CT-216 – Introduction to Communication Systems

Lab Group 2

Project Group 2

Analysis

Soft Decision Decoding

- c - transmitted sequence
- r - received sequence from the demodulator
- c_{jm}- bit at the mth position of the jth branch of transmitted sequence
- r_{jm}- bit at the mth position of the jth branch of received sequence

➢ Unlike Hard decision decoding, where the received sequence will be either 0 or 1 in the output, for soft decision decoding we have:

$$r_{jm} = \sqrt{\varepsilon_c}\left(2c_{jm} - 1\right) + n_{jm}$$

- ε_c - transmitted signal energy
- n_{jm}- noise introduced by the channel

➢ Branch metric (μ_j) for jth branch and ith path is defined as:

$$\mu_j = \log\left(P(Y_j|C_j^{(i)})\right)$$

- Path metric ($PM^{(i)}$) for the i[th] path of B branches through the trellis:

$$PM^{(i)} = \sum_{j=1}^{B} \mu_j^{(i)}$$

- The demodulator output is described statistically by the probability density function:

$$p(r_{jm}|c_{jm}^{(i)}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{r_{jm}-\sqrt{\varepsilon_c}(2c_{jm}-1)}{\sigma}\right)^2}$$

  - Where variance of the noise is $\sigma^2 = \frac{N_0}{2}$
  - Here, n_{jm} follows normal distribution with mean 0 and variance $\frac{N_0}{2}$ (i.e. n_{jm} ~ $N(0, N_0/2)$).

- We can neglect the terms that are common to all the branch metrics:

$$\mu^{(i)} = \sum_{m=1}^{n} r_{jm}(2c_{jm}^{(i)} - 1)$$

- Correlation Metric ($CM^{(i)}$) is given as:

$$CM^{(i)} = \sum_{j=1}^{B} \mu_j^{(i)} = \sum_{j=1}^{B}\sum_{m=1}^{n} r_{jm}(2c_{jm}^{(i)} - 1)$$

- To derive the probability of error, we will assume that the all-zero sequence is transmitted, and we determine the probability of error in favor of another sequence.

- Let, i=0 denote the all-zero path.
  So, c_{jm}= 0 ∀ {j,m}

$$CM^{(0)} = \sum_{j=1}^{B}\sum_{m=1}^{n} r_{jm}(2c_{jm}^{(0)} - 1)$$

$$\therefore CM^{(0)} = \sum_{j=1}^{B}\sum_{m=1}^{n} (\sqrt{\varepsilon_c}(2(0) - 1) + n_{jm})(2(0) - 1)$$

$$\therefore CM^{(0)} = \sum_{j=1}^{B}\sum_{m=1}^{n} (-\sqrt{\varepsilon_c} + n_{jm})(-1)$$

$$\therefore CM^{(0)} = \sqrt{\varepsilon_c}nB - \sum_{j=1}^{B}\sum_{m=1}^{n} n_{jm}$$

➢ The first event probability is defined as the probability that another path that merges with all zero path at node B has a metric that exceeds the metric of all zero path for the first time. It is denoted as $P_2(d)$**.**

- Suppose the incorrect path, say i = 1, that merges with all zero path differs from all zero path in d bits.
- Now for the correct path $CM^{(i)} < CM^{(0)}$ but error will occur when the $CM^{(i)} \geq CM^{(0)}$

$$\therefore P_2(d) = P\left(CM^{(i)} \geq CM^{(0)}\right)$$

$$= P\left(CM^{(i)} - CM^{(0)} \geq 0\right) \;..equation(1)$$

- $CM^{(0)}$ and $CM^{(i)}$ can be written as:

$$CM^{(0)} = \sum_{j=1}^{B}\sum_{m=1}^{n} r_{jm}(2c_{jm}^{(0)} - 1)$$

$$CM^{(i)} = \sum_{j=1}^{B}\sum_{m=1}^{n} r_{jm}(2c_{jm}^{(i)} - 1)$$

- Now applying it in the above equation,

$$P\left(\sum_{j=1}^{B}\sum_{m=1}^{n} r_{jm}\left(2c_{jm}^{(i)} - 1\right) - \sum_{j=1}^{B}\sum_{m=1}^{n} r_{jm}(2c_{jm}^{(0)} - 1) \geq 0\right)$$

$$= P\left(2\sum_{j=1}^{B}\sum_{m=1}^{n} r_{jm}\left(c_{jm}^{(i)} - c_{jm}^{(0)}\right) \geq 0\right)$$

