Lecture 13 Oct 20 Summary

(1) 2 indep sample

$$\underbrace{\begin{pmatrix} \underline{Y}^{(1)} \\ \hline \underline{Y}^{(2)} \end{pmatrix}}_{\underline{Y}} = \underbrace{\left(\begin{array}{c|c} \mathbb{1} & \underline{0} \\ \hline \underline{0} & \mathbb{1} \end{array}\right)}_{X} \underbrace{\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}}_{\underline{\beta}} + \underbrace{\begin{pmatrix} \underline{\varepsilon}^{(1)} \\ \hline \underline{\varepsilon}^{(2)} \end{pmatrix}}_{\underline{\epsilon}}$$

where

$$\underline{\epsilon} \sim N(\underline{0}, I \otimes \sigma^2)$$

Estimands:

$$\mu_1 = (1 \quad 0)\, \underline{\beta}$$

$$\mu_2 =$$

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① 2 indep sample

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where

$$\underline{\epsilon} \sim N(\underline{0}, I \otimes \sigma^2)$$

Estimands:

$$\mu_1 = (1 \quad 0)\, \underline{\beta}$$

$$\mu_2 = (0 \quad 1)\, \underline{\beta}$$

$$\delta = \mu_1 - \mu_2 = (1 \quad -1)\, \underline{\beta}$$

$$\theta = \underline{c}\, \underline{\beta}$$

Estimator:

$$\left.\begin{aligned} \hat{\theta} &= \underline{c} \cdot \hat{\underline{\beta}} \\ &= \underline{c} \left((X'X)^{-1} X' \underline{Y} \right) \end{aligned}\right] \quad \hat{\underline{\beta}} \sim N(\underline{\beta}, (X'X)^{-1}\sigma^2)$$

$$\hat{\theta} \sim N\Big(E(\hat{\theta}) = \underset{\sim}{c}\underset{\sim}{\beta} = \theta,$$

$$Var(\hat{\theta}) = \underset{\sim}{c}\, Var(\hat{\underset{\sim}{\beta}})\, \underset{\sim}{c}'$$

$$= \underset{\sim}{c}(X'X)^{-1}\underset{\sim}{c}'\sigma^2\Big)$$

95% CI for θ

$$\hat{\theta} \sim N\left(\theta,\ \underset{\sim}{c}(X'X)^{-1}\underset{\sim}{c}'\sigma^2\right)$$

$$\Rightarrow \quad \frac{\hat{\theta} - \theta}{\sqrt{\underset{\sim}{c}(X'X)^{-1}\underset{\sim}{c}'\sigma^2}} \sim N(0,1)$$

If σ^2 is known: find (τ_1, τ_2) st:

$$P\left[\tau_1 < \frac{\hat{\theta}-\theta}{\sqrt{\underset{\sim}{c}(X'X)^{-1}\underset{\sim}{c}'\sigma^2}} < \tau_2\right] = 0.95$$

Choose $\tau_1 = -1.96 \quad \tau_2 = +1.96$

$$\Rightarrow P\left[-1.96 < \frac{\hat{\theta}-\theta}{\sqrt{\underset{\sim}{c}(X'X)^{-1}\underset{\sim}{c}'\sigma^2}} < +1.96\right] = 0.95$$

$\vdots$

$$\Rightarrow \quad P\Big[\underbrace{\hat{\theta} - 1.96\sqrt{\underline{c}(X'X)^{-1}\underline{c}'\sigma^2}}_{L} < \theta < \underbrace{\hat{\theta} + 1.96\sqrt{\underline{c}(X'X)^{-1}\underline{c}'\sigma^2}}_{U}\Big] = 0.95$$

$$\Rightarrow \quad P\Big[\ \theta \in [L, U]\ \Big] = 0.95$$

A 95% CI $\boxed{\text{estimator}}$ for θ is: $\qquad$ L, U random

$$\hat{\theta}(\underline{Y}) \pm 1.96\sqrt{\underline{c}(X'X)^{-1}\underline{c}'\sigma^2}$$

A 95% CI estimate based on the observed data $(\underline{y})$ for θ is:

$$\underbrace{\hat{\theta}(\underline{y})}_{\underline{c}(X'X)^{-1}X'\underline{y}} \pm 1.96\sqrt{\underline{c}(X'X)^{-1}\underline{c}'\sigma^2}$$

