Lecture 10

Matrix population models

WILD3810 (Spring 2020)

Readings

> Mills 98-103

Assumptions of the B-D models

Over the coming weeks, we will learn about why and how to relax assumption 3:

1) Population closed to immigration and emigration

2) Model pertains to only the limiting sex, usually females

3) Birth and death rates are independent of an individual's age or biological stage

4) Birth and death rates are constant

Age-structured demography

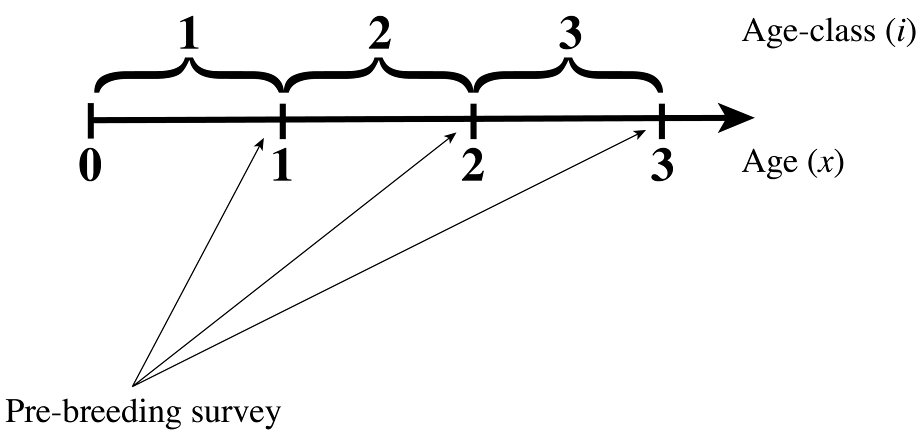
Consider the following life table:

x	m_x	P_x
0	0	P_0
1	m_1	P_1
2	m_2	P_2
3	m_3	0

Age-structred demography

List population abundance in discrete 1-year age classes (n_i)

- e.g., n_1, n_2, n_3
- n_i is the number of individuals about to experience their i^{th} birthday



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Age-structred demography

x	m_x	P_x
0	0	P_0
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2	m_2	P_2
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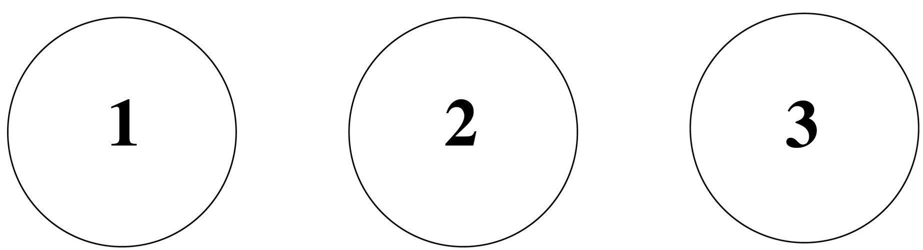

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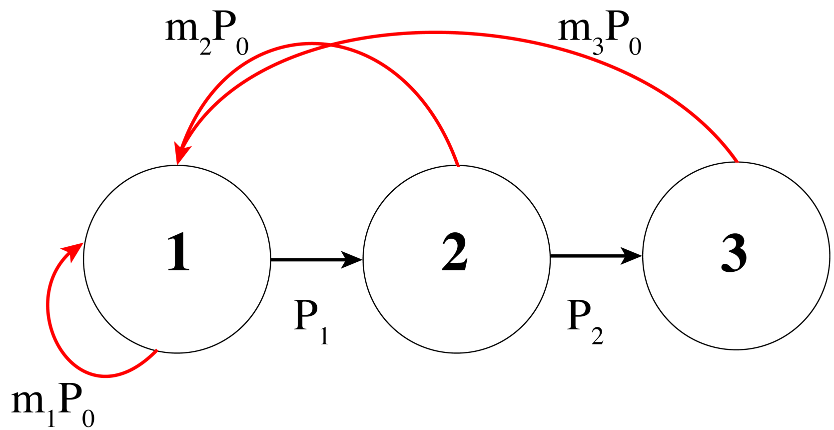
Age-structred demography

How many individuals will be in the population next year?



