Lecture 15 Oct 25

Test H_0: $\Delta_1^2 = 0$ and $\Delta_1^3 = 0$

vs H_1: At least one of Δ_1^2 and Δ_1^3 is not zero.

F_{STAT}

Suppose that H_0 is rejected!

Post-hoc comparisons

<u>Global</u> 95% CI for $\underline{\Delta_1^2}$ and $\underline{\Delta_1^3}$

$$\hat{\underline{\beta}} = \begin{pmatrix} \hat{\beta}_0^1 \\ \hat{\beta}_1^1 \\ \hat{\Delta}_0^2 \\ \hat{\Delta}_1^2 \\ \hat{\Delta}_0^3 \\ \hat{\Delta}_1^3 \\ \hat{\Delta}_0^4 \\ \hat{\Delta}_1^4 \end{pmatrix} \begin{matrix} \\ \\ \\ \longleftarrow \\ \\ \longleftarrow \\ \\ \\ \end{matrix}$$

$$\Delta_1^2 = \underbrace{(0 \ldots 0, 1, 0 \ldots 0)}_{\underline{c}_2 \beta} \underline{\beta}$$

$$\Delta_1^3 = \underbrace{(0 \ldots 0, 1, 0 \ldots 0)}_{\underline{c}_3 \beta} \underline{\beta}$$

$$\hat{\underline{\beta}} \sim N(\underline{\beta}, \overbrace{(X'X)^{-1} \otimes \sigma^2}^{V_{\hat{\underline{\beta}}}})$$

$$\hat{\Delta}_1^2 = \underline{s}_2 \cdot \hat{\beta} \sim N(\Delta_1^2,\ V_2)$$

where $V_2 = \underline{s}_2 V_{\hat{\beta}} \cdot \underline{s}_2'$

$$= \underline{s}_2 (X'X)^{-1} \underline{s}_2' \otimes \sigma^2$$

$$\hat{\Delta}_1^3 \sim N(\Delta_1^3,\ V_3)$$

where $V_3 = \underline{s}_3 (X'X)^{-1} \underline{s}_3' \otimes \sigma^2$

$n_1 + n_2 + n_3 + n_4 = N$

$\vdots$

$$\Rightarrow \quad \frac{\hat{\Delta}_1^2 - \Delta_1^2}{\sqrt{\underline{s}_2 (X'X)^{-1} \underline{s}_2' \otimes \hat{\sigma}^2}} \sim t(df = N-8)$$

where $\hat{\sigma}^2 = \dfrac{\|\underline{e}\|_2}{N-8}$

and
$$\frac{\hat{\Delta}_1^3 - \Delta_1^3}{\sqrt{\underline{s}_3 (X'X)^{-1} \underline{s}_3' \otimes \hat{\sigma}^2}} \sim t(df = N-8)$$

From these pivotal quantities we form individual CI's for Δ_1^2 and Δ_1^3.

Global confidence level : 95%

$\alpha = 5\%$

$\alpha_2 = .04$ $\qquad$ $\alpha_3 = .01$

individual confidence level $\qquad$ 96% $\qquad$ 99%

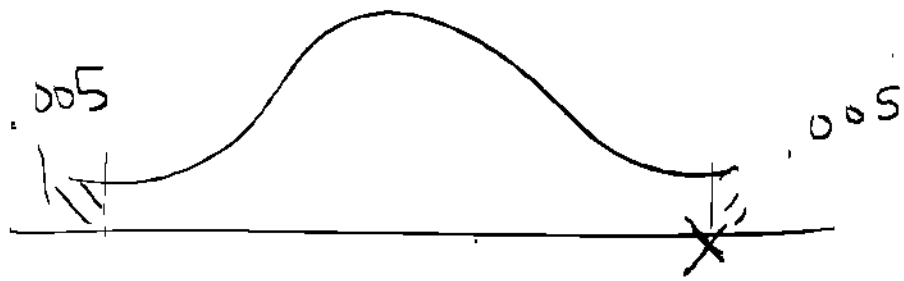
(Bonferonni Correction for multiple testing)

