



Lecture 10

Matrix population models

WILD3810 (Spring 2020)

Readings

Mills 98-103

Assumptions of the B-D models

Over the coming weeks, we will learn about why and how to relax assumption 3:

- 1) Population closed to immigration and emigration
- 2) Model pertains to only the limiting sex, usually females
- 3) Birth and death rates are independent of an individual's age or biological stage
- 4) Birth and death rates are constant

How can we model population dynamics of populations with complex age structure?

Age-structured demography

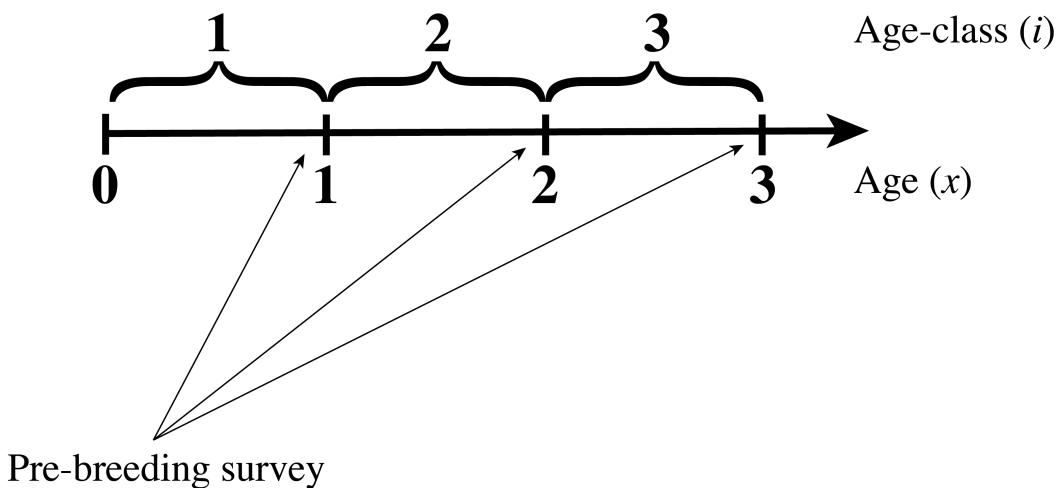
Consider the following life table:

x	m_x	P_x
0	0	P_0
1	m_1	P_1
2	m_2	P_2
3	m_3	0

Age-structured demography

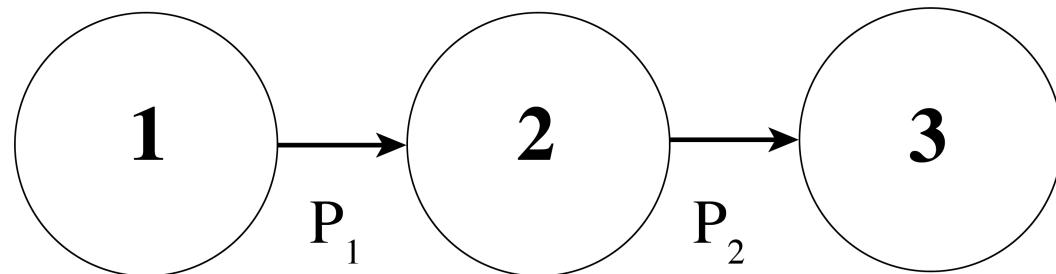
List population abundance in discrete 1-year age classes (n_i)

- e.g., n_1, n_2, n_3
- n_i is the number of individuals about to experience their i^{th} birthday



