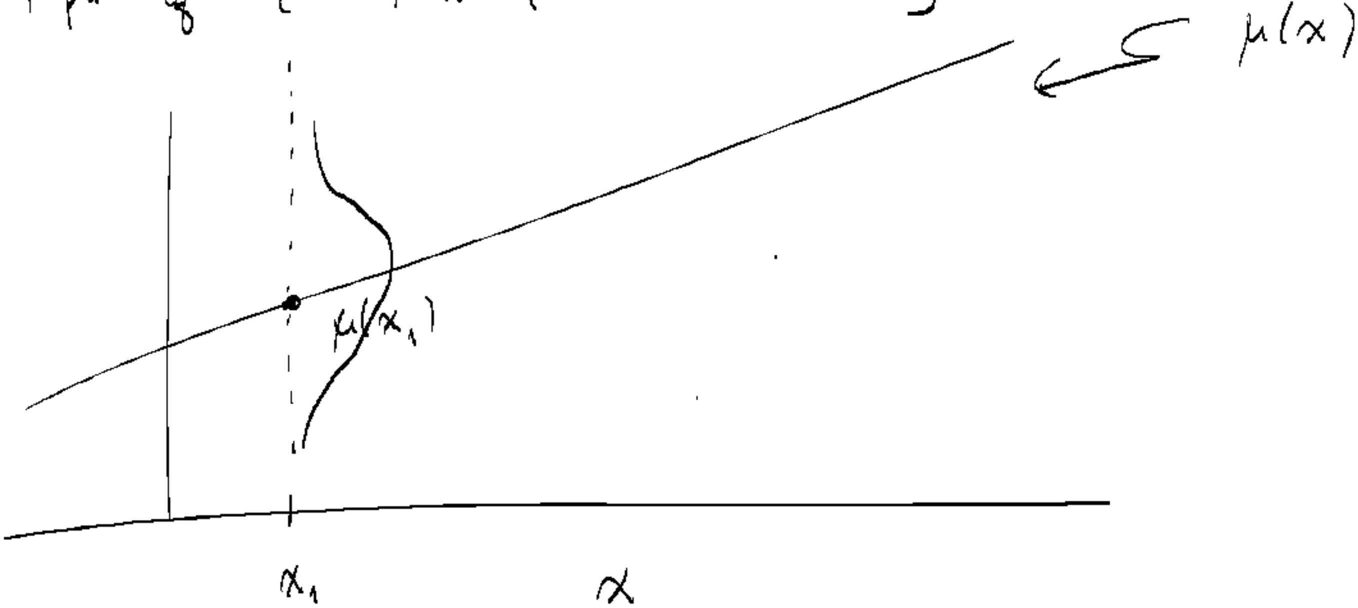
Lecture 08 Oct 11

Pop'n of $\{(x_i, Y_i),\ i = 1 \ldots \infty\}$



$$Y_i \mid x_i \sim N(\mu(x_i), \sigma^2)$$

$$\mu(x_i) = \beta_0 + \beta_1 x_i$$

$$\beta_1 = \mu(x+1) - \mu(x)$$

The LS estimator of β_1 from $\{(x_i, Y_i),\ i = 1 \ldots n\}$

(MLE of β_1) is:

