Problem 4: A Glass of Juice

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Abstract. Using analytical methods or (computer-assisted) numerical bounds we treat extensively first four glasses (cylinder, hemisphere, parabola, and cone). A gallery of integrals used is appended at the end of the paper.

Introduction to juice spilling and C_1-type glasses

As we will consider only solids of revolution of curves we have to notice that actually for given a, b it depends only on α. We can also rotate our glass (let us call it $\mathcal{G}$) around z-axis and then spill some juice, but that will not change anything because our solids are revolutions of curves so rotations around z-axis are useless. In this way we can only to consider rotations around y and x-axis. But let us notice, that we want to know correlation between α and volume of juice that hasn't been spilled yet. Notice that rotation about angle β around x-axis and then about γ around y-axis in this order is equivalent to this two rotations in the opposite order, as angle are two dimensional argument and rotation around x-axis is independent of rotation around y-axis. So we can assume that we rotate our glass only around y-axis about angle α.

Let $f : [0, a] \times \mathbb{R} \to \mathbb{R}$ be given function and $g : [0, a] \to \mathbb{R}$ and assume some properties:

1. $f(0, 0) = 0$
2. $\forall_{x \in [0,a]} \exists_c \; f(x, c) = 0.$
3. $\forall_{x \in [0,a]} \; f(x, a_1) = f(x, a_2) \implies a_1 = a_2.$
4. $\forall_{x \in [0,a]} \; f(x, z) = 0 \implies g(x) = z$
5. g is continuous and convex up on the interval $[0, a]$
6. $\exists_{G:[0,a] \to \mathbb{R}} \; G(x) = \int g(x) \, \mathrm{d}x$

In a clear way we can interpret g as distance between curve and line $x = 0$ for given x. Notice that $\forall_{x \in [0,a]} g(x) \geq 0$, as g is convex up and $g(0) = 0$. Also let us call our glass to be of C_1-type if f is defined as above and there exists g and G like in 5. and 6.

Let's rotate our glass around y-axis about angle α clockwise ($(0, 0, 1)$ will be rotated into points $(1, 0, 0)$, $(0, 0, -1)$ and $(-1, 0, 0)$ in this order) and denote $A' = (a, 0, p)$ where $A' \in \mathcal{G}$. Denote Γ as a plane made by the surface of the juice. Notice that $A' \in \Gamma \cap \mathcal{G}$. Also as Γ is the the surface of the juice, it is parallel to plane $z = 0$. So now let us rotate back glass and Γ to initial position of the glass. As $A' \in \Gamma$ and angle that Γ was rotated back is equal α, we know that equation of Γ is $z = \tan \alpha (x - a) + p$. where p is height of glass in point A'. As g is a function of height, then equation of Γ is equal $z = \tan \alpha (x - b) + g(a)$.

Now let us notice that volume between some two objects $\mathcal{A}$ and $\mathcal{B}$ is just a double integral for every pair of points (x, y) such the equation of height of $\mathcal{A}$ has a bigger value than the equation of height of $\mathcal{B}$. In our case $\mathcal{A}$ is Γ and $\mathcal{B}$ is $\mathcal{G}$. We know the equation of height for Γ and it is equal $\tan \alpha (x - a) + f(a)$. Now we want to find height of the glass for some point (a, b), but we can use g and we easily have that height of this point on the glass is equal to $g\left(\sqrt{x^2 + y^2}\right)$. In this case $V(\alpha)$ has the equation:

$$V(\alpha) = \iint_X \tan \alpha (x - a) + g(a) - g\left(\sqrt{x^2 + y^2}\right) \, \mathrm{d}x \, \mathrm{d}y$$

Where $X = \left\{(x, y) : \tan \alpha (x - a) + g(a) \geq g\left(\sqrt{x^2 + y^2}\right)\right\}$

Now we want to find a volume of the glass and we can notice that volume of glass is equal to cylinder that volume is $ag(a)\pi$ minus volume by cylinder method[1] for g in this way:

$$V = ag(a)\pi - 2\pi \int_0^a xf(x)\,\mathrm{d}x$$

And in the end our formula is equal to:

$$\frac{V(\alpha)}{V} = \frac{\iint_X \tan\alpha(x-a) + g(a) - g\left(\sqrt{x^2+y^2}\right)\,\mathrm{d}x\,\mathrm{d}y}{ag(a)\pi - 2\pi\int_0^a xf(x)\,\mathrm{d}x}$$

