Department of Computer Science
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Introdution to Quantitative Finance

Assignment 2

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Question 1

We can solve this problem by linear programming. We can define the variables as follows:

$$\begin{aligned} x &= \text{Number of Super Dark bars} \\ y &= \text{Number of Special Dark bars} \end{aligned}$$

The objective is to maximize the profit, which is given by:

$$P = 1x + 2y$$

The constraints are:

$$\begin{aligned} 90x + 80y &\leq 1260 \\ 10x + 20y &\leq 240 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

By solving this, we find that the chocolatier should create 0 bars of Super Dark and 12 bars of Special Dark. ($x = 0, y = 12$)

Question 2

We have the following variables:

$$x_i = \text{Binary variable indicating if project } i \text{ is selected (1 if selected, 0 otherwise)}$$

The objective is to maximize the total returns:

$$\text{Maximize} \sum_{i=1}^{5} r_i x_i \qquad (\text{where} \quad r_i \text{ is the return for project } i)$$

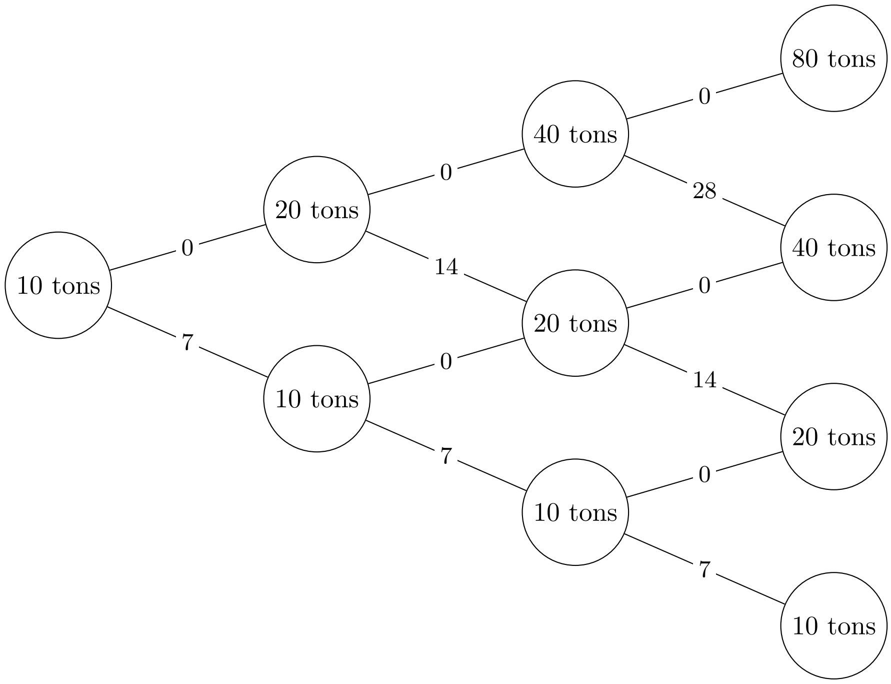
The constraints are:

$$\sum_{i=1}^{5} e_{j,i} x_i \leq 25 \quad \forall j \in \{1, 2, 3\} \quad (\text{where } e_{j,i} \text{ is the expenditure for project } i \text{ in Year } j)$$