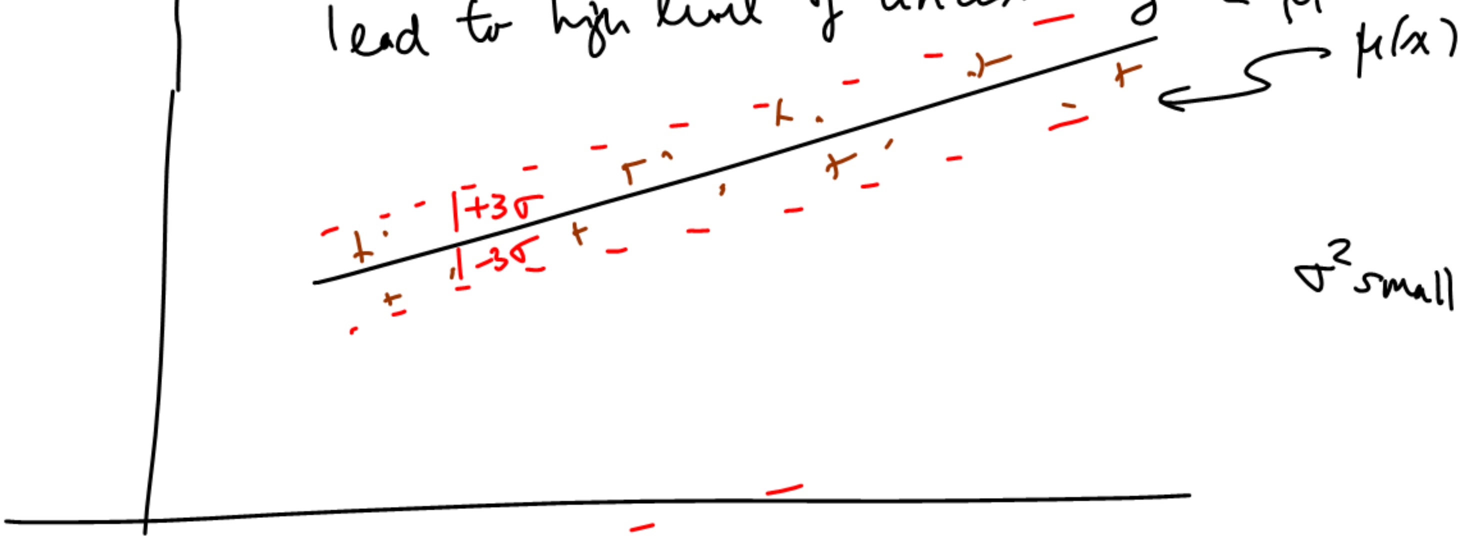
$$\hat{\beta}_1 = \frac{S_{XY}}{S_{XX}}$$