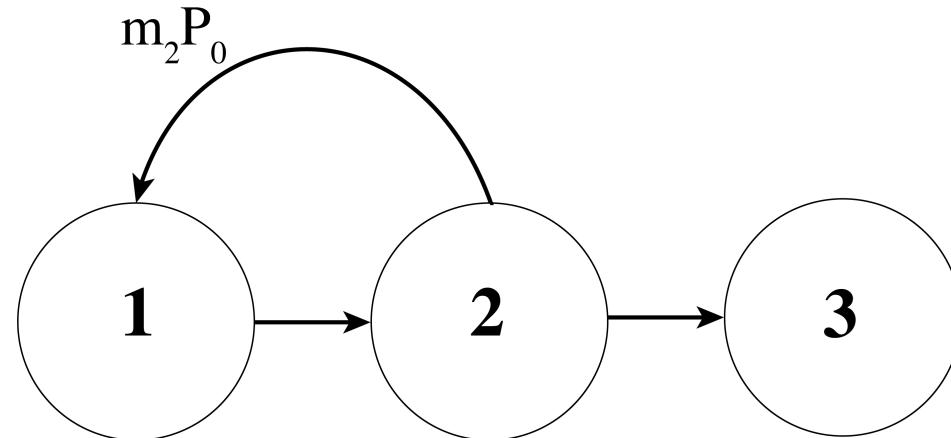
Age-structured demography

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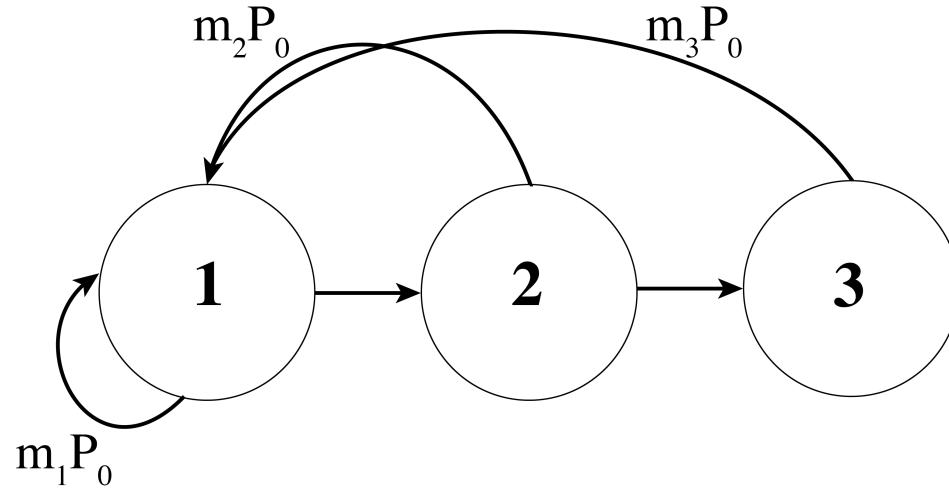
Age-structured demography

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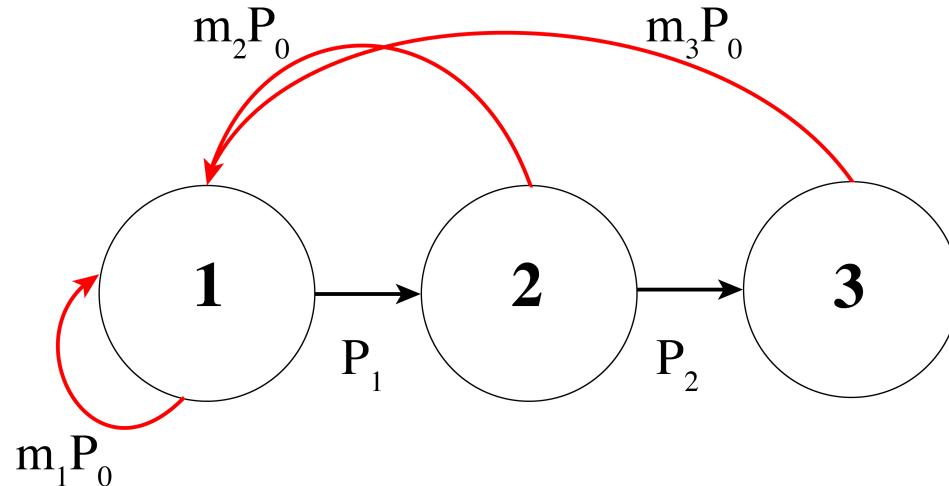
Age-structured demography

