Lecture 3 Sept 27

Popu 1 $N(\mu_1, \sigma^2)$ $\qquad Y_1^1, \dots, Y_{n_1}^1$ iid $N(\mu_1, \sigma^2)$

Popn 2 $N(\mu_2, \sigma^2)$ $\qquad Y_1^2, \dots Y_{n_2}^2$ iid $N(\mu_2, \sigma^2)$

$$Y_i^1 \sim N(\mu_1, \sigma^2)$$

$$\Leftrightarrow \quad Y_i^1 = \mu_1 + \varepsilon_i^1, \qquad \varepsilon_i^1 \sim N(0, \sigma^2)$$

$$Y_i^2 \sim N(\mu_2, \sigma^2)$$

$$Y_i^2 = \mu_2 + \varepsilon_i^2, \qquad \varepsilon_i^2 \sim N(0, \sigma^2)$$

$$\begin{bmatrix} Y_1^1 \\ \vdots \\ Y_{n_1}^1 \\ \hline Y_1^2 \\ \vdots \\ Y_{n_2}^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_1 \\ \hline \mu_2 \\ \vdots \\ \mu_2 \end{bmatrix}} + \begin{bmatrix} \varepsilon_1^1 \\ \vdots \\ \varepsilon_{n_1}^1 \\ \hline \varepsilon_1^2 \\ \vdots \\ \varepsilon_{n_2}^2 \end{bmatrix}$$

$$\underline{Y} = \underbrace{\begin{bmatrix} \mathbb{1}_{n_1} & \underline{0}_{n_2} \\ \underline{0}_{n_2} & \mathbb{1}_{n_2} \end{bmatrix}}_{X} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \underline{\varepsilon}$$

Linear Model

$$\underline{Y} = X \underline{\beta} + \underline{\varepsilon}, \quad \underline{\varepsilon} \sim N(\underline{0}, \underset{N \times N}{I} \otimes \sigma^2)$$

Estimate $\underline{\beta} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

Inference on $\underline{\beta}$: $\mu_1 - \mu_2$ etc...

Least Squares Estimator of $\underline{\beta}$

$$\underline{\beta} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathbb{R}^2$$

Let $\underline{b}$ be an estimator for $\underline{\beta}$

$$E\,\underline{Y} = E(X\underline{\beta} + \underline{\varepsilon}) = \underbrace{X\underline{\beta}}_{\text{True mean of } \underline{Y}}$$

For any candidate estimator $\underline{b}$, the expected squared error:

$$E\|\underline{Y} - X\underline{b}\|^2$$

Squared Criterion Error:

$$C(\underline{b}) = \|\underline{Y} - X\underline{b}\|^2$$

$$= (\underline{Y} - X\underline{b})'(\underline{Y} - X\underline{b})$$

<u>Notation</u>: $\underline{A} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\|A\|^2 = \sum_{i=1}^{n} a_i^2$

$$= \underline{A}'\underline{A}$$

The least squares estimate of β, denoted by $\hat{\beta}$ satisfies:

$$\hat{\beta} = \text{minimizer of } C(\underline{b})$$

$(y - xb)^2$

$2(-x)(y - xb)$

$$\rightarrow C(\underline{b}) = (\underline{Y} - X\underline{b})'(\underline{Y} - X\underline{b})$$

$$\frac{\partial C(\underline{b})}{\partial \underline{b}} = -2X'(\underline{Y} - X\underline{b}).$$

The LSE $\hat{\beta}$ must satisfy:

$$\left.\frac{\partial C(\underline{b})}{\partial \underline{b}}\right|_{\hat{\beta}} = \underline{0}.$$

$$\Rightarrow -\cancel{2}X'(\underline{Y} - X\hat{\beta}) = \underline{0}$$

$$\Rightarrow \quad X'(\underline{Y} - X\hat{\beta}) = \underline{0}$$

$\star$ normal EQUATIONS

$$\Rightarrow \quad X'\underline{Y} - (X'X)\hat{\beta} = \underline{0}$$

$$\Rightarrow \quad (X'X)\hat{\beta} = X'\underline{Y}$$

$$\Rightarrow \quad \hat{\beta} = \underline{(X'X)^{-1}X'\underline{Y}}$$

Least Squares estimator for β

$$X = \left(\begin{array}{c|c} \underline{1}_{n_1} & \underline{0}_{n_1} \\ \hline \underline{0}_{n_2} & \underline{1}_{n_2} \end{array}\right)_{N\times 2} \qquad (X'X) = \left(\begin{array}{c|c} n_1 & 0 \\ \hline 0 & n_2 \end{array}\right)$$

