

# From linear models to GLMs

# Linear models

*response = deterministic part + stochastic part*

$$\underbrace{\mu_i = \beta_0 + \beta_1 \times x_i}_{\text{Deterministic}}$$

$$\underbrace{y_i \sim \text{normal}(\mu_i, \tau)}_{\text{Stochastic}}$$

# Linear models under the hood

## Variations on the deterministic model

# A simple linear model: the t-test

$$\mu_i = \beta_0 + \beta_1 \times x_i$$

$$x_i \in [0, 1]$$

# Under the hood: The t-test

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \cdot \\ \cdot \\ \cdot \\ \mu_N \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \cdot & \\ \cdot & \\ \cdot & \\ 1 & x_N \end{bmatrix} \times \begin{matrix} \beta_0 \\ \beta_1 \end{matrix}$$

$$\mu_1 = \beta_0 \times 1 + \beta_1 \times x_1$$

$$\mu_2 = \beta_0 \times 1 + \beta_2 \times x_1$$

# Under the hood: The t-test

```
model.matrix(lm(mpg ~ am, data = mtcars))[c(3,1,5),]
```

```
##                (Intercept) am1
## Datsun 710                1    1
## Mazda RX4                 1    1
## Hornet Sportabout         1    0
```

$\beta_0$  is the mean mpg for automatic transmissions

$\beta_1$  is the *difference* between automatic and manual transmissions

# Under the hood: The t-test

```
lm(mpg ~ am, data = mtcars)
```

term	estimate	std.error	statistic	p.value
(Intercept)	17.147	1.125	15.248	0e+00
am1	7.245	1.764	4.106	3e-04

# Under the hood: The t-test

```
model.matrix(lm(mpg ~ as.factor(am) - 1, data = mtcars))[c(3,1,5),]
```

```
##              am0 am1
## Datsun 710      0  1
## Mazda RX4      0  1
## Hornet Sportabout 1  0
```

$\beta_0$  is the mean mpg for automatic transmissions

$\beta_1$  is the mean mpg for manual transmissions



# Under the hood: The t-test

```
lm(mpg ~ as.factor(am) - 1, data = mtcars)
```

term	estimate	std.error	statistic	p.value
am0	17.15	1.125	15.25	0
am1	24.39	1.360	17.94	0

# The t-test becomes an ANOVA

$$y_i = \beta_0 + \beta_{1[j]} \times x_i$$

$$j \in [1, 2, 3, \dots, J - 1]$$

$$x_i \in [0, 1]$$

# Under the hood: ANOVA

```
model.matrix(lm(mpg ~ as.factor(cyl), data = mtcars))[c(3,1,5),]
```

```
##                (Intercept) cyl6 cyl8
## Datsun 710                1     0     0
## Mazda RX4                 1     1     0
## Hornet Sportabout         1     0     1
```

$\beta_0$  is the mean mpg for 4-cylinders

$\beta_{1[6-cyl]}$  is the *difference* between 4-cyl & 6-cyl

$\beta_{1[8-cyl]}$  is the *difference* between 4-cyl & 8-cyl

# Under the hood: ANOVA

## Effects parameterization

```
lm(mpg ~ as.factor(cyl) , data = mtcars)
```

term	estimate	std.error	statistic	p.value
(Intercept)	26.664	0.9718	27.437	0e+00
cyl6	-6.921	1.5583	-4.441	1e-04
cyl8	-11.564	1.2986	-8.905	0e+00

# Under the hood: ANOVA

## Means parameterization

$$\mu_i = \beta_{0[j]}$$

```
lm(mpg ~ as.factor(cyl) - 1, data = mtcars)
```

term	estimate	std.error	statistic	p.value
cyl4	26.66	0.9718	27.44	0
cyl6	19.74	1.2182	16.21	0
cyl8	15.10	0.8614	17.53	0

# The ANOVA becomes an ANCOVA

$$y_i = \beta_0 + \beta_{1[j]} \times x1_i + \beta_2 \times x2_i$$

$$x1_i \in [0, 1]$$

$$x2_i \in [-\infty, \infty]$$

# Under the hood: ANCOVA

```
model.matrix(lm(mpg ~ as.factor(cyl) + hp, data = mtcars))[c(3,1,5),]
```

##	(Intercept)	cyl6	cyl8	hp
## Datsun 710	1	0	0	93
## Mazda RX4	1	1	0	110
## Hornet Sportabout	1	0	1	175

$\beta_0$  is the mean mpg for 4-cylinders @ 0hp

$\beta_{1[6-cyl]}$  is the *difference* between 4-cyl & 6-cyl @ 0hp

$\beta_{1[8-cyl]}$  is the *difference* between 4-cyl & 8-cyl @ 0hp

$\beta_2$  is the effect of hp on mpg

# Under the hood: ANCOVA

```
lm(mpg ~ as.factor(cyl) + hp, data = mtcars)
```

term	estimate	std.error	statistic	p.value
(Intercept)	28.650	1.5878	18.044	0.0000
cyl6	-5.968	1.6393	-3.640	0.0011
cyl8	-8.521	2.3261	-3.663	0.0010
hp	-0.024	0.0154	-1.560	0.1300