1. For all solids of revolution of a curve, $\frac{V(\alpha}{V}$ depends only on α.
2. In this case we want to find value of $\frac{V(\frac{\pi}{6})}{V}$ for $a = 1$ and $b = 3$. Notice that if $\forall_{x\in[0,a]} g(x) \leq \tan\alpha$ then Γ lies below glass and then $\mathcal{G}$ is empty.
3. In this case for given a, b we want to find an α such that $\frac{V(\alpha)}{V} = \frac{1}{2}$.

Cylindrical glass

That is not actually a C_1-type glass but we can just consider it in a similar way, as a volume between Γ and plane $z = 0$. In this case equation of Γ is $z = \tan\alpha(x-a) + b$

First of all we are going to find a formula for possible values of $\frac{V(\alpha)}{V}$ in terms of a, b and α when $\tan\alpha \leq \frac{b}{2a}$. Let β be an angle between base (plane $z = 0$) and plane $z_1 = \frac{b}{2a}(x-a)+b$. In this way $\tan\beta = \frac{b}{2a}$. As $\beta \in [0, \frac{\pi}{2}[$, then for $\alpha \in [0, \frac{\pi}{2}[$ if $90° - \alpha \geq 90° - \beta \iff \alpha \leq \beta$, then also $\tan\alpha \leq \tan\beta$, so if $\tan\alpha \leq \frac{b}{2a}$ then our formula is equal to:

$$\frac{\iint_D \tan\alpha(x-a) + b\,\mathrm{d}x\,\mathrm{d}y}{\pi a^2 b}$$

where D $= \{(x,y):\ \ x^2+y^2 \leq a^2\}$. We can easily change it into polar coordinates. In this way we have:

$$\frac{V(\alpha)}{V} = \frac{\iint_D \tan\alpha(x-a) + b\,\mathrm{d}x\,\mathrm{d}y}{\pi a^2 b} = \frac{\int_0^a\int_0^{2\pi}(\tan\alpha(r\cos\varphi - a) + b)r\,\mathrm{d}\varphi\,\mathrm{d}r}{\pi a^2 b} =$$
$$= \frac{1}{\pi a^2 b}\cdot\int_0^a \left[r^2\tan\alpha\sin\varphi - \varphi ar\tan\alpha + \varphi br\right]_0^{2\pi}\mathrm{d}r =$$
$$= \frac{1}{\pi a^2 b}\cdot\int_0^a 2\pi br - 2\pi ar\tan\alpha\,\mathrm{d}r = \frac{1}{\pi a^2 b}\cdot\left[\pi br^2 - \pi ar^2\tan\alpha\right]_0^a =$$
$$= \frac{\pi a^2 b - \pi a^3\tan\alpha}{\pi a^2 b} = 1 - \frac{a\tan\alpha}{b} \quad (1)$$

1. As the cylinder is a solid of revolution, we can easily say that remaining volume of juice depends only on α

2. As $\frac{\sqrt{3}}{3} = \tan(\frac{\pi}{6}) \leq \frac{3}{2}$ we know we can use formula (1):

$$\frac{V(\frac{\pi}{6})}{V} = 1 - \frac{\frac{\sqrt{3}}{3}}{3} = 1 - \frac{\sqrt{3}}{9}$$

3. We will show α that $\tan\alpha = \frac{3}{2}$ is the one for which $\frac{V(\alpha)}{V} = \frac{1}{2}$. Let us notice that $\tan\alpha = \frac{b}{2a}$ so we can simply use formula (1):

$$\frac{V(\alpha)}{V} = 1 - \frac{a\tan\alpha}{b} = 1 - \frac{\frac{3}{2}}{3} = \frac{1}{2}$$