$$n_{1,t+1} = m_1 P_0 n_{1,t} + m_2 P_0 n_{2,t} + m_3 P_0 n_{3,t}$$

Age-structred demography

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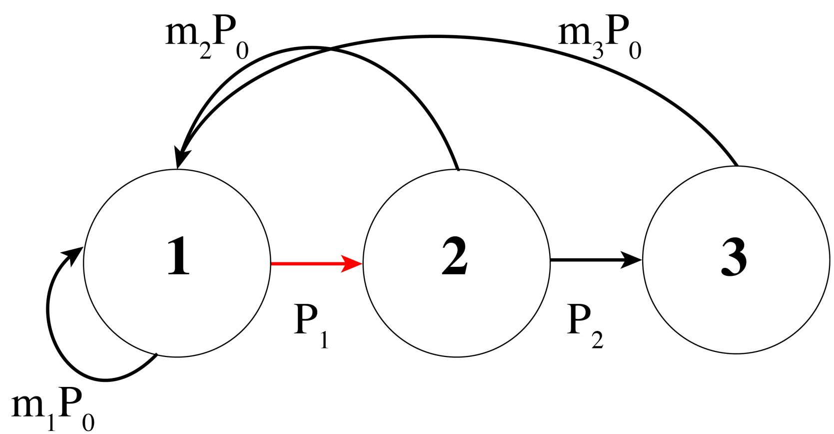

$$n_{1,t+1} = m_1P_0n_{1,t} + m_2P_0n_{2,t} + m_3P_0n_{3,t}$$

$$n_{2,t+1} = P_1n_{1,t}$$

Age-structred demography

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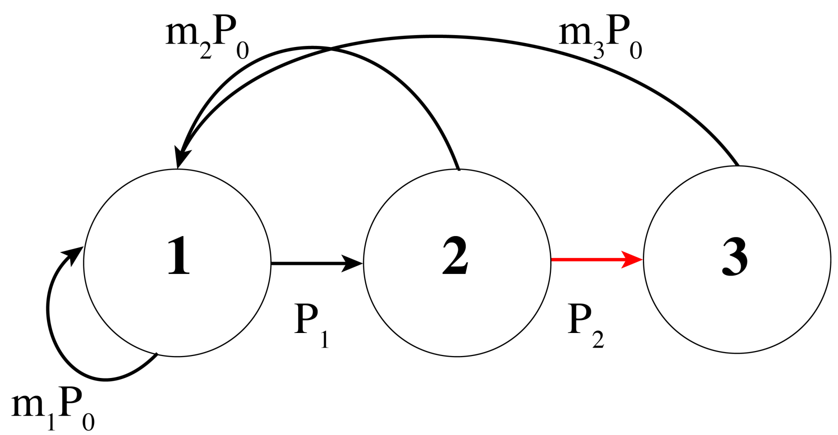

$$n_{1,t+1} = m_1 P_0 n_{1,t} + m_2 P_0 n_{2,t} + m_3 P_0 n_{3,t}$$

$$n_{2,t+1} = P_1 n_{1,t}$$

$$n_{3,t+1} = P_2 n_{2,t}$$

Leslie matrix model

Rather than modeling the dynamics using the previous equations, we can use **matrix projection models**

- defined by square matrix that summarizes the demography of age-specific life cycles
- one column for each age class
- developed by Sir Patrick H. Leslie for application to population biology
- a matrix with age-specific birth and survival rates is called a **Leslie matrix**

Review of matrix algebra

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Review of matrix algebra

Matrix addition

- Matrices must be of the same dimension
- Add corresponding elements of the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 4 & 6 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & 5 \\ 6 & 13 \end{bmatrix}$$

Review of matrix algebra

Matrix subtraction

- Matrices must be of the same dimension
- Subtract corresponding elements of the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 4 & 6 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1 & -5 \\ 2 & -1 \end{bmatrix}$$

Review of matrix algebra

Matrix multiplication

- Dimensions are specified as **rows** by **columns**
- Matrix multiplication does not require matrices to have the same dimensions
- But they must have the same **inner** dimension
    - for example, a 3x3 matrix can be multiplied by a 3x1 matrix

$$3 \times 3 \qquad\qquad\qquad 3 \times 1$$

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Review of matrix algebra

Matrix multiplication

- Dimensions are specified as **rows** by **columns**
- Matrix multiplication does not require matrices to have the same dimensions
- But they must have the same **inner** dimension
    - but a 3x3 matrix **cannnot** be multiplied by a 1x3 matrix

$$3 \times 3 \qquad\qquad\qquad 1 \times 3$$

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \;\; \mathbf{B} = \begin{bmatrix} x & y & z \end{bmatrix}$$

Review of matrix algebra

How does matrix multiplication work?