- Now, the bit difference is d so $(c_{jm}^{(i)} - c_{jm}^{(0)})$ will be 0.
- So, the above formula can be reduced to,

$$P\left(\sum_{l=1}^{d} r_l{}' \geq 0\right)$$

- Where l runs over the d bits in which the two paths differs ε_c the set $\{r_l{}'\}$
- For all zero path we took c_{jm}= 0 ∀ {j,m}
- The expectation of $r_l{}'$ can be calculated as follows:

$$E\{r_l{}'\} = E\{\sqrt{\varepsilon_c}(0 - 1) + n_{jm}\}$$

$$\therefore E\{r_l{}'\} = E\{-\sqrt{\varepsilon_c} + n_{jm}\}$$

$$\therefore E\{r_l{}'\} = E\{-\sqrt{\varepsilon_c}\} + E\{n_{jm}\}$$

$$\therefore E\{r_l{}'\} = -\sqrt{\varepsilon_c}$$

- The variance of $r_l{}'$ can be calculated as follows:

$$var\{r_l{}'\} = var\{-\sqrt{\varepsilon_c} + n_{jm}\}$$

$$\therefore var\{r_l{}'\} = 0 + var\{n_{jm}\}$$

$$\therefore var\{r_l{}'\} = \frac{N_0}{2}$$

- Therefore, $\{r_l'\}$ ~ $\mathrm{N}(-\sqrt{\varepsilon_c}, N_0/2)$ i.e identically and independently Gaussian distributed variable
- Now, using CLT (Central Limit Theorem) for the d bits

$$\sum_{l=1}^{d} r_l' \sim N\left(-d\sqrt{\varepsilon_c}\,, \frac{dN_0}{2}\right)$$

- Converting it into standard normal variate $Z \sim N(0,1)$

$$P\left(\sum_{l=1}^{d} r_l' \geq 0\right) = P\left(\frac{\sum_{l=1}^{d} r_l' + d\sqrt{\varepsilon_c}}{\sqrt{\frac{dN_0}{2}}} \geq \frac{d\sqrt{\varepsilon_c}}{\sqrt{\frac{dN_0}{2}}}\right)$$

$$= P\left(Z \geq \sqrt{\frac{2\varepsilon_c d}{N_0}}\right)$$

$$P\left(z \geq \sqrt{\frac{2\varepsilon_c d}{N_0}}\right) = \frac{1}{\sqrt{2\pi}\sigma} \int_{\sqrt{\frac{2\varepsilon_c d}{N_0}}}^{\infty} e^{-\frac{z^2}{2}}\, dz$$

- Here, $\sigma = 1$ and $\mu = 0$ because we have converted it into standard normal variate.

$$\therefore\ P\left(z \geq \sqrt{\frac{2\varepsilon_c d}{N_0}}\right) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2\varepsilon_c d}{N_0}}}^{\infty} e^{-\frac{z^2}{2}}\, dz$$

$$\therefore P_2(d) = P = Q\left(\sqrt{\frac{2\varepsilon_c d}{N_0}}\right)$$

$$\because \left[Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{u^2}{2}}\, du\right]$$

- Now, $SNR(\gamma_b) = \frac{\varepsilon_b}{N_0}$
- And, $\varepsilon_b = \frac{\varepsilon_c}{R_c}$

$$\therefore \left(\frac{\varepsilon_b}{N_0} = \frac{\varepsilon_c}{R_c N_0} = \gamma_b\right)$$

- ε_b- Energy per bit
- ε_c- Transmitted symbol energy
- R_c - Code Rate

$$\boxed{P_2(d) = Q\left(\sqrt{2\gamma_b R_c d}\right)} \qquad ..equation(2)$$

➢ There can be more than one incorrect path that merges with all zero path.

- Thus, we can sum up the error probability over all possible path d that can merge with all zero path. Then, we can obtain the upper bound on the first event probability as:

$$P_e \leq \sum_{d=d_{free}}^{\infty} a_d P_2(d)$$

- Here, a_d is number of paths having distance d from the all zero path
- Now, using $equation(2)$, we get,

$$\boxed{P_e \leq \sum_{d=d_{free}}^{\infty} a_d Q\left(\sqrt{2\gamma_b R_c d}\right)} \qquad ..equation(3)$$

- Now, the upper-bound of the Q function is given as:

$$Q\left(\sqrt{2\gamma_b R_c d}\right) \leq e^{-\gamma_b R_c d}$$

- Replacing the Q function with the upper bound in the $equation(3)$:

$$P_e \leq \sum_{d=d_{free}}^{\infty} a_d \; e^{-\gamma_b R_c d}$$

$$\therefore P_e \leq \sum_{d=d_{free}}^{\infty} a_d \; D^d \text{ , where } D = e^{-\gamma_b R_c}$$

- Now,

$$\sum_{d=d_{free}}^{\infty} a_d \; D^d \; = T(D)$$

Here, $T(D)$ is the transfer function.

$$\therefore P_e \leq T(D), \text{ where } D = e^{-\gamma_b R_c}$$

➢ Now, for determining bit error probability, we can do the following:

- Consider transfer function $T(D, N)$ where the exponent of N shows the bit errors in selecting an incorrect path that merges with the all zero path at some particular node B.

$$T(D, N) = \sum_{d=d_{free}}^{\infty} a_d \; D^d \; N^{f(d)}$$

- Here, $f(d)$ denotes the exponent of N as a function of d.
- Taking the derivative of $T(D, N)$ w.r.t N and then placing N = 1.