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Age-structured demography

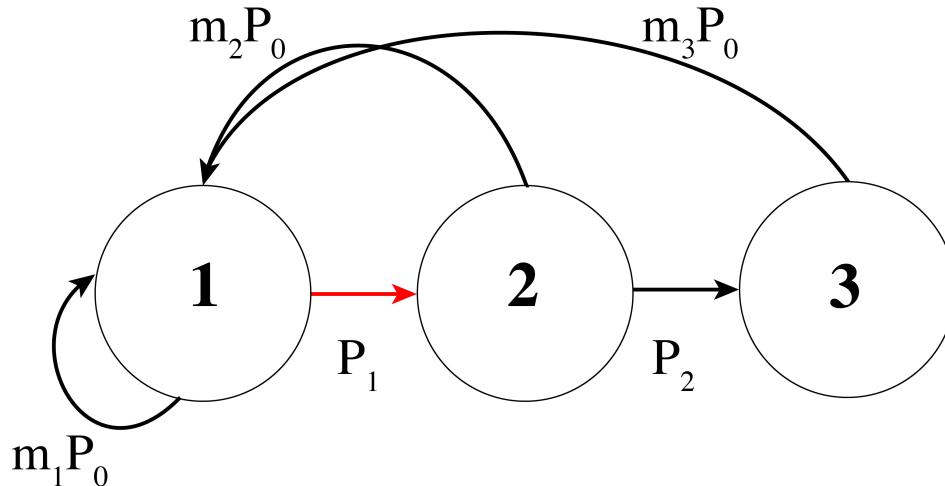
How many individuals will be in the population next year?



$$n_{1,t+1} = m_1 P_0 n_{1,t} + m_2 P_0 n_{2,t} + m_3 P_0 n_{3,t}$$

Age-structured demography

How many individuals will be in the population next year?

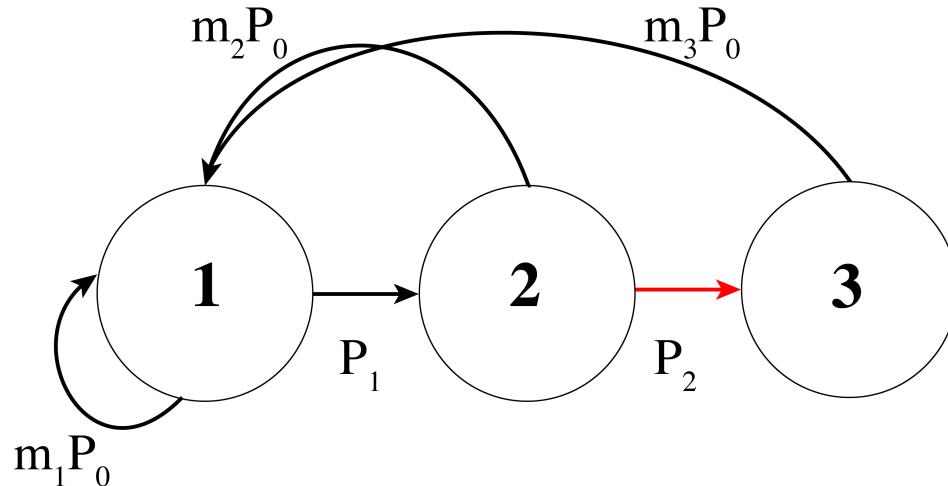


$$n_{1,t+1} = m_1 P_0 n_{1,t} + m_2 P_0 n_{2,t} + m_3 P_0 n_{3,t}$$

$$n_{2,t+1} = P_1 n_{1,t}$$

Age-structured demography

How many individuals will be in the population next year?



$$n_{1,t+1} = m_1 P_0 n_{1,t} + m_2 P_0 n_{2,t} + m_3 P_0 n_{3,t}$$

$$n_{2,t+1} = P_1 n_{1,t}$$

$$n_{3,t+1} = P_2 n_{2,t}$$

Leslie matrix model

Rather than modeling the dynamics using the previous equations, we can use **matrix projection models**