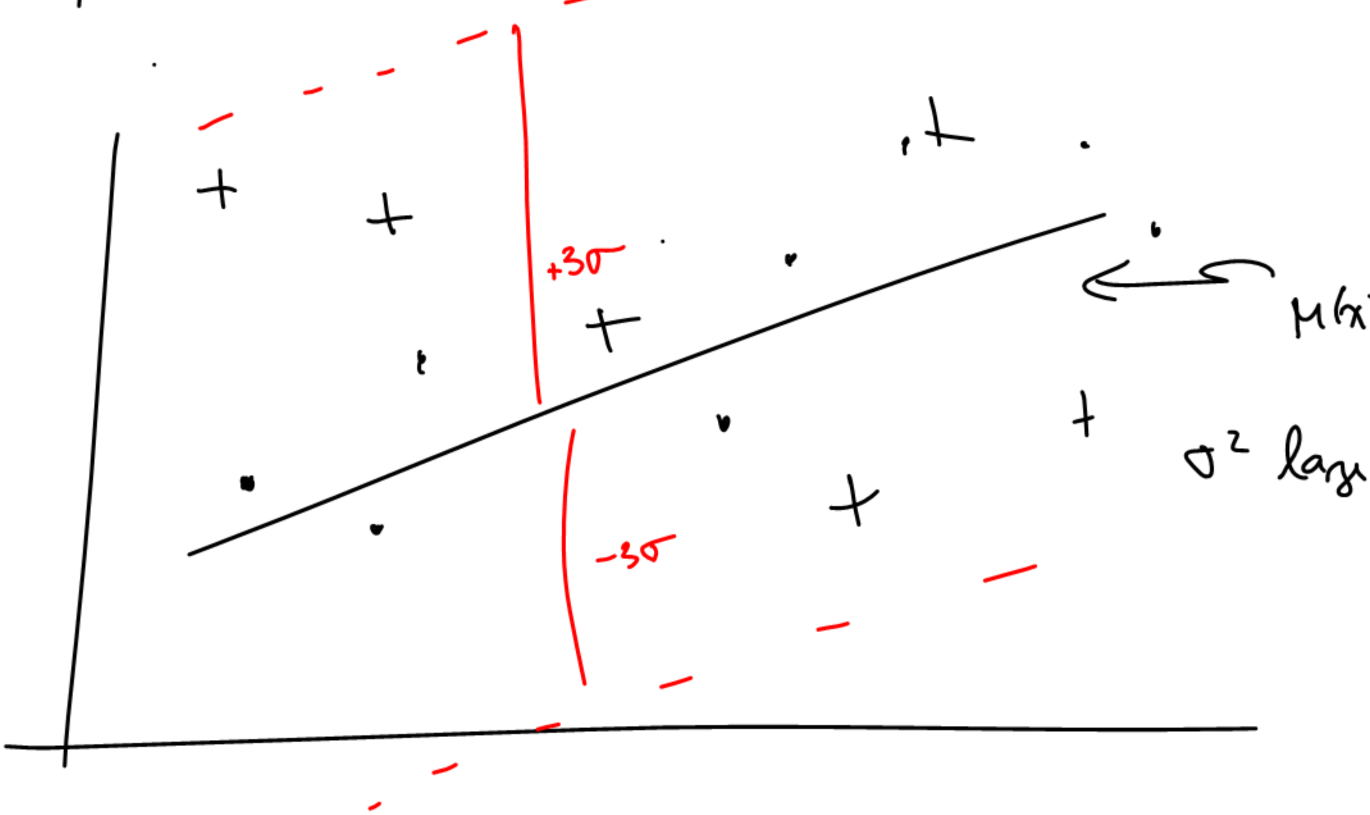
$$= \sum_{i=1}^{n} w_i Y_i, \qquad w_i = \frac{x_i - \bar{x}}{\sum_{j=1}^{n} (x_j - \bar{x})^2}$$

$$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\overset{\Downarrow}{\sigma^2}}{\sum (x_i - \bar{x})^2}\right)$$

