n=0}^{\infty} t_n x^n \\
&= 1 + x + 5x^2 + 11x^3 + x(T(x) - 1 - x - 5x^2) \\
&\quad + 5x^2(T(x) - 1 - x) + x^3(T(x) - 1) - x^4 T(x) \\
&= 1 + x + 5x^2 + 11x^3 + xT(x) - x - x^2 - 5x^3 \\
&\quad + 5x^2 T(x) - 5x^2 - 5x^3 + x^3 T(x) - x^3 - x^4 T(x)
\end{aligned}$$

$$\begin{aligned}
T(x)(1 - x - 5x^2 - x^3 + x^4) &= 1 - x^2 \\
T(x) &= \frac{1 - x^2}{1 - x - 5x^2 - x^3 + x^4}
\end{aligned}$$

(d) Verify for $n = 6$.
For $n = 6$:

$$\begin{aligned}
t_6 &= t_5 + 5t_4 + t_3 - t_2 \\
&= (t_4 + 5t_3 + t_2 - t_1) + 5t_4 + t_3 - t_2 \\
&= 6t_4 + 6t_3 - t_1 \\
&= 6(36) + 6(11) - 1 = 281
\end{aligned}$$

Using taylor's expansion for $T(x)$, and taking the coefficient of x^6, we get: $t_6 = 281$.

So, the G.F. is correct.

5. (a)

$$\begin{aligned} r^n - 3r^{n-1} - 4r^{n-2} &= 0 \\ r^2 - 3r - 4 &= 0 \\ (r-4)(r+1) &= 0 \end{aligned}$$

So, $r = 4, -1 \to a_n = A(4)^n + B(-1)^n$.

Using the initial conditions, we get:

$$\begin{aligned} a_0 &= A + B = 1 \\ a_1 &= 4A - B = 1 \end{aligned}$$

Using $B = 1 - A$, we get $4A - 1 + A = 1 \to 5A = 2 \to A = \frac{2}{5}$.
So, $B = 1 - \frac{2}{5} = \frac{3}{5}$.

So, $a_n = \frac{2}{5}4^n + \frac{3}{5}(-1)^n$.

(b) For this recurrence, we will create 2 components, a_n^p and a_n^h.

First, lets look at the homogeneous part.

$$\begin{aligned} r^n - 2r^{n-1} &= 0 \\ r^{n-1}(r-2) &= 0 \end{aligned}$$

So, $r = 0, 2 \to a_n^h = A(0)^n + B(2)^n = B(2)^n$.

Now, lets look at the particular part.
Let $a_n^p = X + Yn + Zn^2 + Wn^3$.

Substituting in the equation, we get:

$$\begin{aligned} X + Y(n+1) + Z(n+1)^2 + W(n+1)^3 - 2(X + Yn + Zn^2 + Wn^3) &= 7(n+1)^3 \\ X + Y + Z(n^2 + 2n + 1) + W(n^3 + 3n^2 + 3n + 1) - 2X - Yn - 2Zn^2 - 2Wn^3 &= 7(n+1)^3 \\ Y + 2Zn + Z + 3Wn^2 + 3Wn + W - X - Yn - Zn^2 - Wn^3 &= 7(n+1)^3 \\ X + (Yn - Y) + (Zn^2 - 2Zn - Z) + (Wn^3 - 3Wn^2 - 3Wn - W) &= -7(n+1)^3 \\ X + Y(n-1) + Z(n^2 - 2n - 1) + W(n^3 - 3n^2 - 3n - 1) &= -7(n+1)^3 \end{aligned}$$

Now, using $n = 0, 1, 2, 3-$, we get:

$$\begin{aligned} X - Y - Z - W &= -7 \\ X + 0 - 2Z - 6W &= -56 \\ X + Y - Z - 11W &= -189 \\ X + 2Y + 2Z - 10W &= -448 \end{aligned}$$

Solving, we get, $X = -182, Y = -126, Z = -42, W = -7$.

Now, since $a_n = a_n^h + a_n^p$, we get:

$$a_n = B(2)^n - 182 - 126n - 42n^2 - 7n^3$$

Again, using $a_0 = 2$, we get $2 = B - 182 \to B = 184$.

So, $a_n = 184(2)^n - 182 - 126n - 42n^2 - 7n^3$.

(c) Since we have a non-homogeneous recurrence, we will have to solve for the particular and homogeneous parts separately.

Generally, $a_n = a_n^h + a_n^p$.

First, lets look at the homogeneous part.

$$\begin{aligned} r^n - 5r^{n-1} + 6r^{n-2} &= 0 \\ r^{n-2}(r^2 - 5r + 6) &= 0 \\ (r-3)(r-2) &= 0 \end{aligned}$$

So, $r = 3, 2 \rightarrow a_n^h = A(3)^n + B(2)^n$.