- defined by square matrix that summarizes the demography of age-specific life cycles
- one column for each age class
- developed by Sir Patrick H. Leslie for application to population biology
- a matrix with age-specific birth and survival rates is called a **Leslie matrix**

$$\mathbf{A} = \begin{bmatrix} m_1 P_0 & m_2 P_0 & m_3 P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

Review of matrix algebra

Review of matrix algebra

Matrix addition

- Matrices must be of the same dimension
- Add corresponding elements of the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 4 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & 5 \\ 6 & 13 \end{bmatrix}$$

Review of matrix algebra

Matrix subtraction

- Matrices must be of the same dimension
- Subtract corresponding elements of the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 4 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1 & -5 \\ 2 & -1 \end{bmatrix}$$

Review of matrix algebra

Matrix multiplication

- Dimensions are specified as **rows** by **columns**
- Matrix multiplication does not require matrices to have the same dimensions
- But they must have the same **inner dimension**
 - for example, a 3x3 matrix can be multiplied by a 3x1 matrix

$$\mathbf{A} = \begin{matrix} 3 \times 3 \\ \left[\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix} \right] \end{matrix}, \quad \mathbf{B} = \begin{matrix} 3 \times 1 \\ \left[\begin{matrix} x \\ y \\ z \end{matrix} \right] \end{matrix}$$

Review of matrix algebra

Matrix multiplication

- Dimensions are specified as **rows** by **columns**
- Matrix multiplication does not require matrices to have the same dimensions
- But they must have the same **inner dimension**
 - but a 3x3 matrix **cannnot** be multiplied by a 1x3 matrix

$$\begin{matrix} 3 \times 3 & & 1 \times 3 \\ \mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \mathbf{B} = [x & y & z] \end{matrix}$$

Review of matrix algebra

How does matrix multiplication work?

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a \times x + b \times y + c \times z \\ d \times x + e \times y + f \times z \\ g \times x + h \times y + i \times z \end{bmatrix}$$

Review of matrix algebra

How does matrix multiplication work?

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 4 & 3 \\ 2 & 6 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 \times 3 + 0 \times 2 + 5 \times 1 \\ 0 \times 3 + 4 \times 2 + 3 \times 1 \\ 2 \times 3 + 6 \times 2 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 18 \end{bmatrix}$$

Review of matrix algebra

Transpose of a matrix

$$\mathbf{B} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{B}^T = [3 \quad 2 \quad 1]$$

$$\mathbf{C} = [3 \quad 2 \quad 1], \quad \mathbf{C}^T = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Review of matrix algebra

Transpose of a matrix

- is the following allowed?

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 4 & 3 \\ 2 & 6 & 0 \end{bmatrix}, \quad \mathbf{B} = [3 \quad 2 \quad 1]$$

$$A \times B$$

Review of matrix algebra

Transpose of a matrix

- is the following allowed?

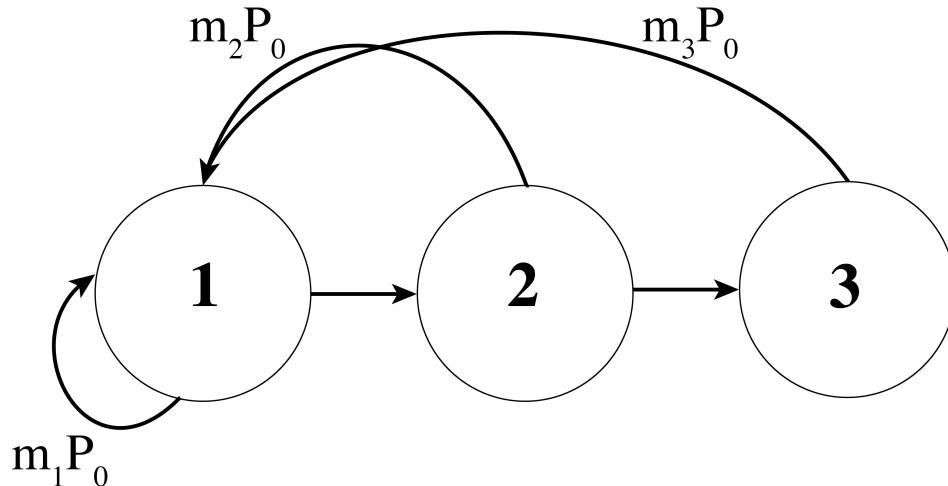
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 4 & 3 \\ 2 & 6 & 0 \end{bmatrix}, \quad \mathbf{B} = [3 \quad 2 \quad 1]$$