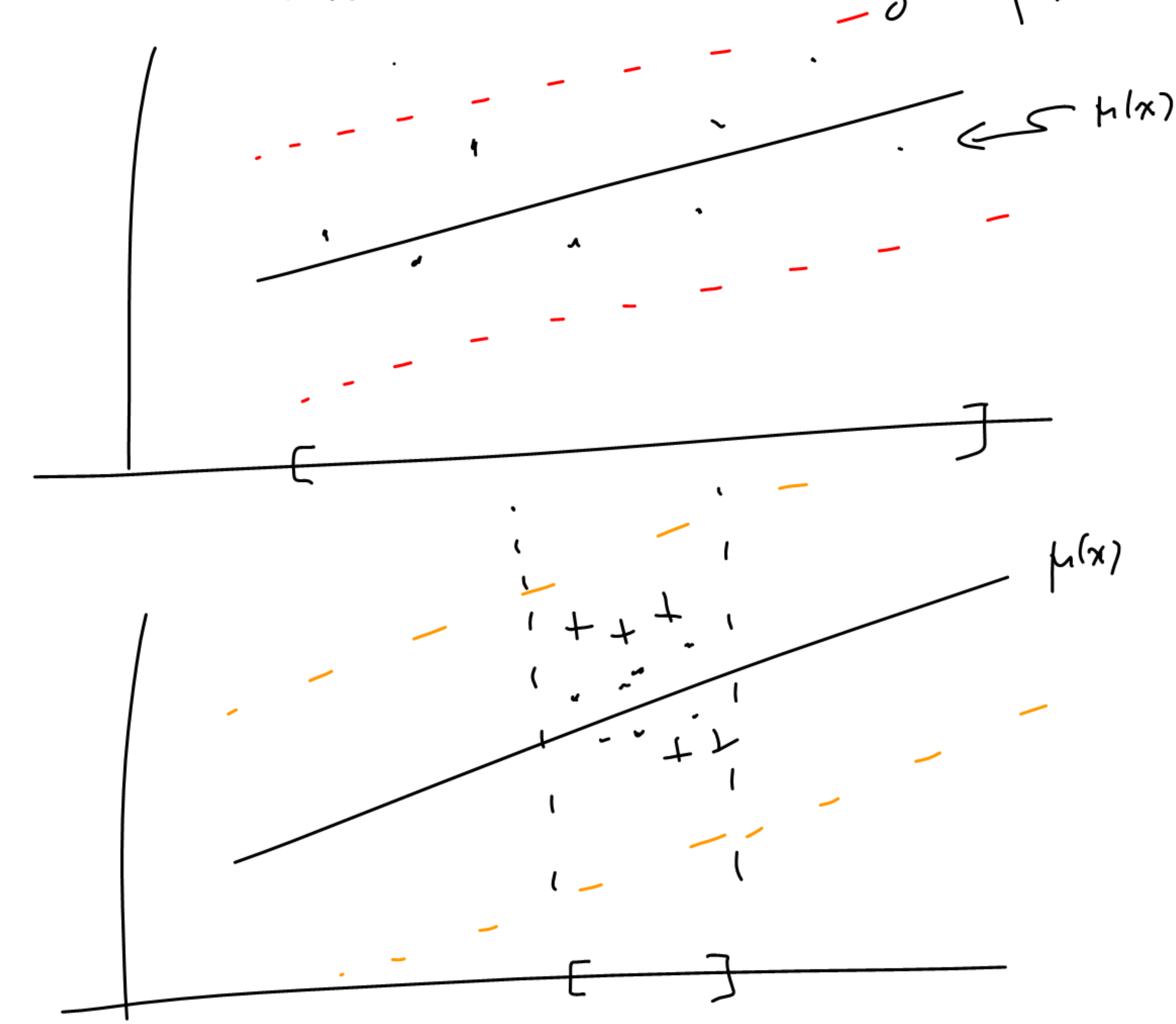
$\Uparrow$

<u>Rem 1</u>. Large σ^2 (i.e. high variability in Y_i's) lead to high level of uncertainty in $\hat{\beta}_1$





Rem 2: When the x values are more spread out then there is less uncertainty in $\hat{\beta}_1$.



Inference on β_1

(a) $H_0: \beta_1 = 0$ vs $H_1: \beta_1 \neq 0$

(b) Confidence interval on β_1.

$$\Downarrow$$

(1) $\hat{\beta}_1 \sim N\left(\beta_1, \sigma^2 / \sum (x_j - \bar{x})^2\right) \Leftrightarrow \dfrac{\hat{\beta}_1 - \beta_1}{\sqrt{\sigma^2 / \sum (x_j - \bar{x})^2}} \sim N(0,1)$

(2) $$\frac{(n-2)\hat{\sigma}^2}{\sigma^2} = \frac{\sum \left(Y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)\right)^2}{\sigma^2}$$

$$\sim \chi^2 (df = n-2)$$

(3) $\hat{\beta}_1 \perp \hat{\sigma}^2$ indep.

$$\Rightarrow \frac{\dfrac{\hat{\beta}_1 - \beta_1}{\sqrt{\cancel{\sigma^2} / \sum (x_j - \bar{x})^2}}}{\sqrt{\dfrac{\cancel{(n-2)}\hat{\sigma}^2}{\cancel{\sigma^2}} \Big/ \cancel{(n-2)}}} \sim t(df = n-2)$$

$$\Rightarrow \quad \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{\sigma}^2 / \Sigma (x_j - \bar{x})^2}} \sim t\,(df = n-2)$$