If σ^2 is not known:

$$\underline{\hat{Y}} = X\underline{\hat{\beta}} = X\begin{pmatrix}\bar{Y}_1\\ \bar{Y}_2\end{pmatrix} = \begin{pmatrix}\mathbb{1}\cdot\bar{Y}_1\\ \hline \mathbb{1}\cdot\bar{Y}_2\end{pmatrix}$$

$$\underline{R} = \underline{Y} - \underline{\hat{Y}}$$

$$= \begin{pmatrix}\underline{Y}^{(1)}\\ \hline \underline{Y}^{(2)}\end{pmatrix} - \begin{pmatrix}\mathbb{1}\,\bar{Y}_1\\ \hline \mathbb{1}\,\bar{Y}_2\end{pmatrix}$$

$$\hat{\sigma}^2 = \frac{\|\underline{R}\|_2}{(n_1+n_2)-2} = \frac{\left(\sum_{i=1}^{n_1}(Y_i^1-\bar{Y}_1)^2 + \sum_{j=1}^{n_2}(Y_j^2-\bar{Y}_2)^2\right)}{n_1+n_2-2}$$

$$\underline{c}\,\underline{\hat{\beta}} \sim N\left(\theta,\ \underline{c}(X'X)^{-1}\underline{c}'\sigma^2\right)$$

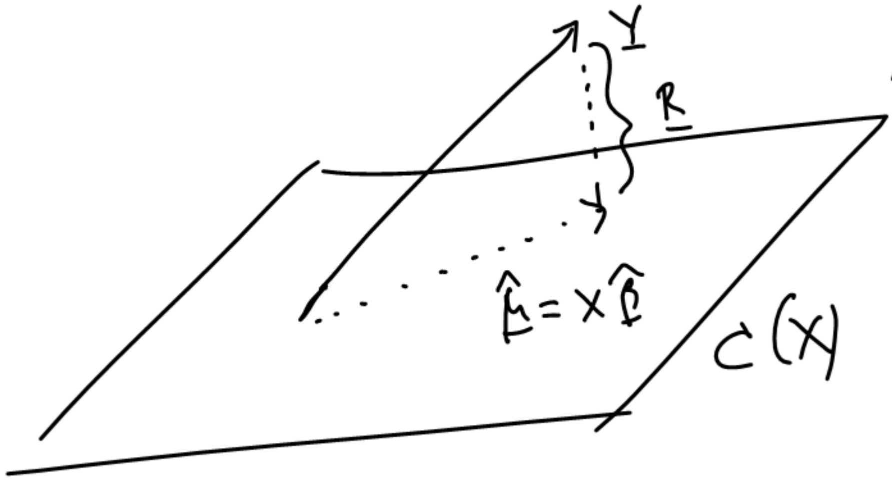
Recall

(1) $$\frac{\underline{c}\,\underline{\hat{\beta}} - \theta}{\sqrt{\underline{c}(X'X)^{-1}\underline{c}'\sigma^2}} \sim N(0,1)$$

(2) $$\frac{((n_1+n_2)-2)\,\hat{\sigma}^2}{\sigma^2} = \frac{\|\underline{R}\|_2}{\sigma^2} \sim \chi^2$$

$$\chi^2(df = n_1 + n_2 - 2)$$

(3) $\underline{\hat{\beta}}$ & $\underline{R}$ independent



$$\underline{\mu} = \underbrace{E(\underline{Y})} = X\underline{\beta}$$

$$X = [\underline{x}_1 | \dots | \underline{x}_p]$$

$$\underline{\beta} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$$

$$X\underline{\beta} = \sum_{k=1}^{p} \underline{x}_k \beta_k$$

$$\in C(X)$$

$$\operatorname{cov}(\underbrace{X\hat{\underline{\beta}}}_{\hat{\underline{\mu}}}, \underline{R}) = 0$$

$$\hat{\underline{\mu}} = X\hat{\underline{\beta}} = \underbrace{X(X'X)^{-1}X'}_{M}\,\underline{Y}$$

$$= M\,\underline{Y}.$$

Here, M is symmetric ($M = M'$)

and idempotent (i.e., $MM = M$)

$\therefore$ M is the projection matrix onto $C(X)$.