$$A \times B^T$$

Age-structured matrix models

Age-structured matrix models

How many individuals will be in the population next year?



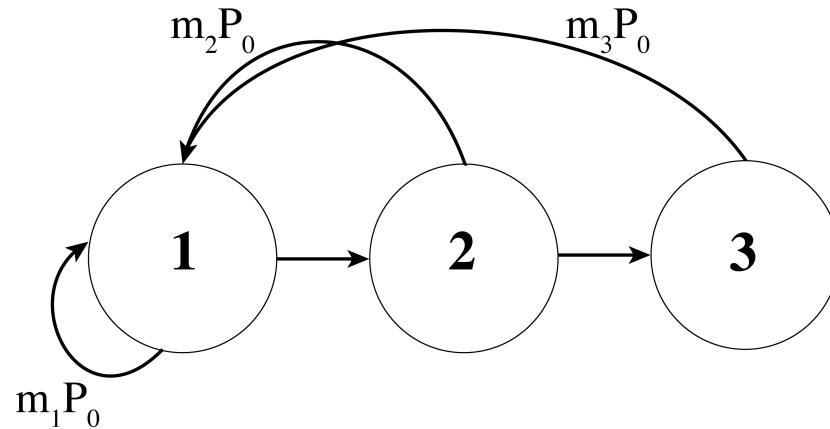
$$n_{1,t+1} = m_1 P_0 n_{1,t} + m_2 P_0 n_{2,t} + m_3 P_0 n_{3,t}$$

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Age-structured matrix models

How many individuals will be in the population next year?

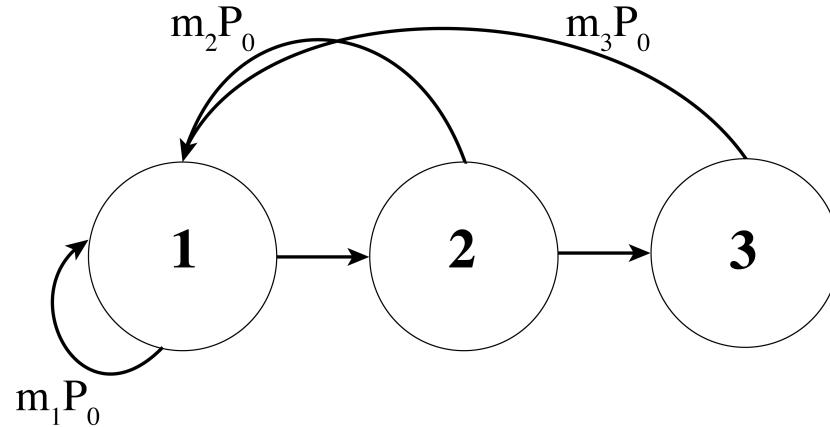


$$A = \begin{bmatrix} m_1 P_0 & m_2 P_0 & m_3 P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

from 1
to 1

Age-structured matrix models

How many individuals will be in the population next year?