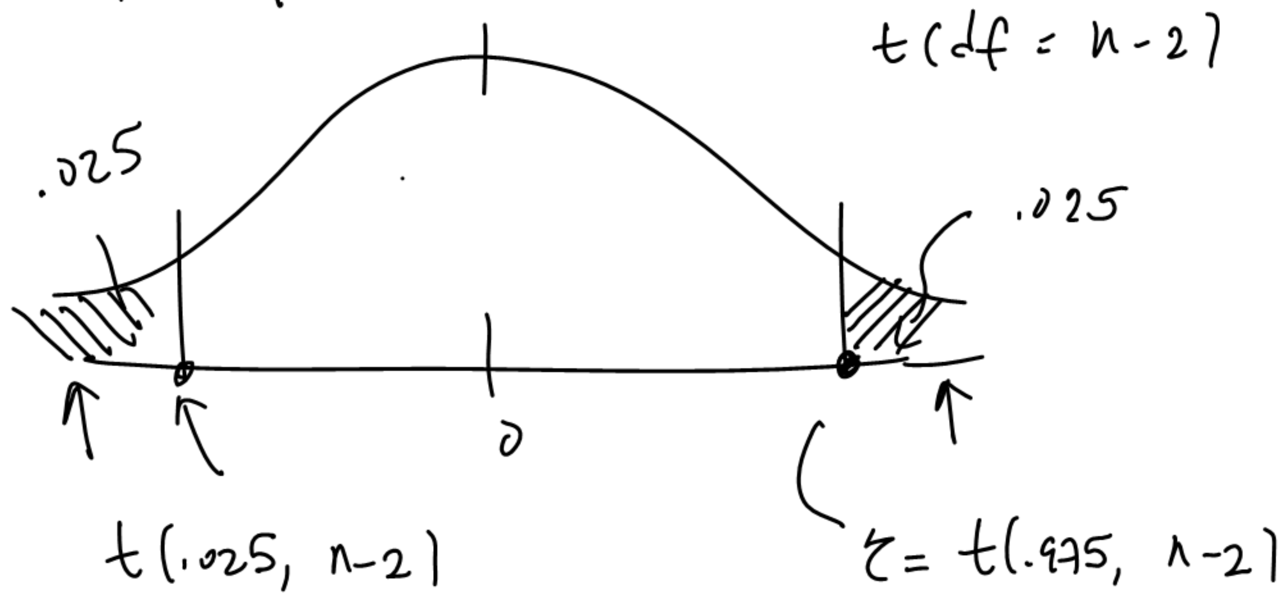
pivotal quantity

<u>Test Statistic</u>:

$$\frac{\hat{\beta}_1}{\sqrt{\hat{\sigma}^2 / \Sigma (x_j - \bar{x})^2}} \overset{H_0}{\sim} t\,(df = n-2)$$

$$P(\text{Type I error}) = \alpha = .05$$

$$P(\text{Reject } H_0 \mid H_0 \text{ true}) = .05$$



Decision Rule: Reject H_0 if

$$|T| > c = t(.975, n-2)$$

CI estimation

β_1 unknown parameter

$[L, U]$ random interval computed from $\{(x_i, Y_i), i = 1, \dots, n\}$

95% CI for β_1:

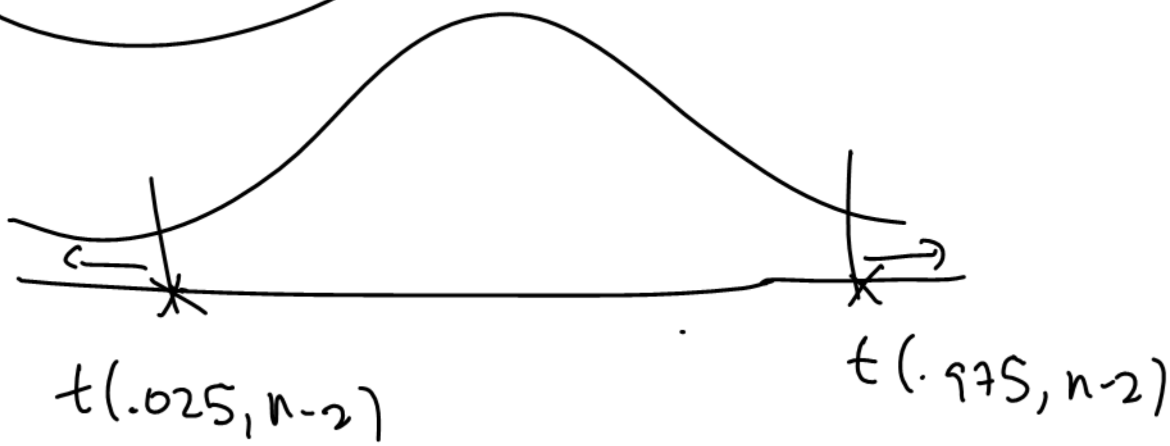
$$P\left(\beta_1 \in [L, U]\right) = 0.95$$