# Under the hood: Interactions

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \cdot \\ \cdot \\ \cdot \\ \mu_N \end{bmatrix} = \begin{bmatrix} 1 & x1_1 & x2_1 & x1_1 * x2_1 \\ 1 & x1_2 & x2_2 & x1_2 * x2_2 \\ 1 & x1_3 & x2_3 & x1_3 * x2_3 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & x1_N & x2_N & x1_N * x2_N \end{bmatrix} \times \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

# Under the hood: Interactions

```
model.matrix(lm(mpg~as.factor(cyl)*hp, data = mtcars))[c(3,1,5),]
```

##	(Intercept)	cyl6	cyl8	hp	cyl6:hp	cyl8:hp
## Datsun 710	1	0	0	93	0	0
## Mazda RX4	1	1	0	110	110	0
## Hornet Sportabout	1	0	1	175	0	175

$\beta_0$  is the mean mpg for 4-cylinders @ 0hp

$\beta_{1[6-cyl]}$  is the *difference* between 4-cyl & 6-cyl @ 0hp

$\beta_{1[8-cyl]}$  is the *difference* between 4-cyl & 8-cyl @ 0hp

$\beta_2$  is the effect of hp on mpg for 4-cylinders

$\beta_{3[6-cyl]}$  is the *difference* between the effect of hp in 4-cyl vs 6-cyl

$\beta_{3[8-cyl]}$  is the *difference* between the effect of hp in 4-cyl vs 8-cyl

# Under the hood: Interactions

```
lm(mpg ~ as.factor(cyl) * hp, data = mtcars)
```

term	estimate	std.error	statistic	p.value
(Intercept)	35.9830	3.8891	9.252	0.0000
cyl6	-15.3092	7.4346	-2.059	0.0496
cyl8	-17.9030	5.2596	-3.404	0.0022
hp	-0.1128	0.0457	-2.465	0.0206
cyl6:hp	0.1052	0.0685	1.536	0.1367
cyl8:hp	0.0985	0.0486	2.026	0.0531

# Under the hood: Interactions

```
model.matrix(lm(mpg ~ as.factor(cyl) * hp - 1 - hp, data = mtcars))[,c(
```

##	cyl4	cyl6	cyl8	cyl4:hp	cyl6:hp	cyl8:hp
## Datsun 710	1	0	0	93	0	0
## Mazda RX4	0	1	0	0	110	0
## Hornet Sportabout	0	0	1	0	0	175

$\beta_{0[4-cyl]}$  is the mean mpg for 4-cylinders @ 0hp

$\beta_{0[6-cyl]}$  is the mean mpg for 6-cylinders @ 0hp

$\beta_{0[8-cyl]}$  is the mean mpg for 8-cylinders @ 0hp

# Under the hood: Interactions

```
model.matrix(lm(mpg ~ as.factor(cyl) * hp - 1 - hp, data = mtcars))[,c(
```

##	cyl4	cyl6	cyl8	cyl4:hp	cyl6:hp	cyl8:hp
## Datsun 710	1	0	0	93	0	0
## Mazda RX4	0	1	0	0	110	0
## Hornet Sportabout	0	0	1	0	0	175