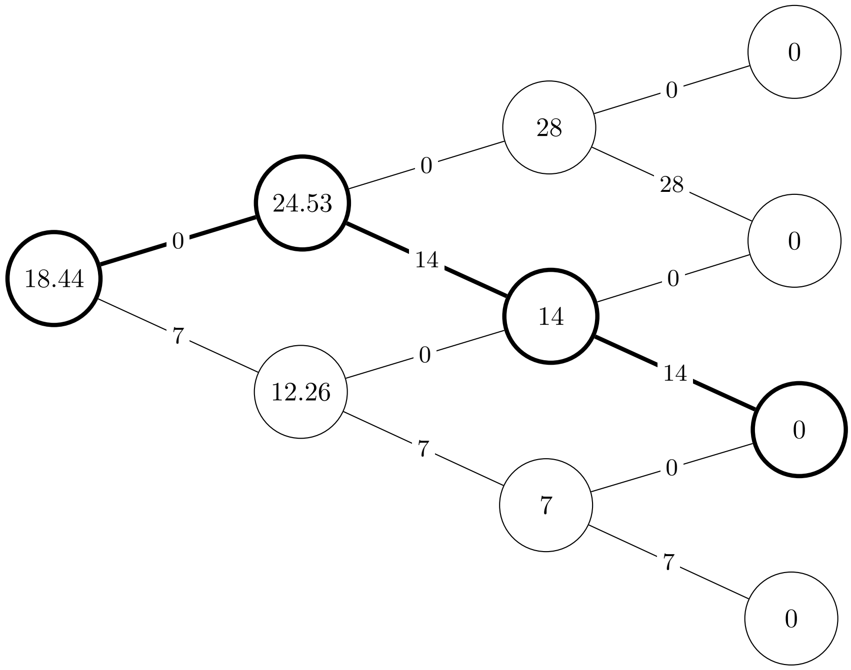
Using the PuLP solver, we find that the projects that should be selected are Projects 1, 2, 3, and 4. This allows us to earn a maximum profit of Rs. 95 Cr.

Question 3

We first need to make a decision tree, where each node represents a state (the number of fish in the lake at the start of the season) and each edge represents a decision (to fish or not), and the value of the edge is the profit for that decision. We can then use the decision tree to find the optimal strategy. In this method, we also use discounting, since when we add the NPV of the future profits, we multiply the profit of that decision by a discount factor, in this case $\frac{1}{1.33}$.



Now, we can use dynamic programming to find the optimal strategy. We can start at the end of the tree and work our way back to the start. While doing this, at each node, we will store the highest NPV possible from that node to the end of the tree.



Question 4

We know that the 2 bullets are loaded consecutively, and in the current situation, we have already shot an empty round. The first round being empty means that the first bullet was not in the first or the last slot. This means that we must be in one of the 4 slots initially.

The probability that the next round is a bullet is then the probability that the first bullet is in one of the even slots, which is $\frac{1}{4}$.

If we decide to spin the barrel before shooting, the probability of the bullet being in the next chamber is $\frac{2}{6} = \frac{1}{3}$.

Hence, spinning the barrel before shooting is worse, as it lowers the probability of the bullet being in the next chamber.

Question 5

(a) To determine c, we need to ensure that the integral of the joint PDF over the entire range equals 1.

$$\int_0^2 \int_1^3 c(x^2 + xy + y^2)\, dy\, dx = 1$$

We can solve this by first integrating with respect to y:

$$\int_0^2 \left[c\left(x^2 y + \frac{xy^2}{2} + \frac{y^3}{3} \right) \right]_1^3 dx = 1$$

Solving the integral, we get:

$$\int_0^2 c\left(x^2(3-1) + \frac{x(3^2 - 1^2)}{2} + \frac{3^3 - 1^3}{3} \right) dx = \int_0^2 c\left(2x^2 + 4x + \frac{26}{3} \right) dx = 1$$

Now, integrating with respect to x, we get:

$$c\left(\frac{2x^3}{3} + 2x^2 + \frac{26x}{3} \right) \Bigg|_0^2 = 1$$

Solving this, we get:

$$c\left(\frac{16}{3} + 8 + \frac{52}{3} \right) = 1 \implies c\left(\frac{16 + 24 + 52}{3} \right) = 1 \implies c\left(\frac{92}{3} \right) = 1 \implies c = \frac{3}{92}$$