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a \times x + b \times y + c \times z \\ d \times x + e \times y + f \times z \\ g \times x + h \times y + i \times z \end{bmatrix}$$

Review of matrix algebra

How does matrix multiplication work?

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 4 & 3 \\ 2 & 6 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 \times 3 + 0 \times 2 + 5 \times 1 \\ 0 \times 3 + 4 \times 2 + 3 \times 1 \\ 2 \times 3 + 6 \times 2 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 18 \end{bmatrix}$$

Review of matrix algebra

Transpose of a matrix

$$\mathbf{B} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{B}^T = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Review of matrix algebra

Transpose of a matrix

- is the following allowed?

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 4 & 3 \\ 2 & 6 & 0 \end{bmatrix}, \;\; \mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$$A \times B$$

Review of matrix algebra

Transpose of a matrix

- is the following allowed?

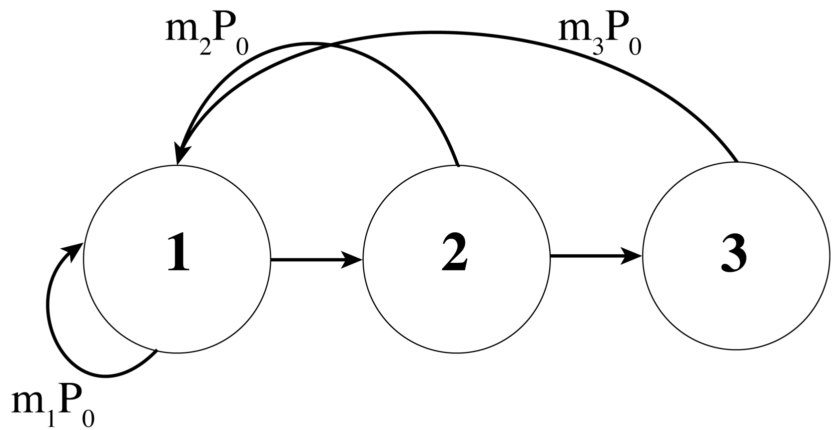
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 4 & 3 \\ 2 & 6 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$$A \times B^T$$

Age-structured matrix models

Age-structured matrix models

How many individuals will be in the population next year?



$$n_{1,t+1} = m_1 P_0 n_{1,t} + m_2 P_0 n_{2,t} + m_3 P_0 n_{3,t}$$

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Age-structured matrix models

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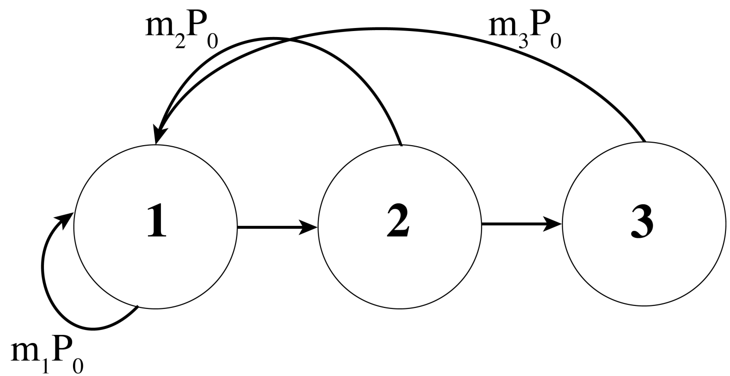
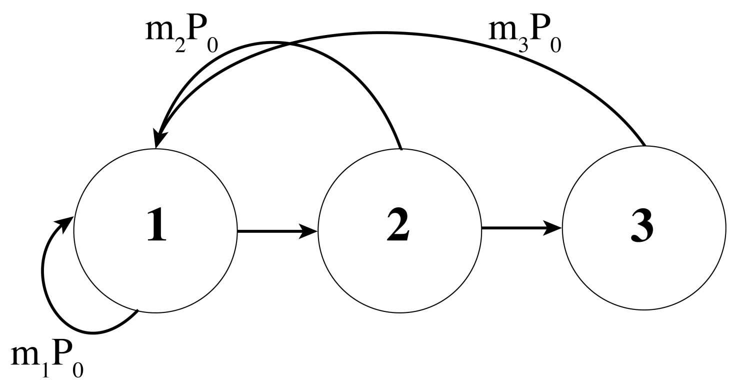