$\beta_{0[4-cyl]}$  is the mean mpg for 4-cylinders @ 0hp

$\beta_{0[6-cyl]}$  is the mean mpg for 6-cylinders @ 0hp

$\beta_{0[8-cyl]}$  is the mean mpg for 8-cylinders @ 0hp

$\beta_{1[4-cyl]}$  is the effect of hp on mpg for 4-cylinders

$\beta_{1[6-cyl]}$  is the effect of hp on mpg for 6-cylinders

$\beta_{1[8-cyl]}$  is the effect of hp on mpg for 8-cylinders

## Under the hood: Interactions

```
lm(mpg ~ as.factor(cyl) * hp - 1, data = mtcars)
```

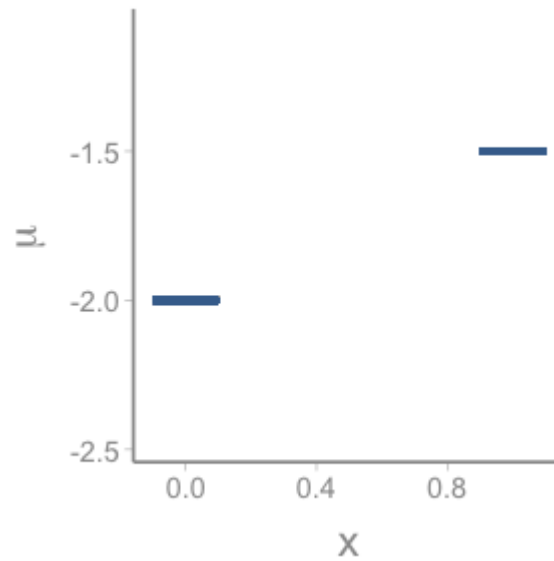
term	estimate	std.error	statistic	p.value
cyl4	35.9830	3.8891	9.2523	0.0000
cyl6	20.6739	6.3362	3.2628	0.0031
cyl8	18.0801	3.5410	5.1059	0.0000
cyl4:hp	-0.1128	0.0457	-2.4652	0.0206
cyl6:hp	-0.0076	0.0510	-0.1494	0.8824
cyl8:hp	-0.0142	0.0165	-0.8645	0.3952

