Mod 3

Due on March 20, 2025

Tuesday/Thursday 11:00-12:15, Warner 209

Mark Floryan - Section 001

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#1

For each of the languages below, provide a context-free grammar that generates it (Note that some of these might also be regular languages, but we still want a grammar for each). For all parts, $\Sigma = \{0, 1\}$:

- Strings that contain exactly two 1's OR exactly two 0's

  $S \to A|X$

  $A \to 0A|1B$
  $B \to 0B|1C$
  $C \to 0C|\epsilon$

  $X \to 1X|0Y$
  $Y \to 1Y|0Z$
  $Z \to 1Z|\epsilon$

- Strings of even length that contain 1100 directly in the center (i.e., $w1100u \mid |w| = |u|$)

  $S \to T1100T$
  $T \to 0T|1T|\epsilon$

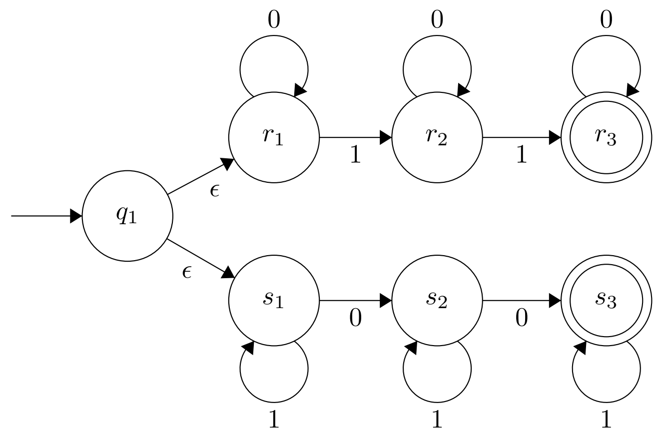
- $ww^Ruu^R \mid w \in \Sigma^* \wedge u \in \Sigma^*$

  $S \to AA$
  $A \to 0A0|1A1|\epsilon$

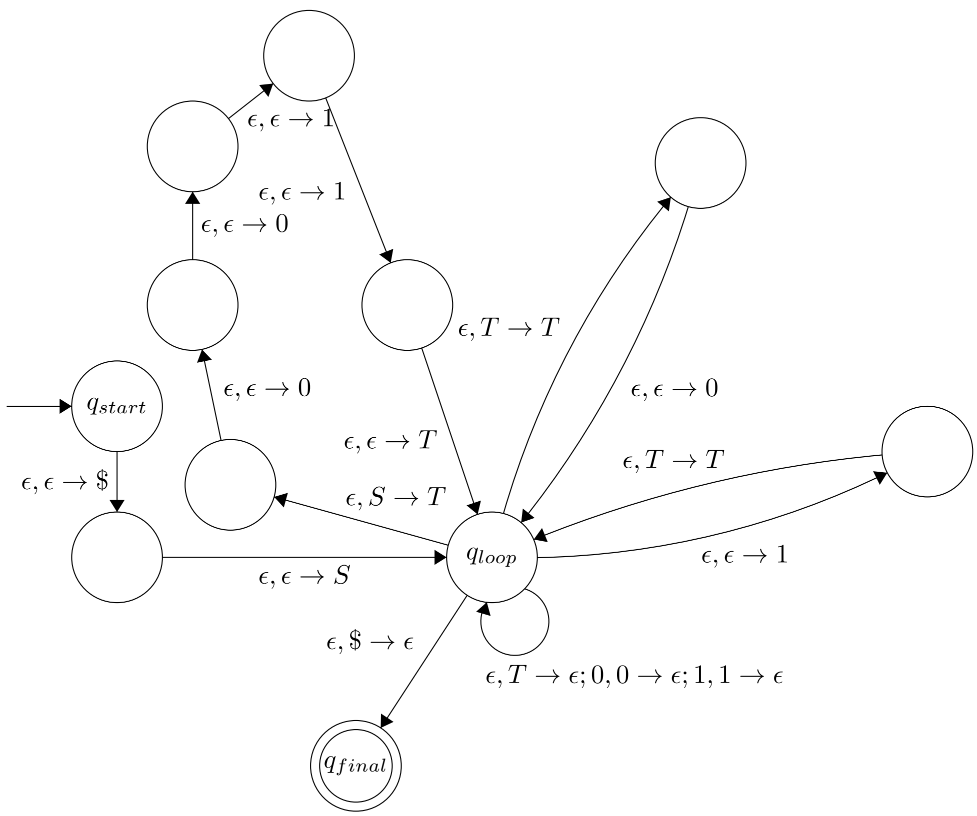
#2

Draw PDAs for each of the languages in the previous exercise (note that you can draw a DFA / NFA if the language happens to be regular).