(b)

$$\begin{aligned}
E[X] &= \int_0^2 \int_1^3 x \cdot c(x^2 + xy + y^2)\, dy\, dx \\
&= \int_0^2 \int_1^3 x \cdot \frac{3}{92}(x^2 + xy + y^2)\, dy\, dx \\
&= \frac{3}{92} \int_0^2 \left(2x^3 + 4x^2 + \frac{26x}{3} \right) dx \\
&= \frac{3}{92} \left(\frac{2x^4}{4} + \frac{4x^3}{3} + \frac{26x^2}{6} \right) \Bigg|_0^2 \\
&= \frac{3}{92} \left(8 + \frac{32}{3} + \frac{26 \cdot 4}{6} \right) \\
&= 1.174
\end{aligned}$$

$$
\begin{aligned}
E[Y] &= \int_0^2 \int_1^3 y \cdot c(x^2 + xy + y^2)\, dy\, dx \\
&= \frac{3}{92} \int_0^2 \int_1^3 \left(x^2 y + xy^2 + y^3\right)\, dy\, dx \\
&= \frac{3}{92} \int_0^2 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3} + \frac{y^4}{4}\right) \Bigg|_1^3 dx \\
&= \frac{3}{92} \int_0^2 \left(4x^2 + \frac{26x}{3} + 20\right)\, dx \\
&= \frac{3}{92} \left(\frac{4x^3}{3} + \frac{26x^2}{6} + 20x\right) \Bigg|_0^2 \\
&= \frac{3}{92} \left(\frac{32}{3} + \frac{52}{3} + 40\right) \\
&= 2.217
\end{aligned}
$$

$$
\begin{aligned}
E[XY] &= \int_0^2 \int_1^3 xy \cdot c(x^2 + xy + y^2)\, dy\, dx \\
&= \frac{3}{92} \int_0^2 x \int_1^3 \left(x^2 y + xy^2 + y^3\right)\, dy\, dx \\
&= \frac{3}{92} \int_0^2 \left(4x^3 + \frac{26x^2}{3} + 20x\right)\, dx \\
&= \frac{3}{92} \left(x^4 + \frac{26x^3}{9} + 10x^2\right) \Bigg|_0^2 \\
&= \frac{3}{92} \left(16 + \frac{26 \cdot 8}{9} + 40\right) \\
&= 2.580
\end{aligned}
$$

(c) For X and Y to be statistically independent, $E[XY] = E[X]E[Y]$. We can see that this is not the case here, since $E[XY] = 2.58 \neq 2.603 = 1.174 \times 2.217 = E[X]E[Y]$.

(d) We first begin by calculating the probability distribution of X given $Y = y_0$. We can do this by using the formula for conditional probability:

$$
g(x) = \frac{f_{X,Y}(x, y_0)}{f_Y(y_0)}
$$

Note that in this case, $y_0 = E[Y] = 2.217$.

$$
\begin{aligned}
f_Y(y_0) &= \frac{3}{92} \int_0^2 (x^2 + xy_0 + y_0^2)\, dx \\
&= \frac{3}{92} \left(\frac{8}{3} + 2y_0 + 2y_0^2\right) \\
&= \frac{3}{92} \left(\frac{8}{3} + 2 \cdot 2.217 + 2 \cdot (2.217)^2\right) \\
&= \frac{3}{92} \left(\frac{8}{3} + 4.434 + 10.23\right) \\
&= \frac{3}{92} \cdot 17.997 \\
&= 0.552
\end{aligned}
$$

$$
\begin{aligned}
f_{X,Y}(x, y_0) &= c(x^2 + xy_0 + y_0^2) \\
&= \frac{3}{92}(x^2 + 2.217x + (2.217)^2)
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= \frac{f_{X,Y}(x, y_0)}{f_Y(y_0)} \\
&= \frac{3}{92} \cdot \frac{x^2 + 2.217x + 4.915}{0.552}
\end{aligned}
$$

Now, to find $E[X|Y = y_0]$, we can simply evaluate the following integral:

$$
\begin{aligned}
E[X|Y = y_0] &= \int_0^2 x \cdot g(x)\, dx \\
&= \int_0^2 x \cdot \frac{3}{92} \cdot \frac{x^2 + 2.217x + 4.915}{0.552}\, dx = 1.166
\end{aligned}
$$