$$\mathbf{A} = \begin{array}{cc} & \text{from 1} \\ \text{to 1} & \left[\begin{array}{ccc} m_1P_0 & m_2P_0 & m_3P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{array}\right] \end{array}$$

Age-structured matrix models

How many individuals will be in the population next year?



$$\mathbf{A} = \begin{array}{cc} & \begin{array}{ccc} & \text{from 2} & \end{array} \\ \begin{array}{c} \text{to 1} \\ \\ \end{array} & \begin{bmatrix} m_1P_0 & m_2P_0 & m_3P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \end{array}$$

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Age-structured matrix models

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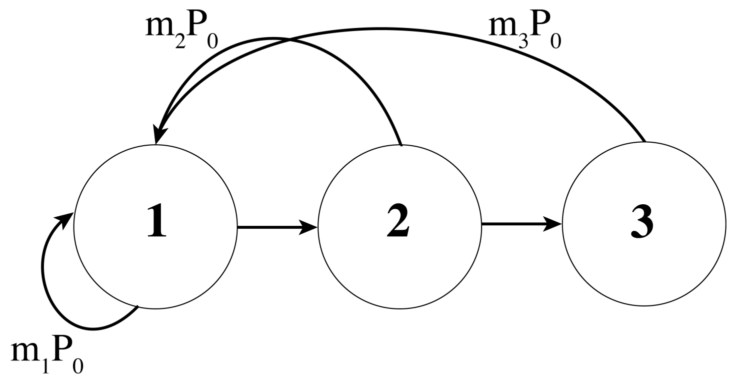

$$\mathbf{A} = \begin{array}{rc} & \begin{array}{ccc} & & \textbf{from 3} \end{array} \\ \textbf{to 1} & \begin{bmatrix} m_1P_0 & m_2P_0 & m_3P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \end{array}$$

Age-structured matrix models

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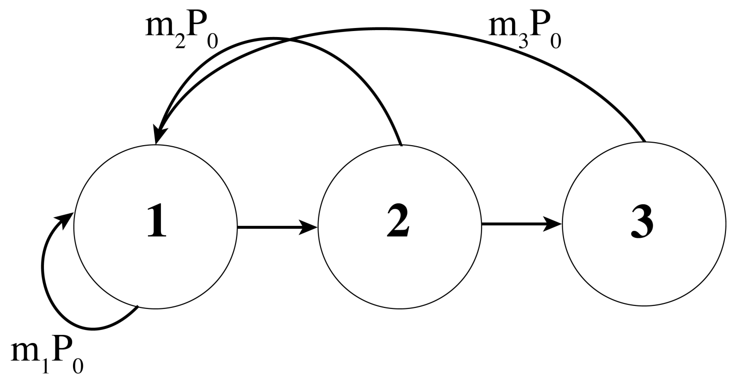

$$\mathbf{A} = \begin{array}{cc} & \begin{array}{ccc}\text{from 1} & & \end{array} \\ \begin{array}{c} \\ \text{to 2} \\ \\ \end{array} & \begin{bmatrix} m_1P_0 & m_2P_0 & m_3P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \end{array}$$