$$\underline{(X'X)^{-1} = \left(\begin{array}{c|c} 1/n_1 & 0 \\ \hline 0 & 1/n_2 \end{array}\right)}.$$

$$(X'\underline{Y}) = \begin{pmatrix} \underline{1}'_{n_1} & \underline{0}'_{n_2} \\ \hline \underline{0}'_{n_1} & \underline{1}'_{n_2} \end{pmatrix} \begin{bmatrix} Y_1^1 \\ \vdots \\ Y_{n_1}^1 \\ \hline Y_1^2 \\ \vdots \\ Y_{n_2}^2 \end{bmatrix}$$

$$= \underbrace{\begin{pmatrix} \sum_{i=1}^{n_1} Y_i^1 \\ \sum_{i=1}^{n_2} Y_i^2 \end{pmatrix}}.$$

$$\hat{\underline{\beta}} = \begin{pmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{pmatrix}$$

$$\Downarrow$$

let $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

<u>VERIFY</u>: $$C(\underline{b}) = \sum_{i=1}^{n_1} (Y_i^1 - b_1)^2 + \sum_{i=1}^{n_2} (Y_i^2 - b_2)^2$$

$$\left.\begin{matrix} \dfrac{\partial C(\underline{b})}{\partial b_1} \\ \\ \dfrac{\partial C(\underline{b})}{\partial b_2} \end{matrix}\right|_{\hat{\underline{\beta}}} = \underline{0} \Rightarrow \hat{\underline{\beta}} = \begin{pmatrix} \bar{Y}^1 \\ \bar{Y}^2 \end{pmatrix}$$

Two Sample Problem

- t-test
- Linear Model
  Estimator β LSE
- Likelihood

Likelihood

$Y^1_1, \dots, Y^1_{n_1}$ iid $N(\mu_1, \sigma^2)$ and

$Y^2_1, \dots, Y^2_{n_2}$ iid $N(\mu_2, \sigma^2)$

Assume $Y^1_i \perp Y^2_j \quad \forall\, i, j$

The parameters: $\underline{\theta} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \sigma^2 \end{pmatrix} \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$

The likelihood function of $\underline{\theta}$ given the data $\underline{Y} = (Y^1_1 \dots Y^1_{n_1}, \dots Y^2_{n_2})$ is:

$\star$

$\boxed{\text{ASIDE}}$ Likelihood function

$$Y_i = \begin{cases} 1, & \text{wp } \theta \\ 0, & \text{wp } 1-\theta \end{cases}, \quad i = 1 \ldots, 10$$

Given the data:

$$\underbrace{Y_1 = \ldots Y_6 = 1}_{6 \text{ Heads}}, \quad \underbrace{Y_7 = Y_8 = \ldots Y_{10} = 0}_{4 \text{ Tails}}$$

Likelihood function

$$L(\theta) = \prod_{i=1}^{10} \theta^{Y_i} (1-\theta)^{1-Y_i} = \theta^{\Sigma Y_i} (1-\theta)^{10 - \Sigma Y_i}$$

$$, \theta \in \Theta \subseteq [0, 1]$$

$$\Downarrow$$

Here: $\Theta = \{0.2, 0.5, 0.6, 0.9\}$

If $\theta = 0.2$, the probability of observing the event : 6 Heads & 4 Tails is :

$$L(0.2) = (0.2)^6 (0.8)^4$$

If $\theta = 0.5$

$$L(0.5) = (0.5)^6 (0.5)^4$$

If $\theta = 0.6 \quad L(0.6)$

$\theta = 0.9 \quad L(0.9)$

$$\hat{\theta}_{MLE} = \underset{\theta \in \{0.2, 0.5, 0.6, 0.9\}}{\arg\max} L(\theta)$$

(NORMAL RV data)

The likelihood function of $\underline{\theta}$ given $\underline{Y}$

$$L(\underline{\theta} | \underline{Y}) = f(\underline{Y} | \underline{\theta})$$

$$= \prod_{i=1}^{n_1} f(Y_i^1 | \underline{\theta}) \cdot \prod_{i=1}^{n_2} f(Y_i^2 | \underline{\theta})$$

$$= \prod_{i=1}^{n_1} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left\{\frac{-1}{2}\frac{1}{\sigma^2}(Y_i^1 - \mu_1)^2\right\} \cdot$$

$$\prod_{i=1}^{n_2} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left\{\frac{-1}{2}\frac{1}{\sigma^2}(Y_i^2 - \mu_2)^2\right\}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{N = n_1 + n_2} \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2}\left[\sum_{i=1}^{n_1}(Y_i^1 - \mu_1)^2 + \sum_{i=1}^{n_2}(Y_i^2 - \mu_2)^2\right]\right\}$$