# Linear models under the hood

## Variations on the stochastic model

# Stochasticity in the linear model

$$\underbrace{\mu_i = -2 + 0.5 \times x_i}_{\text{Deterministic}}$$





# Stochasticity in the linear model

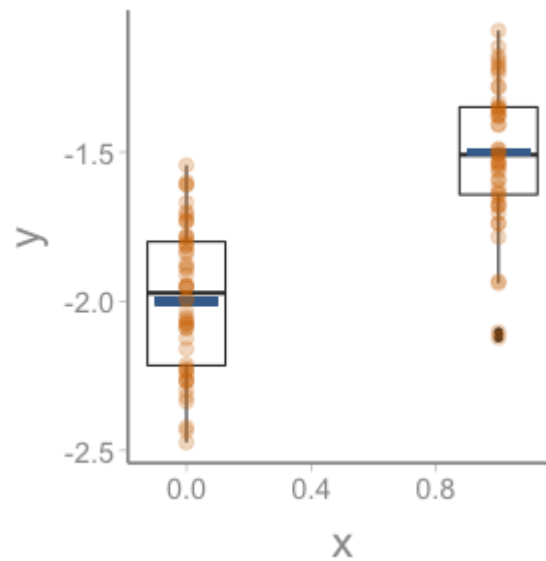
$$\mu_i = -2 + 0.5 \times x_i$$

$$\underbrace{y_i \sim \text{normal}(\mu_i, \tau)}_{\text{Stochastic}}$$

# Stochasticity in the linear model

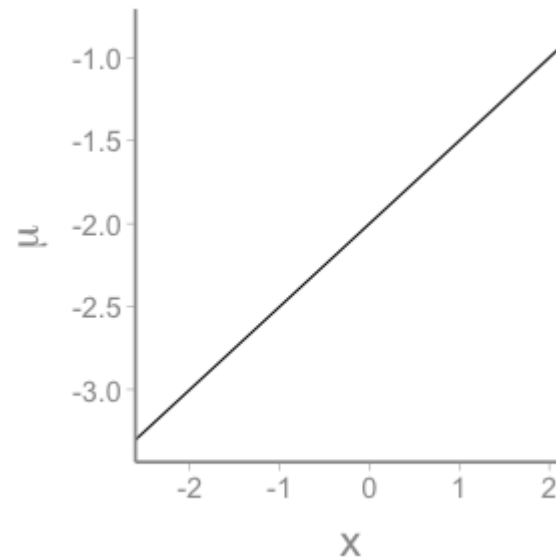
$$\mu_i = -2 + 0.5 \times x_i$$

$$\underbrace{y_i \sim \text{normal}(\mu_i, \tau)}_{\text{Stochastic}}$$



# Stochasticity in the linear model

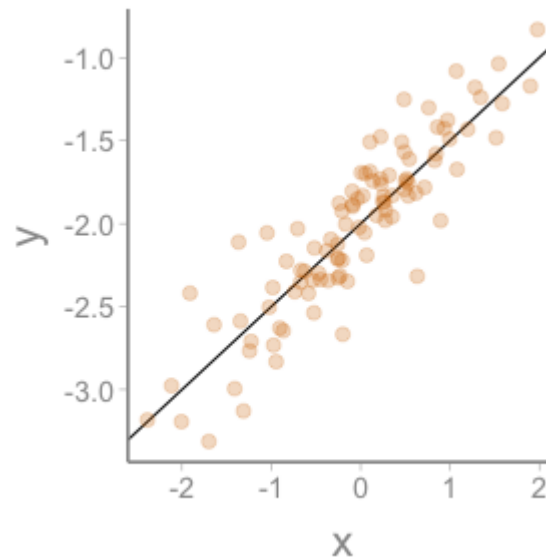
$$\mu_i = -2 + 0.5 \times x_i$$



# Stochasticity in the linear model

$$\mu_i = -2 + 0.5 \times x_i$$

$$y_i \sim \text{normal}(\mu_i, \tau)$$



# Components of the linear model

# Components of the linear model

## 1) Distribution

$$y_i \sim \text{normal}(\mu_i, \tau)$$

## 2) Link function

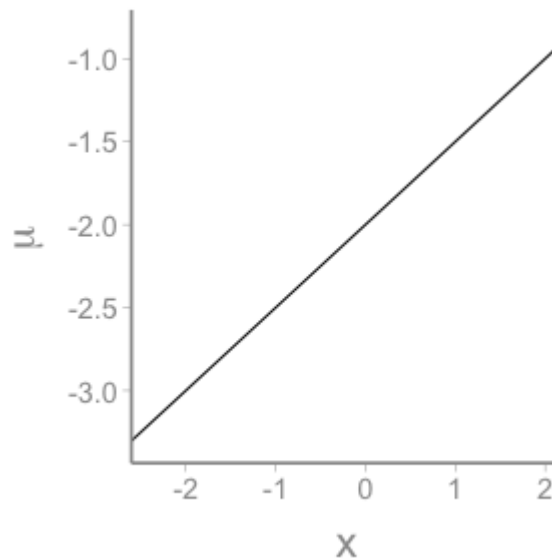
$$\mu_i = E(y_i) = \text{linear predictor}$$

## 3) Linear predictor

$$\beta_0 + \beta_1 \times x_i$$

# Stochasticity in the linear model

What happens if  $0 \leq y_i$ ?



# Components of the generalized linear model

## 1) Distribution

$$y_i \sim \textit{Poisson}(\lambda_i)$$

## 2) Link function

$$\log(\lambda_i) = \log(E(y_i)) = \textit{linear predictor}$$



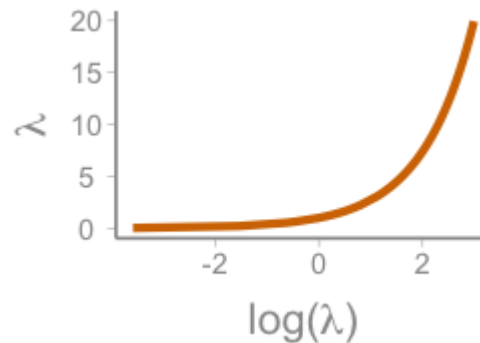
# Components of the generalized linear model

## 1) Distribution

$$y_i \sim \text{Poisson}(\lambda_i)$$

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# Components of the generalized linear model

## 1) Distribution

$$y_i \sim \textit{Poisson}(\lambda_i)$$

## 2) Link function

$$\log(\lambda_i) = \log(E(y_i)) = \textit{linear predictor}$$

## 3) Linear predictor

$$\beta_0 + \beta_1 \times x_i$$

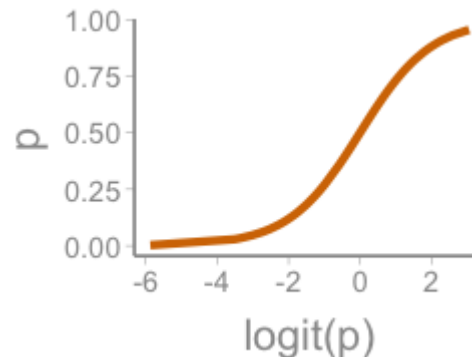
# Components of the generalized linear model

## 1) Distribution

$$y_i \sim \text{Bernoulli}(p_i)$$

## 2) Link function

$$\text{logit}(p_i) = \log \frac{p_i}{1 - p_i} = \text{linear predictor}$$



# Components of the generalized linear model

## 1) Distribution

$$y_i \sim \textit{Bernoulli}(p_i)$$

## 2) Link function

$$\textit{logit}(p_i) = \log \frac{p_i}{1 - p_0} = \textit{linear predictor}$$

## 3) Linear predictor

$$\beta_0 + \beta_1 \times x_i$$

# Components of the generalized linear model

## 1) Distribution

$$y_i \sim \text{binomial}(N, p_i)$$

## 2) Link function

$$\text{logit}(p_i) = \log \frac{p_i}{1 - p_0} = \text{linear predictor}$$

## 3) Linear predictor

$$\beta_0 + \beta_1 \times x_i$$

# Generalized linear models

- Flexible method to model observations arising from different probability distributions
- Link many classical tests under unified framework
- Underlie nearly all common ecological models