To find the variance of X given $Y = y_0$, we can use the following formula:

$$
\begin{aligned}
Var(X|Y = y_0) &= E[X^2|Y = y_0] - (E[X|Y = y_0])^2 \\
E[X^2|Y = y_0] &= \int_0^2 x^2 \cdot g(x)\, dx \\
&= \int_0^2 x^2 \cdot \frac{3}{92} \cdot \frac{x^2 + 2.217x + 4.915}{0.552}\, dx = 1.676
\end{aligned}
$$

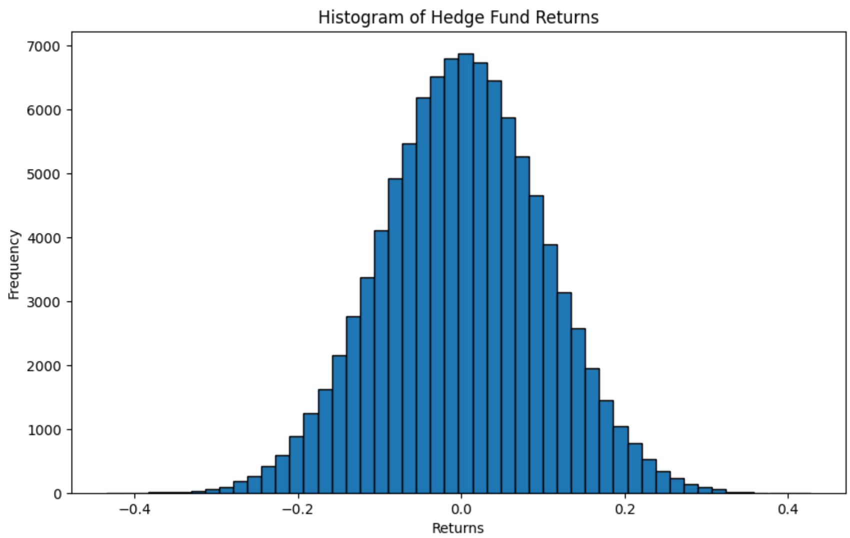
Therefore, $Var(X|Y = y_0) = 1.676 - (1.166)^2 = 0.3164$

Question 6

(a) Mean return $= 0.0022$, Risk $= 0.0999$

(b) Within $\sigma = 68.33\%$, Within $2\sigma = 95.36\%$, Within $3\sigma = 99.74\%$

(c) Histogram of returns:



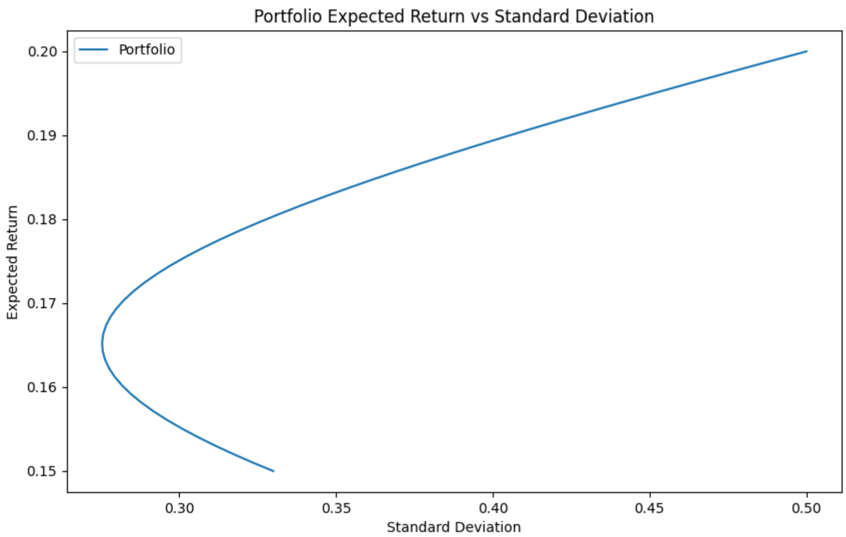
Question 7

(a) $E(r_p) = 0.1 \cdot 0.2 + 0.9 \cdot 0.15 = 0.155$

$\sigma_p = \sqrt{0.1^2 \cdot 0.5^2 + 0.9^2 \cdot 0.33^2} = 0.3012$ (Since correlation coefficient $= 0$)