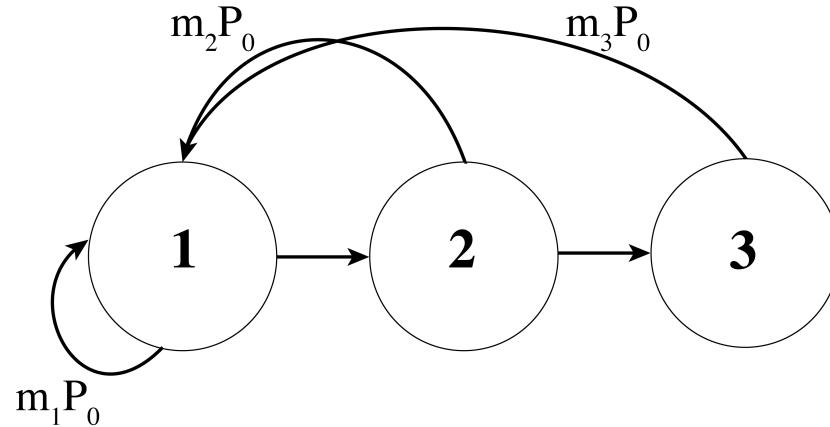


$$A = \begin{bmatrix} m_1 P_0 & m_2 P_0 & m_3 P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

to 1 from 2

Age-structured matrix models

How many individuals will be in the population next year?

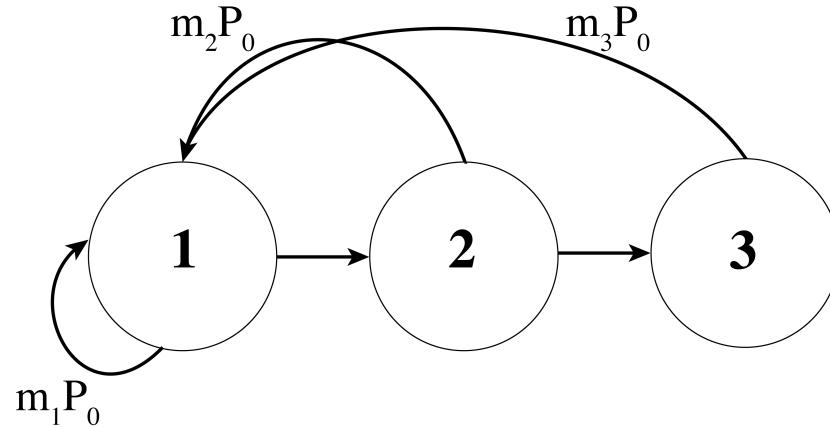


$$A = \begin{bmatrix} m_1 P_0 & m_2 P_0 & m_3 P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

to 1 from 3

Age-structured matrix models

How many individuals will be in the population next year?

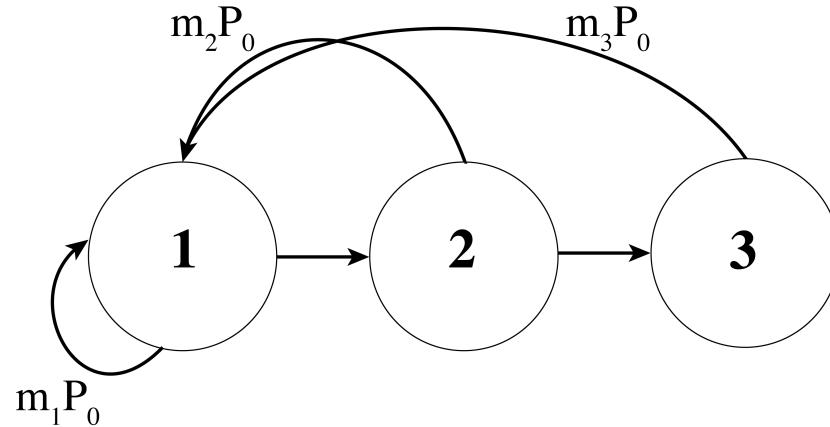


$$A = \begin{bmatrix} m_1 P_0 & m_2 P_0 & m_3 P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

from 1
to 2

Age-structured matrix models

How many individuals will be in the population next year?



$$A = \begin{bmatrix} m_1 P_0 & m_2 P_0 & m_3 P_0 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

from 2

to 3

Age-structured matrix models

Rather than separately modeling fecundity and juvenile survival, Leslie matrices often include **recruitment**