$$\underline{R} = \underline{Y} - \hat{\underline{Y}} = \underline{Y} - \underbrace{X\hat{\underline{\beta}}}$$

$$= \underline{Y} - M\underline{Y}$$

$$= (I - M)\underline{Y}$$

$$\operatorname{cov}(\underbrace{X\hat{\underline{\beta}}}_{\hat{\underline{\mu}}}, \underline{R}) = \operatorname{cov}(M\underline{Y}, (I-M)\underline{Y})$$

$$= M \underbrace{\operatorname{cov}(\underline{Y}, \underline{Y})}_{I \otimes \sigma^2} (I-M)'$$

$$= M(I-M)' \otimes \sigma^2$$

$$= M(I-M) \otimes \sigma^2$$

$$= (M - M^2) \otimes \sigma^2 = 0$$

$$\Rightarrow \quad \underbrace{\operatorname{cov}(X\hat{\beta}, \underline{R}) = 0}$$

$$\Rightarrow \quad X\hat{\beta} \text{ and } \underline{R} \text{ independent}$$

$$\Rightarrow \quad X \underbrace{\operatorname{cov}(\hat{\beta}, \underline{R})} = 0$$

$$\Rightarrow \quad \operatorname{cov}(\hat{\beta}, \underline{R}) = 0$$

$$\Rightarrow \quad \hat{\beta} \perp \underline{R}$$

Finally :

$$\frac{\dfrac{\overset{\hat{\theta}}{\underline{c}\hat{\beta} - \theta}}{\sqrt{\underline{c}(X'X)^{-1}\underline{c}'\cancel{\sigma^2}}}}{\sqrt{\dfrac{\cancel{(n_1+n_2-2)}\hat{\sigma}^2}{\cancel{\sigma^2}} \Big/ \cancel{(n_1+n_2-2)}}} \sim t(df = n_1+n_2-2)$$

$$\Rightarrow \quad \frac{\hat{\theta} - \theta}{\sqrt{\hat{\sigma}^2 \underline{s} (X'X)^{-1} \underline{s}'}} \sim t\,(df = n_1 + n_2 - 2)$$

Conduct tests of hypothesis : $H_0 : \theta = a$

Simple Regression Setting

(Random Data) : $\{ (x_i, Y_i),\ i = 1, \dots, n \}$

<u>Model</u> : $Y_i \mid x_i \sim N(\mu(x_i), \sigma^2)$

$$\mu(x_i) = \beta_0 + \beta_1 x_i$$

<u>Equivalently</u>
$$Y_i = \mu(x_i) + \varepsilon_i,$$
$$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$$
$$\mu(x_i) = \beta_0 + \beta_1 x_i$$

$$\underline{Y} = X \underline{\beta} + \underline{\varepsilon}$$

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

Estimand : $\underline{\beta}$

$\underline{c}\,\underline{\beta}$ e.g. $(1 \quad x^*)\,\underline{\beta}$

$= \beta_0 + \beta_1 x^*$

$= \mu(x^*)$

$= E(Y \mid x^*)$

LSE

Estimator for $\underline{\beta}$

$$\hat{\underline{\beta}} = (X'X)^{-1}X'\underline{Y}$$

Estimator for $\mu(x^*) = \underline{c}\,\underline{\beta} = (1 \quad x^*)\,\underline{\beta}$

is: $\widehat{\mu}(x^*) = (1 \quad x^*)\,\hat{\underline{\beta}}$

95% CI's for $\underline{c}\,\underline{\beta}$

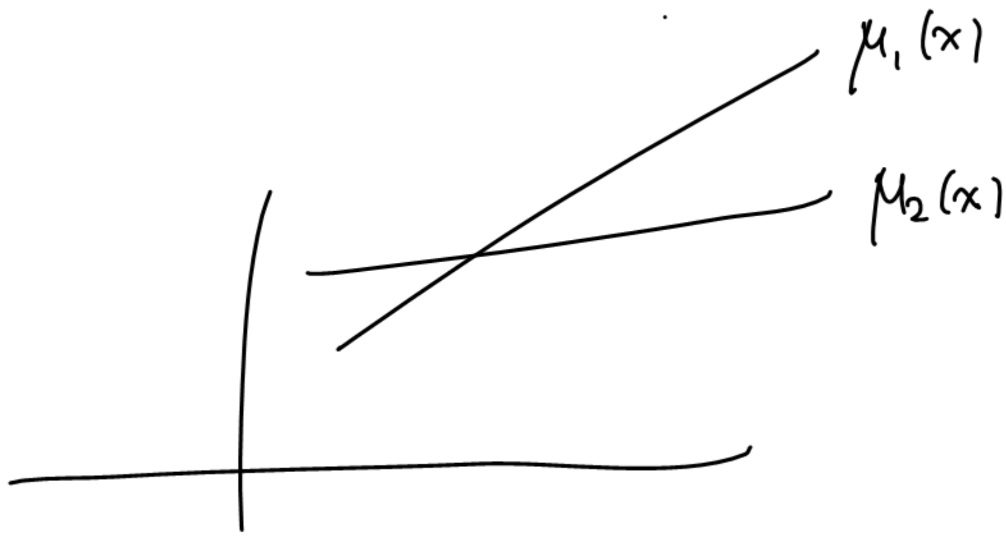
Conduct tests of hypothesis for $\underline{c}\,\underline{\beta}$