Now, lets look at the particular part.
Let $a_n^p = X3^n n + Yn + Z$.

Substituting in the equation, we get:

$$\begin{aligned} a_n^p - 5a_{n-1}^p + 6a_{n-2}^p &= 3^n + 2n \\ X3^n n + Yn + Z - 5(X3^{n-1}(n-1) + Y(n-1) + Z) + 6(X3^{n-2}(n-2) + Y(n-2) + Z) &= 3^n + 2n \\ X3^n n + Yn + Z - 5X3^{n-1}(n-1) - 5Y(n-1) - 5Z + 6X3^{n-2}(n-2) + 6Y(n-2) + 6Z &= 3^n + 2n \\ X(3^n n - 3(3^{n-1})n + 3(3^{n-1})) + Y(2n-7) + 2Z &= 3^n + 2n \\ X(3^{n-1}) + Y(2n-7) + 2Z &= 3^n + 2n \end{aligned}$$

Now, using $n = 0, 1, 2$, we get:

$$\begin{aligned} \frac{1}{3}X - 7Y + 2Z &= 1 \\ X - 5Y + 2Z &= 5 \\ 3X - 3Y + 2Z &= 13 \end{aligned}$$

Solving, we get $X = 3, Y = 1, Z = \frac{7}{2}$.
So, $a_n^p = 3(3)^n n + n + \frac{7}{2} = 3^{n+1}n + n + \frac{7}{2}$.

Now, since $a_n = a_n^h + a_n^p$, we get:

$$a_n = A(3)^n + B(2)^n + 3^{n+1}n + n + \frac{7}{2}$$

Using $a_0 = 1, a_1 = 2$, we get:

$$\begin{aligned} 1 &= A + B + \frac{7}{2} \\ 2 &= 3A + 2B + \frac{27}{2} \end{aligned}$$

Solving, we get $A = -6.5, B = 4$.

So, $a_n = -6.5(3)^n + (2)^{n+2} + 3^{n+1}n + n + \frac{7}{2}$.

2 ¬Straightforward

1. (a) For a given n days, lets start by considering the placement of the first day's menu. It can be places into any of the n days, except the first day. This means, there are $n-1$ choices for the first day.

   Now, consider some arbitrary day k. If the first day's menu is placed on day k, and the menu on day k is placed on the first day, then we have to derange the remaining $n-2$ days. This means, there are $D(n-2)$ ways to do this.

   Otherwise, if the first day's menu is placed on day k, and the menu on day k is placed on some other day, then we have to derange the remaining $n-1$ days. This means, there are $D(n-1)$ ways to do this.

   By the principle of exclusion and inclusion, we get:

   $$D(n) = (n-1)(D(n-1) + D(n-2))$$

   Using the base cases, $D(1) = 0, D(2) = 1$, we can calculate $D(n)$ for any n.

   (b) We need to compute:

   $$P = \frac{D(7)}{7!}$$

   since there are 7! ways to arrange the menu, and $D(7)$ ways to arrange it according to the TA's demands.

   $D(7) = 1854$, so $P = \frac{1854}{5040} = \frac{309}{840} = \frac{103}{280}$.

2. By definition, the Ducas number is the coefficient of x^n in the product of the generating function of Lucas numbers with itself.

   So:

   $$D(x) = L(x) \cdot L(x)$$

   This means, the coefficient of x^n in $D(x)$ is the Ducas number, which is given by:

   $$d_n = \sum_{i=0}^{n} l_i \cdot l_{n-i}$$

   where l_i is the i^{th} Lucas number and d_n is the n^{th} Ducas number.

   Now, we know the closed form of the generating function of Lucas numbers is:

   $$l_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

   For simplicity, let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. So:

   $$l_n = a^n + b^n$$

   By using the generating function of Ducas numbers, we get:

   $$\begin{aligned} d_n &= \sum_{i=0}^{n} (a^i + b^i)(a^{n-i} + b^{n-i}) \\ &= \sum_{i=0}^{n} (a^n + b^n + a^i b^{n-i} + a^{n-i} b^i) \end{aligned}$$

3 Bonus

1. The coefficient of the term x^k in the expansion of $(1-x)^{-n}$ is given by:

$$\binom{-n}{k} = \frac{(-n)(-n-1)(-n-2)\dots(-n-k+1)}{k!}$$

We can simplify this as:

$$\binom{-n}{k} = \frac{(-1)^k n(n+1)(n+2)\dots(n+k-1)}{k!} = (-1)^k \binom{n+k-1}{k}$$

This results in:

$$\binom{-n}{k}(-x)^k = \binom{n+k-1}{k} x^k$$

Hence, the series expansion of $(1-x)^{-n}$ can be rewritten using positive terms as:

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

This is the generating function.