$$\Leftrightarrow \quad P\left(L < \beta_1 < U\right) = 0.95$$

Here: $\dfrac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{\sigma}^2 / \sum (x_j - \bar{x})^2}}$ is a pivotal quantity

because it is a function of β_1 and $\hat{\beta}_1$ and its distribution is known and does not depend on any unknown parameter.

$$\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{\sigma}^2 / \sum (x_i - \bar{x})^2}} \sim t \, (df = n-2)$$



(*)

$$\Rightarrow P\left(t(.025, n-2) < \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{\sigma}^2 / \sum (x_j - \bar{x})^2}} < t(.975, n-2) \right) = 0.95$$

$\vdots$

$$P\left(L < \beta_1 < U \right) = 0.95$$

From (*) above:

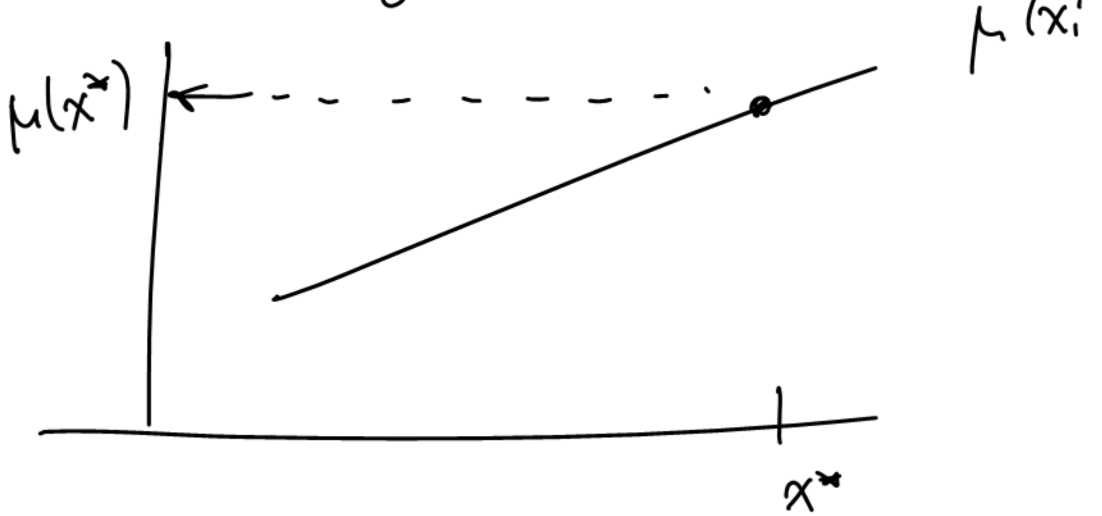
$$P\left(t(.025, n-2)\sqrt{\hat{\sigma}^2 / \Sigma(x_j-\bar{x})^2} < \hat{\beta}_1 - \beta_1 < t(.975, n-2)\sqrt{\hat{\sigma}^2 / \Sigma(x_j-\bar{x})^2} \right) = 0.95$$

$$\Rightarrow P\left(\hat{\beta}_1 - t(.975, n-2)\sqrt{\hat{\sigma}^2 / C} < \beta_1 < \hat{\beta}_1 - t(.025, n-2)\sqrt{\hat{\sigma}^2 / C} \right) = 0.95$$