Then $\alpha = \tan^{-1}\left(\frac{3}{2}\right)$

4. As long as $\tan\alpha \leq \frac{b}{2a}$ we know that our formula is:

$$\frac{V(\alpha)}{V} = 1 - \frac{a\tan\alpha}{b}$$

Also when $\alpha \in [\frac{\pi}{2}, 2\pi]$ then we have $\frac{V(\alpha)}{V} = 0$

What in case when $\tan\alpha > \frac{b}{2a}$ and $\alpha \in [0, \frac{\pi}{2}[$? First we have to find the equation of intersection of Γ and $z = 0$.

$$\begin{aligned} 0 &= (\tan\alpha)\cdot(x-a) + b \\ -b &= (\tan\alpha)(x-a) \\ -\frac{b}{\tan\alpha} &= x - a \\ x &= a - \frac{b}{\tan\alpha} \end{aligned}$$

As b and $\tan\alpha$ are some positive values, $a - \frac{b}{\tan\alpha} < a$ then our formula will be:

$$\frac{V(\alpha)}{V} = \frac{\int_{a-\frac{b}{\tan\alpha}}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \tan\alpha(x-a) + b \, \mathrm{d}y \, \mathrm{d}x}{\pi a^2 b} =$$

$$= \frac{1}{\pi a^2 b} \cdot \int_{a-\frac{b}{\tan\alpha}}^{a} \left[y(\tan\alpha(x-a)+b)\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \mathrm{d}x =$$

$$= \frac{2}{\pi a^2 b}\left(\int_{a-\frac{b}{\tan\alpha}}^{a} x\tan\alpha\cdot\sqrt{a^2-x^2}\,\mathrm{d}x + \int_{a-\frac{b}{\tan\alpha}}^{a} (b - a\tan\alpha)\cdot\sqrt{a^2-x^2}\,\mathrm{d}x\right) =$$

$$= \frac{2}{\pi a^2 b}\left(\tan\alpha\int_{a-\frac{b}{\tan\alpha}}^{a} x\cdot\sqrt{a^2-x^2}\,\mathrm{d}x + (b - a\tan\alpha)\int_{a-\frac{b}{\tan\alpha}}^{a} \sqrt{a^2-x^2}\,\mathrm{d}x\right) =$$

$$= \frac{2\tan\alpha\left(a^2 - \left(a - \frac{b}{\tan\alpha}\right)^2\right)^{\frac{3}{2}}}{3\pi a^2 b} + \frac{2\pi a^2(b - a\tan\alpha)}{4\pi a^2 b} +$$

$$+\frac{-2(b-a\tan\alpha)}{\pi a^2 b}\cdot\frac{\left(a-\frac{b}{\tan\alpha}\right)\sqrt{a^2-\left(a-\frac{b}{\tan\alpha}\right)^2}}{2}+$$
$$+\frac{-2(b-a\tan\alpha)}{\pi a^2 b}\cdot\frac{a^2}{2}\arcsin\left(\frac{\left(a-\frac{b}{\tan\alpha}\right)}{a}\right)=$$
$$=\frac{2\tan\alpha\left(a^2-\left(a-\frac{b}{\tan\alpha}\right)^2\right)^{\frac{3}{2}}}{3\pi a^2 b}+\frac{b-a\tan\alpha}{2b}+$$
$$+\frac{(a\tan\alpha-b)^2\sqrt{a^2-\left(a-\frac{b}{\tan\alpha}\right)^2}}{\tan\alpha\cdot\pi a^2 b}+$$
$$+\frac{a\tan\alpha-b}{\pi b}\cdot\arcsin\left(\frac{a\tan\alpha-b}{a\tan\alpha}\right)$$

And we stop at this point. Our formula for $\tan\alpha>\frac{b}{2a}$ is:

$$\frac{V(\alpha)}{V}=\frac{2\tan\alpha\left(a^2-\left(a-\frac{b}{\tan\alpha}\right)^2\right)^{\frac{3}{2}}}{3\pi a^2 b}+\frac{b-a\tan\alpha}{2b}+$$
$$+\frac{(a\tan\alpha-b)^2\sqrt{a^2-\left(a-\frac{b}{\tan\alpha}\right)^2}}{\tan\alpha\cdot\pi a^2 b}+\frac{a\tan\alpha-b}{\pi b}\cdot\arcsin\left(\frac{a\tan\alpha-b}{a\tan\alpha}\right)$$

Hemispherical glass

First of all, let us redefine $f(x,z)=-z+a-\sqrt{a^2-x^2}$ and $0\le x\le a$. Now it is clearly C_1-type glass, it is still the equation of hemisphere, but it has an easier solution and we will not use integrals here.