Age-structured matrix models

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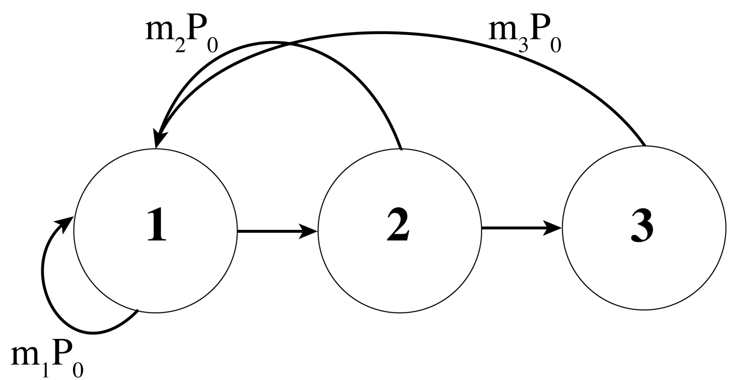

$$\mathbf{A} = \begin{array}{cc} & \begin{array}{ccc} & \textbf{from 2} & \end{array} \\ \begin{array}{c} \\ \\ \textbf{to 3} \end{array} & \begin{bmatrix} m_1P_0 & m_2P_0 & m_3P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \end{array}$$

Age-structured matrix models

Rather than separately modeling fecundity and juvenile survival, Leslie matrices often include **recruitment**

- F_x: Recruitment (or sometimes **F**ertility)
- $F_x = m_x P_0$

$$\mathbf{A} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

Age-structured matrix models

Recruitment doesn't have to be estimated from the life table

Birds:

- Clutch size
- Nest survival
- Chick survival
- Juvenile survival

$$F = cs \times ns \times chs \times js$$

Age-structured matrix models

Recruitment doesn't have to be estimated from the life table

Plants:

- Seed production
- Seed survival
- Germination rate
- Seedling survival

$$F = sp \times sds \times gr \times sls$$

Projecting abundance

The Leslie matrix can be used to project abundance

$$\mathbf{N}_t = \begin{bmatrix} n_{1,t} \\ n_{2,t} \\ n_{3,t} \end{bmatrix}$$

Projecting abundance

The Leslie matrix can be used to project abundance

- multiply the Leslie matrix

$$\mathbf{A} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

- by the abundance matrix

Projecting abundance

Population abundance is projected through time using matrix multiplication

$$\mathbf{N}_{t+1} = \mathbf{A} \times \mathbf{N}_t$$

$$\begin{bmatrix} n_{1,t+1} \\ n_{2,t+1} \\ n_{3,t+1} \end{bmatrix} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \times \begin{bmatrix} n_{1,t} \\ n_{2,t} \\ n_{3,t} \end{bmatrix}$$

Notice that the inner dimensions match

Projecting abundance

Population abundance is projected through time using matrix multiplication

$$\begin{bmatrix} n_{1,t+1} \\ n_{2,t+1} \\ n_{3,t+1} \end{bmatrix} = \begin{bmatrix} \boxed{n_1F_{1,t} + n_2F_{2,t} + n_3F_{3,t}} \\ n_{1,t}P_1 \\ n_{2,t}P_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\begin{matrix} n_{1,t} \\ n_{2,t} \\ n_{3,t} \end{matrix}} \end{bmatrix} \times \begin{bmatrix} \boxed{\begin{matrix} F_1 & F_2 & F_3 \end{matrix}} \\ \begin{matrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{matrix} \end{bmatrix}$$

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Projecting abundance

Population abundance is projected through time using matrix multiplication

$$\begin{bmatrix} n_{1,t+1} \\ n_{2,t+1} \\ n_{3,t+1} \end{bmatrix} = \begin{bmatrix} n_1F_{1,t} + n_2F_{2,t} + n_3F_{3,t} \\ \boxed{n_{1,t}P_1} \\ n_{2,t}P_3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} n_{1,t} \\ n_{2,t} \\ n_{3,t} \end{bmatrix}} \times \begin{bmatrix} F_1 & F_2 & F_3 \\ \boxed{P_1 \quad 0 \quad 0} \\ 0 & P_2 & 0 \end{bmatrix}$$

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Projecting abundance

Population abundance is projected through time using matrix multiplication

$$\begin{bmatrix} n_{1,\,t+1} \\ n_{2,\,t+1} \\ n_{3,\,t+1} \end{bmatrix} = \begin{bmatrix} n_1 F_{1,\,t} + n_2 F_{2,\,t} + n_3 F_{3,t} \\ n_{1,t} P_1 \\ \boxed{n_{2,t} P_3} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\begin{matrix} n_{1,\,t} \\ n_{2,\,t} \\ n_{3,\,t} \end{matrix}} \end{bmatrix} \text{x} \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$