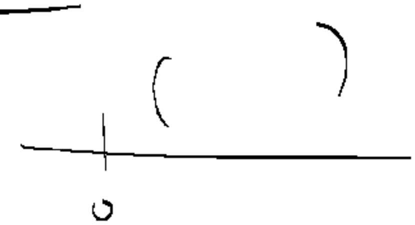
A 96% CI for Δ_1^2 is:

$$\hat{\Delta}_1^2 \pm t(0.98, N-8)\sqrt{\underline{c}_2 (X'X)^{-1}\underline{c}_2' \hat{\sigma}^2}$$

A 99% CI for Δ_1^3 is:

$$\hat{\Delta}_1^3 \pm t(0.995; N-8)\sqrt{c_3 (X'X)^{-1} c_3' \hat{\sigma}^2}$$





$$P\left[\Delta_1^2 \in [L_2, U_2] \text{ AND } \Delta_1^3 \in [L_3, U_3]\right]$$

$$= 0.95$$

ANOTHER WAY TO test for

$H_0: \beta_1^1 = \beta_1^2 = \beta_1^3$ vs

$H_1:$

$H_0: \Delta_1^2 = \beta_1^2 - \beta_1^1 = 0$ AND

$\Delta_1^3 = \beta_1^3 - \beta_1^1 = 0$

Under the full model:

$$\hat{\underline{\beta}} \sim N\left(\underline{\beta},\ V_{\hat{\underline{\beta}}} = (X'X)^{-1} \otimes \sigma^2\right)$$

$$\text{let } \underline{C} = \begin{array}{c} \begin{matrix} \beta_0^1 & \beta_1^1 & \beta_0^2 & \beta_1^2 & \beta_0^3 & \beta_1^3 & \beta_0^4 & \beta_1^4 \end{matrix} \\ \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$\underline{\Delta} = \begin{pmatrix} \Delta_1^2 \\ \Delta_1^3 \end{pmatrix} = \underline{C}\,\underline{\beta}$$

$$\underset{(2\times1)}{\hat{\underline{\Delta}}} = \begin{pmatrix} \hat{\Delta}_1^2 \\ \hat{\Delta}_1^3 \end{pmatrix} = \underset{(2\times8)}{\underline{C}}\ \underset{(8\times1)}{\hat{\underline{\beta}}} = \begin{pmatrix} \hat{\beta}_1^2 - \hat{\beta}_1^1 \\ \hat{\beta}_1^3 - \hat{\beta}_1^1 \end{pmatrix}$$

$$\hat{\underline{\Delta}} \sim N\left(\underline{\Delta},\ V_{\hat{\underline{\Delta}}} = \underline{S}(X'X)^{-1}\underline{S}' \otimes \sigma^2\right)$$

Under H_0:

$$\hat{\underline{\Delta}} \sim N\left(\underline{0},\ \underbrace{V_{\hat{\underline{\Delta}}} = \underline{S}(X'X)^{-1}\underline{S}' \otimes \sigma^2}\right)$$

$$\hat{\underline{\Delta}}\left(\tfrac{1}{\sigma}\right) \sim N\left(\underline{0},\ \underline{S}(X'X)^{-1}\underline{S}'\right)$$

<u>Recall</u>: $\quad U \sim N(0,\ a^2 \cdot \sigma^2)$

$$\Rightarrow \quad \frac{U}{a\sigma} \sim N(0,1)$$

$$\Rightarrow \quad \frac{U^2}{a^2\sigma^2} \sim \chi^2(1)$$

$$(U)'(a^2\sigma^2)^{-1}U \sim \chi^2(1)$$

From above $(\dots)$

$$\underbrace{\hat{\underline{\Delta}}'\left(\underline{S}(X'X)^{-1}\underline{S}' \cdot \sigma^2\right)^{-1}\hat{\underline{\Delta}} \sim \chi^2(2)}$$

Since σ^2 is not Known:

$$\hat{\sigma}^2 = \frac{\|\underline{R}\|^2}{N-8}$$

$$\underbrace{\frac{(N-8)\,\hat{\sigma}^2}{\sigma^2} = \frac{\|\underline{R}\|_2}{\sigma^2} \sim \chi^2(N-8)}$$

$$\Rightarrow \frac{\hat{\underline{\Delta}}' \frac{(\underline{S}(X'X)^{-1}\underline{S}')^{-1}}{\cancel{\sigma^2}} \hat{\underline{\Delta}} \Big/ 2}{\frac{\cancel{(N-8)}\,\hat{\sigma}^2}{\cancel{\sigma^2}} \Big/ \cancel{(N-8)}} \sim F(2, N-8)$$