(b) $E(r_{w_A}) = w_A \cdot 0.2 + (1 - w_A) \cdot 0.15$
$\sigma_{w_A} = \sqrt{w_A^2 \cdot 0.5^2 + (1 - w_A)^2 \cdot 0.33^2}$

The plot below shows the portfolio expected return and standard deviation for all values of $w_A \in [0, 1]$.



(c) The calculate the equation of this line, we need to first find it's slope:

$$m = \frac{0.20 - 0.03}{0.50} = 0.34 \quad \text{(A straight line since no-correlation)}$$

$$\implies r_p = 0.34\sigma_p + 0.03$$

To find this in terms of w_A, we can use the following steps:

$$\sigma_p = \sqrt{w_A^2 \cdot 0.5^2 + (1 - w_A)^2 \cdot 0^2} = 0.5w_A$$

$$\implies r_p = 0.34 \cdot 0.5w_A + 0.03 = 0.17w_A + 0.03$$

(d) To find the portfolio with the maximum slope, we need to find the portfolio with the highest expected return for a given level of risk. This is given by the formula:

$$\frac{E(r_{w_A}) - r_f}{\sigma_{w_A}} = \frac{w_A \cdot 0.2 + (1 - w_A) \cdot 0.15 - 0.03}{\sqrt{w_A^2 \cdot 0.5^2 + (1 - w_A)^2 \cdot 0.33^2}}$$

By taking the derivative and setting it to 0, we can find the value of w_A that maximizes the slope, which is $w_A = 0.3816 = 38.16\%$. The corresponding maximum slope is 0.4978.

(e) The table below shows the risk, return, and utility for different portfolios on the capital allocation line with the maximum slope. We use the same equations as part b, but set asset A as (0.2794, 0.1691) (the portfolio with the maximum slope) and asset B as (0, 0.03) (the risk free asset).

Return	Risk	Utility
0.03	0.0	0.0300
0.09	0.1205	0.0718
0.15	0.2410	0.0774
0.20	0.3415	0.0543

Question 8

(a) Point A is not achievable as it is on the left/above the efficient frontier. This is easy to show, since if Point A was achievable, it would dominate some points on the efficient frontier. However, since the efficient frontier dominates all other points, Point A cannot be achievable.

(b) Points C and G will not be chosen by a rational, risk-averse investor, as it has the same risk as other points on the efficient frontier, but lower returns. This is equivalent to saying that Points C and G are on the right of the efficient frontier, and hence are dominated by other points on the efficient frontier.

(c) A risk neutral investor is one who is only concerned with the expected return of a portfolio. Since Point D has the highest expected return, it is the most suitable for a risk-neutral investor.

(d) Gold is on the inefficient part of the feasible set. Nonetheless, gold is owned by many rational investors as part of a larger portfolio. This is because of 2 major reasons: 1) gold is a safe asset, and hence, it is held by risk averse investors as a diversifier. Moreover, since gold is said to be negatively correlated with stocks, it helps to reduce the overall risk. 2) Gold lies on the inefficient part of the feasible set for this model, but not every investor agrees with the model. This is because the model has 2 major assumptions: i) Every investor is a mean-variance investor, and ii) Each investor has the same information and agree on the same probability distribution. In reality, not every investor agrees with these assumptions, and hence, they may choose to invest in assets that lie on the inefficient part of the feasible set.

(e) The utility of an investor at point P with a risk aversion coefficient of 3 is:

$$U = E(r) - \frac{1}{2}A\sigma^2 = 0.16 - \frac{1}{2} \cdot 3 \cdot 0.17^2 = 0.16 - 0.04335 = 0.11665$$