- F_x : Recruitment (or sometimes Fertility)
- $F_x = m_x P_0$

$$\mathbf{A} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

Age-structured matrix models

Recruitment doesn't have to be estimated from the life table

Birds:

- Clutch size
- Nest survival
- Chick survival
- Juvenile survival

$$F = cs \times ns \times chs \times js$$

$$\mathbf{A} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

Age-structured matrix models

Recruitment doesn't have to be estimated from the life table

Plants:

- Seed production
- Seed survival
- Germination rate
- Seedling survival

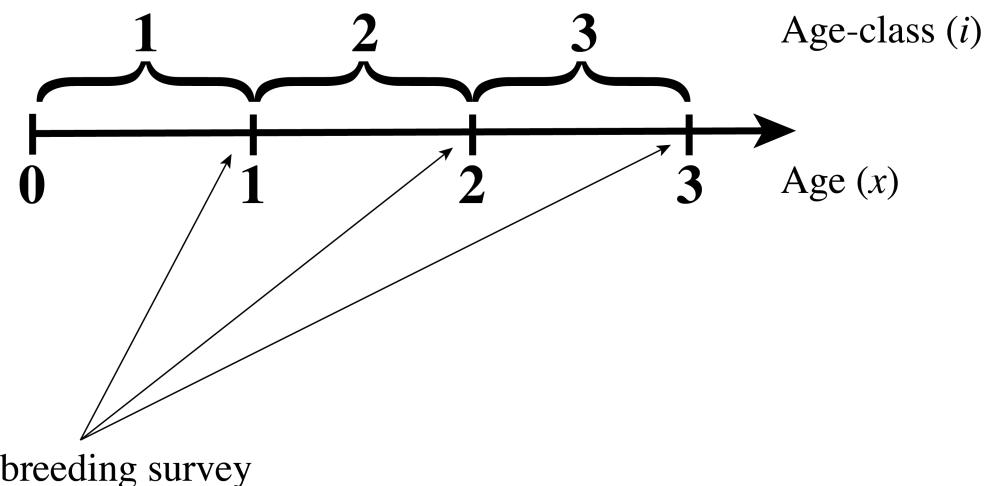
$$F = sp \times sds \times gr \times sls$$

$$\mathbf{A} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

Projecting abundance

The Leslie matrix can be used to project abundance

$$\mathbf{N}_t = \begin{bmatrix} n_{1,t} \\ n_{2,t} \\ n_{3,t} \end{bmatrix}$$



Projecting abundance

The Leslie matrix can be used to project abundance

- multiply the Leslie matrix

$$\mathbf{A} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

- by the abundance matrix

$$\mathbf{N}_t = \begin{bmatrix} n_{1,t} \\ n_{2,t} \\ n_{3,t} \end{bmatrix}$$

using the rules of matrix multiplication

Projecting abundance

Population abundance is projected through time using matrix multiplication

$$\mathbf{N}_{t+1} = \mathbf{A} \times \mathbf{N}_t$$

$$\begin{bmatrix} n_{1,t+1} \\ n_{2,t+1} \\ n_{3,t+1} \end{bmatrix} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \times \begin{bmatrix} n_{1,t} \\ n_{2,t} \\ n_{3,t} \end{bmatrix}$$

Notice that the inner dimensions match

Projecting abundance

Population abundance is projected through time using matrix multiplication

$$\begin{bmatrix} n_{1,t+1} \\ n_{2,t+1} \\ n_{3,t+1} \end{bmatrix} = \begin{bmatrix} n_1 F_{1,t} + n_2 F_{2,t} + n_3 F_{3,t} \\ n_{1,t} P_1 \\ n_{2,t} P_3 \end{bmatrix}$$

$$= \begin{bmatrix} n_{1,t} \\ n_{2,t} \\ n_{3,t} \end{bmatrix} \times \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

Projecting abundance

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Projecting abundance

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