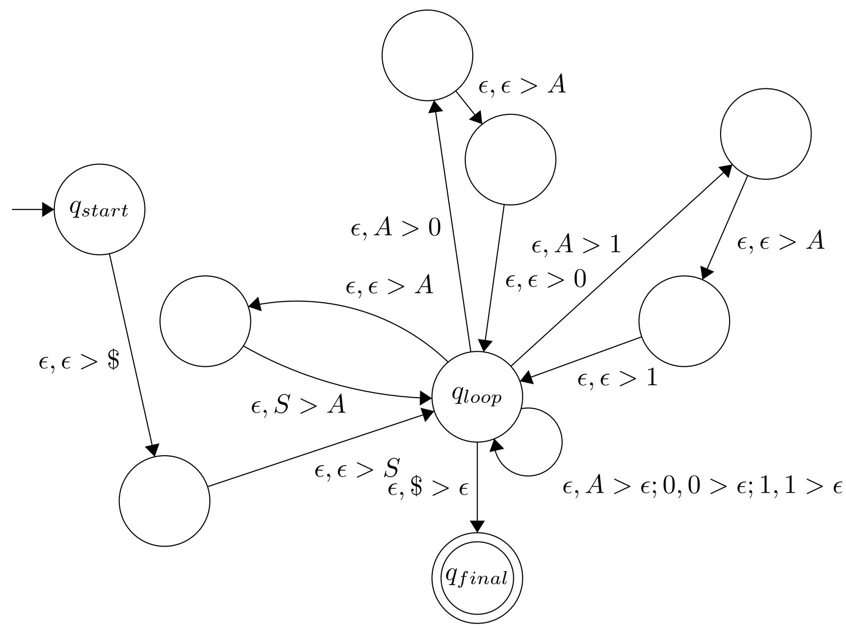
- Strings that contain exactly two 1's OR exactly two 0's



- Strings of even length that contain 1100 directly in the center (i.e., $w1100u \mid |w| = |u|$)



- $ww^R uu^R \mid w \in \Sigma^* \wedge u \in \Sigma^*$



#3

For this question, you will prove that context-free languages are NOT closed under intersection. The alphabet for all languages in this question is $\Sigma = \{a, b, c\}$ Do this by showing the following:

- **Part 1:** First, show that $A = \{a^m b^n c^n | m, n \geq 0\}$ is context-free by producing a context-free grammar that generates it.

  $S \rightarrow AB$
  $A \rightarrow aA|\epsilon$
  $B \rightarrow bBc|\epsilon$

- **Part 2:** Do the same, but for language $B = \{a^n b^n c^m | m, n \geq 0\}$

  $S \rightarrow AB$
  $A \rightarrow aAb|\epsilon$
  $B \rightarrow cB|\epsilon$

- **Part 3:** Lastly, find the intersection of these two sets and use the pumping lemma to show that the intersection language is not context-free.

  $A \cap B = \{a^n b^n c^n | n \geq 0\}$

  Assume $A \cap B$ is context-free.
  Let p be the pumping length.
  Let $s = a^p b^p c^p$.

  By the pumping lemma, s can be divided into $s = uvwxy$ such that:

  1. $|vwx| \leq p$
  2. $|vx| \geq 1$
  3. $uv^i wx^i y \in A \cap B$ for all $i \geq 0$

  Given $s = a^p b^p c^p$, vwx must be contained in the first p characters of s.

  We can divide s into 2 cases:

  1. vwx contains only one type of character.
     In this case, uv^2wx^2y will contain more of one type of character than the other two types.
     Thus, $uv^2wx^2y \notin A \cap B$.
  2. vwx contains two types of characters.
     In this case, uv^2wx^2y will make it such that the characters are out of order (a then b then c).
     Thus, $uv^2wx^2y \notin A \cap B$.

  In both cases, $uv^2wx^2y \notin A \cap B$. Thus, $A \cap B$ is not context-free.

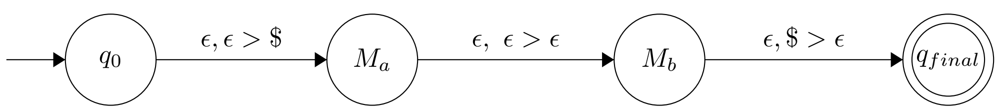
#4 Let us define a new operation using the $\Diamond$ symbol as such: if A and B are languages, then $A\Diamond B = \{xy|x \in A, y \in B, |x| = |y|\}$. Prove that if A and B are regular languages, then $A\Diamond B$ must be a context-free language.
Given that A and B are regular languages, we can construct DFAs M_A and M_B that accept A and B, respectively.

- Let $M_a = (Q_a, \Sigma, \delta_a, q_{0a}, F_a)$ be the DFA that accepts A.
- Let $M_b = (Q_b, \Sigma, \delta_b, q_{0b}, F_b)$ be the DFA that accepts B.