95% Prediction Interval for Y at $x = x^*$

Regression Lines for Several Groups (ANCOVA)

Group 1 :

Group 2 :



$$\mu_1(x_i) = \beta_0^1 + \beta_1^1 x_i$$

$$\mu_2(x_i) = \beta_0^2 + \beta_1^2 x_i$$

Let $G_{1i} = \begin{cases} 1, & \text{if } i\text{th} \in \text{Group } 1 \\ 0, & \text{o/w} \end{cases}$

$G_{2i} = \begin{cases} 1, & \text{if } i\text{th} \in \text{Group } 2 \\ 0, & \text{o/w} \end{cases}$

I $\quad \mu(x_i) = (\beta_0^1 + \beta_1^1 x_i)\, G_{1i} + (\beta_0^2 + \beta_1^2 x_i)\, G_{2i}$

if ith $\in$ Group 1 $\Rightarrow \quad G_{1i} = 1, \; G_{2i} = 0$

$\Rightarrow \quad \mu(x_i) = \beta_0^1 + \beta_1^1 x_i$

(II) Another parameterization is:

$$\mu(x_i) = \underline{(\delta_0 + \delta_1 x_i) G_{1i}} + \underline{\beta_0^2 + \beta_1^2 x_i}$$

if $i^{th} \in$ Group 1 $\Rightarrow G_{1i} = 1$

$$\mu(x_i) = (\beta_0^1 + \beta_1^1 x_i) + (\beta_0^2 + \beta_1^2 x_i)$$

$$= \underbrace{(\beta_0^2 + \delta_0)}_{\substack{\text{intercept} \\ \text{Gp 1} \\ \beta_0^1}} + \underbrace{(\beta_1^2 + \delta_1)}_{\substack{\text{slope} \\ \text{Gp 1} \\ \beta_1^1}} x_i$$

if $i^{th} \in$ Group 2

$$\mu(x_i) = \beta_0^2 + \beta_1^2 x_i$$

Build the linear model $\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$ for parameterization (II)

$$
\begin{array}{ll}
\text{gp1} & \\
\text{gp2} &
\end{array}
\begin{bmatrix} Y_1 \\ \vdots \\ Y_{n_1} \\ \hline Y_{n_1+1} \\ \vdots \\ Y_{n_1+n_2} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} \beta_0^2 & \beta_1^2 & \delta_0 & \delta_1 \end{array} \\
\begin{bmatrix}
1 & x_1 & 1 & x_1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & x_{n_1} & 1 & x_{n_1} \\ \hline
1 & x_{n_1+1} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
1 & x_{n_1+n_2} & 0 & 0
\end{bmatrix}_{(n_1+n_2)\times 4}
\end{array}
\begin{pmatrix} \beta_0^2 \\ \beta_1^2 \\ \delta_0 \\ \delta_1 \end{pmatrix}
+
\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \hline \vdots \\ \varepsilon_{n_1+n_2} \end{pmatrix}
$$

$$\underset{\sim}{Y}$$

$$\underset{\sim}{\hat{\beta}} = (X'X)^{-1} X' \underset{\sim}{Y}$$

Are the two regression lines parallel?

$$H_0: \delta_1 = 0 \quad vs \quad H_1: \delta_1 \neq 0$$

t-test $\qquad \hat{\delta}_1 = \underset{\sim}{c}\hat{\beta} = (0\ 0\ 0\ 1)\hat{\beta}$

ANOVA Test

Full Model

Reduced Model