$$\left(\frac{d}{dN} T(D, N)\right)_{N=1} = \sum_{d=d_{free}}^{\infty} a_d \; D^d \; f(d)$$

$$= \sum_{d=d_{free}}^{\infty} B_d \, D^d$$

$$\text{where}\,, B_d = \; a_d \, f(d)$$

- Now, let P_b be the probability of bit error rate (BER).
- The upper bound for bit error probability for k = 1 is as below:

$$P_b \; < \sum_{d=\,d_{free}}^{\infty} \; a_d \, f(d) P_2(d)$$

$$P_b \; < \sum_{d=\,d_{free}}^{\infty} \; B_d P_2(d)$$

$$\boxed{P_b \; < \sum_{d=\,d_{free}}^{\infty} \; B_d Q\left(\sqrt{2\gamma_b R_c d}\right)} \qquad \ldots .equation(4)$$

- Now,

$$Q\left(\sqrt{2\gamma_b R_c d}\right) \leq D^d$$

$$P_b \; < \sum_{d=\,d_{free}}^{\infty} \; B_d \, D^d = \frac{dT(D,N)}{dN}$$

$$P_b \; < \frac{dT(D,N)}{dN}$$

- Differentiation of transfer function w.r.t N gives the upper bound of bit error probability.

$$P_b \; < \left. \frac{dT(D,N)}{dN} \right|_{N=1, D=e^{-\gamma_b R_c}}$$

Hard Decision Decoding

- The metrics in the Viterbi algorithm are the hamming distances between the received sequence and $2^{k(K-1)}$ surviving sequences in the trellis for the hard decision decoding.
  - As assumed in the soft decision decoding, let the all zero sequence be transmitted. On similar lines, let us find the first event error probability, i.e $P_2(d)$.
  - When d is odd:

$$P_2(d) = \sum_{k=\frac{(d+1)}{2}}^{d} \binom{d}{k} p^k (1-p)^{d-k}$$

  - Here p is the probability of flipping the bit in the binary symmetric channel.
  - From the concept of t_c(error correction capacity)=$\left\lfloor \frac{d-1}{2} \right\rfloor$, if number of errors is greater than t_c, the bits cannot be corrected and hence produces an error.
  - When d is even:

$$P_2(d) = \sum_{k={d}/{2}+1}^{d} \binom{d}{k} p^k (1-p)^{d-k} + \frac{1}{2} \binom{d}{d/2} p^{d/2} (1-p)^{d/2}$$

  - Now on using upper bound to simplify the above equations we get,

$$P_2(d) = \sum_{k=\frac{(d+1)}{2}}^{d} \binom{d}{k} p^k (1-p)^{d-k}$$

$$\leq \sum_{k=\frac{(d+1)}{2}}^{d} \binom{d}{k} p^{d/2} (1-p)^{d/2}$$

$$= p^{d/2}(1-p)^{d/2} \sum_{k=\frac{(d+1)}{2}}^{d} \binom{d}{k}$$

$$\leq p^{d/2}(1-p)^{d/2} \sum_{k=0}^{d} \binom{d}{k}$$

$$\leq \ 2^d p^{d/2}(1-p)^{d/2}$$

$$\therefore P_2(d) < [4p(1-p)]^{d/2}$$

- We will get the same result when d is even.
- Summing up all the first event probabilities as in soft decision decoding, we get the overall first event probability (P_e).

$$P_e < \sum_{d=d_{free}}^{\infty} a_d [4p(1-p)]^{d/2}$$

- Using transfer function $T(D)$, in the above equation we get,

$$P_e < T(D) \ where, D = \sqrt{4p(1-p)}$$

- Let us consider the transfer function $T(D, N)$ where the power of N indicates the number of bit errors in selecting an incorrect path that merges with the all zero path at some node B.

$$T(D, N) = \sum_{d=d_{free}}^{\infty} a_d \ D^d \ N^{f(d)}$$

- Here, $f(d)$ denotes the exponent of N as a function of d.
- Taking the derivative of $T(D, N)$ w.r.t N and then placing N = 1.

$$\frac{d}{dN} T(D, N)|_{(N=1)} = \sum_{d=d_{free}}^{\infty} a_d \ D^d \ f(d)$$

$$= \sum_{d=d_{free}}^{\infty} B_d \ D^d$$

$$where\ , B_d = \ a_d\, f(d)$$

- The measure for the performance of convolutional code is given by bit error probability P_b.
- Bit error probability for k = 1 is given by the below upper bound:

$$P_b \ < \sum_{d=\, d_{free}}^{\infty} B_d P_2(d)$$

- Like the soft decision decoding, bit error probability for hard decision decoding is upper bounded by the differentiation of the transfer function w.r.t N.

$$P_b \ < \left. \frac{dT(D, N)}{dN} \right|_{N=1, D=\sqrt{4p(1-p)}}$$