Then we can construct a PDA M that accepts $A\Diamond B$ as follows:
Given $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ where:

- $Q = q_0, q_{final} \cup Q_a \cup Q_b$
- $\Gamma = \Sigma \cup \{\$\}$
- $F = q_{final}$
- δ is defined as follows:

First, we'll push a $ onto the stack to mark the end empty point and we'll transition to the start state of M_a. At each read input, we'll push a # onto the stack in order to keep track of the length of x. Then at every accept state for M_a we'll perform a ϵ transition to the start state of M_b. At each read input and during this epsilon transition, we'll pop a # from the stack in order to keep track of the length of y. Then at every accept state for M_b we'll perform a ϵ transition to the final state. If the stack is empty (i.e., there is a $ remaining) and we're at the final state, we'll accept the string.
This PDA would look something like this with augmented M_a to push on every transition and M_b to pop on every transition:



This PDA recognizes $A\Diamond B$ because it ensures $x \in A$ by simulating M_a and $y \in B$ by simulating M_b and ensures that $|x| = |y|$ by pushing a # for every character in x and popping a # for every character in y. Since PDAs recognize exactly the context-free languages, and we've constructed a PDA that recognizes $A\Diamond B$, we've shown that $A\Diamond B$ is a context-free language.

#5 Suppose you have a context-free language C and a regular language R. Prove that $C \cap R$ is context-free. Given a PDA P that recognizes C, we can just use the DFA D that recognizes R to simulate the PDA P. We can construct a new PDA P' that recognizes $C \cap R$ as follows by simulating P and D in parallel:

Given $P = (Q_p, \Sigma, \Gamma_p, \delta_p, q_{0p}, F_p)$ and $D = (Q_d, \Sigma, \delta_d, q_{0d}, F_d)$, build $P' = (Q, \Sigma, \Gamma, \delta, q_0, F)$ where:

- $Q = Q_p \times Q_d$
- $\Gamma = \Gamma_p$
- $F = F_p \times F_d$
- δ is defined as follows:

  For each transition in P where $\delta_p(q, a, X) = (p, Y)$, and each transition in D where $\delta_d(r, a) = s$:

$$\delta((q, r), a, X) = ((p, s), Y)$$

Since we are able to construct a PDA for $C \cap R$, we have shown that $C \cap R$ is a context-free language and that the intersection of a CFL and RL is context-free.

#6 Consider the language $A = \{w \mid w \in \{a, b, c\}^* \wedge F(w, a) = F(w, b) = F(w, c)\}$ where $F(w, a)$ counts the number of occurences of character a in string w. Prove that A is not context-free by using the result of the previous question (*Hint: Assume this language is a CFL and intersect it with a regular language of your choice!*)
We'll conduct a proof by contradiction. Assume that A is a context-free language.

Given the results of the previous question we can intersect A with the regular language $B = a^*b^*c^*$ to get $A \cap B = a^n b^n c^n | n \geq 0$.

Now, we've already shown in problem 3, part 3 that due to the pumping lemma that $A \cap B$ is not context-free. This proof has been provided again below.

Assume $A \cap B$ is context-free.
Let p be the pumping length.
Let $s = a^p b^p c^p$.

By the pumping lemma, s can be divided into $s = uvwxy$ such that:

1. $|vwx| \leq p$
2. $|vx| \geq 1$
3. $uv^iwx^iy \in A \cap B$ for all $i \geq 0$

Given $s = a^p b^p c^p$, vwx must be contained in the first p characters of s.

We can divide s into 2 cases:

1. vwx contains only one type of character.
   In this case, uv^2wx^2y will contain more of one type of character than the other two types.
   Thus, $uv^2wx^2y \notin A \cap B$.
2. vwx contains two types of characters.
   In this case, uv^2wx^2y will make it such that the characters are out of order (a then b then c).
   Thus, $uv^2wx^2y \notin A \cap B$.

In both cases, $uv^2wx^2y \notin A \cap B$. Thus, $A \cap B$ is not context-free. Given that $A \cap B$ is not context-free, we have shown that A is not context-free as CFLs are closed under intersection with RLs.

#7

Prove that the language A from the previous question is not context-free again, but this time do so by utilizing the *Pumping Lemma for Context-Free Languages*.

We can solve this problem purely using the pumping lemma for context-free languages through a proof by contradiction. If A is context-free, then there exists a pumping length p $\geq$ 1 such that any string s $\in$ A with $|s| \geq p$ can be divided into five strings $s = uvwxy$ such that:

1. $uv^iwx^iy \in A$ for all $i \geq 0$
2. $|vx| \geq 1$
3. $|vwx| \leq p$

Consider the string $s = a^pb^pc^p$. Since $|s| \geq p$, we can apply the pumping lemma to s. Since the substring vwx must be contained in the first p characters of s, vwx cannot contain all three characters a, b, and c. This leaves us with the following two cases:

1. vwx contains only one type of character.
   In this case, we will end up with more of one type of character than the other two types when we pump v and x. Thus, $uv^2wx^2y \notin A$.
2. vwx contains two types of characters.
   In this case, we will end up with the characters out of order (a then b then c) when we pump v and x. Thus, $uv^2wx^2y \notin A$.

In all cases, we have shown that there exists some value of i such that $uv^iwx^iy \notin A$. Thus, contradicting the first condition of the pumping lemma. Since assuming that A is context-free leads to a contradiction, we have shown that A is not context-free.