$\therefore$ A 95% CI estimator for β_1 is:

$$\left[\underbrace{\hat{\beta}_1 - t(.975, n-2)\sqrt{\hat{\sigma}^2 / \Sigma(x_j-\bar{x})^2}}_{L}, \ \underbrace{\hat{\beta}_1 - t(.025, n-2)\sqrt{\hat{\sigma}^2 / \Sigma(x_j-\bar{x})^2}}_{U} \right]$$

Confidence Interval for $\mu(x^*)$



$$\mu(x^*) = \beta_0 + \beta_1 x^*$$

$$= \begin{pmatrix} 1 & x^* \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$= \underline{c}\,\underline{\beta}$$

Estimate for $\underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$

$$\hat{\beta}_1 = \frac{S_{XY}}{S_{XX}} = \sum w_i Y_i$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$$

Recall $\quad \underline{Y} = X\beta + \underline{\varepsilon}, \quad \varepsilon \sim N(\underline{0}, I \otimes \sigma^2)$

$$\underline{Y} \sim N(\underline{X\beta}, I \otimes \sigma^2)$$

$\underline{U} \sim N(\underline{\mu}, \Sigma)$

$\Rightarrow A\underline{U} \sim N(A\underline{\mu}, A\Sigma A')$

$$\hat{\beta} = \boxed{\underbrace{(X'X)^{-1}X'}\underline{Y}}$$

$$\sim N\Big(E\hat{\beta} = (X'X)^{-1}X' E\underline{Y}$$

$$\mathrm{cov}(\hat{\beta}) = (X'X)^{-1}X'(I \otimes \sigma^2) \underbrace{((X'X)^{-1}X')'}\Big)$$

$$E\hat{\beta} = (X'X)^{-1}X'X\beta = \beta$$

$$\mathrm{Cov}(\hat{\beta}) = (X'X)^{-1}X'X(X'X)^{-1} \otimes \sigma^2$$

$$= (X'X)^{-1} \otimes \sigma^2$$

$$\hat{\beta} \sim N(\beta, (X'X)^{-1} \otimes \sigma^2)$$

CI for $\mu(x^*) = \beta_0 + \beta_1 x^*$
$= \underset{\sim}{c}\, \beta$

An estimator for $\mu(x^*) = \underset{\sim}{c}\, \beta$ is $\underset{\sim}{c}\, \hat{\beta}$

<u>Goal:</u>

$$P\left(\mu(x^*) \in [L, U] \right) = 0.95$$

$$\Leftrightarrow \quad P\left(L \le \mu(x^*) \le U \right) = 0.95$$

$$P\left(L \le c\beta \le U \right) = 0.95$$

$$\hat{\beta} \sim N\left(\beta, \underbrace{(X'X)^{-1} \otimes \sigma^2}_{} \equiv V_{\hat{\beta}} \right)$$

$$\underset{\sim}{c}\hat{\beta} \sim N\left(\underset{\sim}{c}\beta, \underset{\sim}{c}\, \underbrace{V_{\hat{\beta}}}_{\sigma^2}\, \underset{\sim}{c}' \right)$$

$$\Rightarrow \quad \frac{\underset{\sim}{c}\hat{\beta} - \underset{\sim}{c}\beta}{\sqrt{\underset{\sim}{c}\, V_{\hat{\beta}}\, \underset{\sim}{c}'}} \sim N(0, 1)$$

$$\Rightarrow \quad \frac{\underline{c}\,\hat{\beta} - \underline{c}\,\beta}{\sqrt{\underline{c}\,(X'X)^{-1}\underline{c}' \otimes \sigma^2}} \sim N(0,1)$$

σ^2 not known:

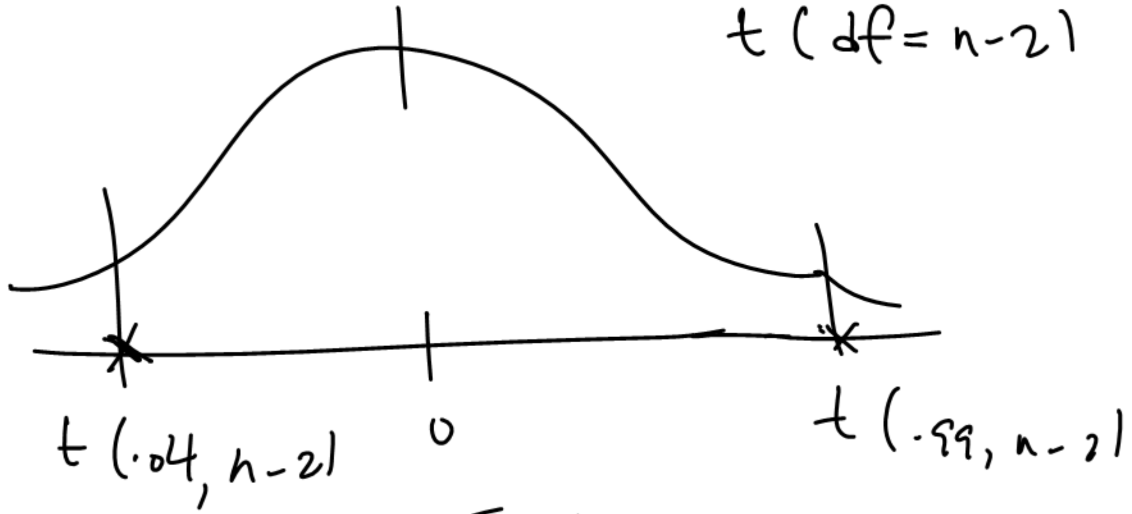
$$\frac{(n-2)\,\hat{\sigma}^2}{\sigma^2} \sim \chi^2(df = n-2)$$

$$\Rightarrow \quad \frac{\dfrac{\underline{c}\,\hat{\beta} - \underline{c}\,\beta}{\sqrt{\underline{c}\,(X'X)^{-1}\underline{c}'\,\cancel{\sigma^2}}}}{\sqrt{\dfrac{\cancel{(n-2)}\,\hat{\sigma}^2}{\cancel{\sigma^2}} \Big/ \cancel{(n-2)}}} \sim t(df = n-2)$$

$$\Rightarrow \quad \frac{\underline{c}\,\hat{\beta} - \underline{c}\,\beta}{\sqrt{\underline{c}\,(X'X)^{-1}\underline{c}'\,\hat{\sigma}^2}} \sim t(df = n-2)$$

$= \mu(x^*)$

pivotal quantity



$$\Rightarrow P\left(t(.04, n-2) < \frac{\underline{c}\hat{\beta} - \mu(x^*)}{\sqrt{\underline{c}(X'X)^{-1}\underline{c}'\hat{\sigma}^2}} < t(.99, n-2) \right) = 0.95$$

$$\vdots$$

$$\Rightarrow P\left(\underline{c}\hat{\beta} - t(.99, n-2)\sqrt{\underline{c}(X'X)^{-1}\underline{c}'\hat{\sigma}^2} < \mu(x^*) < c\hat{\beta} - t(.04, n-2)\sqrt{\underline{c}(X'X)^{-1}\underline{c}'\hat{\sigma}^2} \right)$$

$$= 0.95$$

$\therefore$ A 95% CI estimator for $\mu(\underline{x}^*) = \beta_0 + \beta_1 x^*$:

$$\left[\underbrace{\underline{c}\hat{\beta}}_{\hat{\beta}_0 + \hat{\beta}_1 x^*} - t(.99)\ldots, \quad \hat{\underline{c}}\hat{\beta} - \underline{t(.04)\ldots} \right]$$