Let M be the center of the surface of the hemisphere $\mathcal{G}$ and k be the line that $k\perp\Gamma$; $M\in k$; $A=k\cap\Gamma$; $B=k\cap\mathcal{G}$; $C=(a,0,a)$. Notice that $\widehat{MAC}=90^\circ$ in triangle MAC we have:

$$|AB|=a-|MA|=a-a\sin\alpha=a(1-\sin\alpha)$$

As formula for spherical cap for given radius of the sphere and height of the cap is well known[2] we know that our formula is:

$$V(\alpha)=\pi a^3\left(1-\sin\alpha\right)^2-\frac{\pi}{3}a^3\left(1-\sin\alpha\right)^3$$

and that gives us

$$\frac{V(\alpha)}{V}=\frac{\pi a^3\left(1-\sin\alpha\right)^2-\frac{\pi}{3}a^3\left(1-\sin\alpha\right)^3}{\frac{2}{3}\pi a^3}=\frac{(1-\sin\alpha)^2-\frac{1}{3}(1-\sin\alpha)^3}{\frac{2}{3}}\qquad(2)$$

1. As the hemisphere is a revolving solid, we can easily say that remaining volume of juice depends only on α

2. After using formula (2) we have:

$$\frac{V(\frac{\pi}{6})}{V} = \frac{(1-\sin\frac{\pi}{6})^2 - \frac{1}{3}(1-\sin\frac{\pi}{6})^3}{\frac{2}{3}} = \frac{(\frac{1}{2})^2 - \frac{1}{3}\cdot(\frac{1}{2})^3}{\frac{2}{3}} = \frac{5}{16}$$

3. Define $x = 1 - \sin\alpha$ where $x \in \langle 0, 1\rangle$. After using formula (2) we have:

$$\frac{1}{2} = \frac{V(\alpha)}{V} = \frac{x^2 - \frac{1}{3}x^3}{\frac{2}{3}}$$

$$\frac{1}{3} = x^2 - \frac{1}{3}x^3$$

$$x^3 - 3x^2 + 1 = 0$$

We can solve $(x^3 - 3x^2 + 1)' = 0$ to find out there is only one solution where $x \in [0, 2]$, (actually $x \in [0, 1]$, you can notice it in firther part) and it is

$$x = 1 - \frac{1 - i\sqrt{3}}{2^{\frac{2}{3}}\sqrt[3]{1+i\sqrt{3}}} - \frac{\left(1+i\sqrt{3}\right)^{\frac{4}{3}}}{2\sqrt[3]{2}} \implies$$

$$\implies \sin\alpha = \frac{1 - i\sqrt{3}}{2^{\frac{2}{3}}\sqrt[3]{1+i\sqrt{3}}} + \frac{\left(1+i\sqrt{3}\right)^{\frac{4}{3}}}{2\sqrt[3]{2}} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{\frac{4}{3}} + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{\frac{4}{3}} =$$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{4}{3}} + \left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)^{\frac{4}{3}} =$$

$$= \cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9} + \cos\frac{-4\pi}{9} + i\sin\frac{-4\pi}{9} =$$

$$= 2\cos\frac{4\pi}{9} = 2\sin\frac{\pi}{18}$$

Where $i^2 = -1$. As $\sin\alpha = 2\sin\frac{\pi}{18} < 2\sin\frac{\pi}{6} = 1$ then also $x = 1 - \sin\alpha < 1$. And in this way we got $\alpha = \arcsin\left(2\sin\frac{\pi}{18}\right)$.

Parabolic glass

This is clearly the C_1-type glass where Γ has an equation $z = \tan\alpha(x-b)+b^2$ and $0 \leq x \leq b$. Let us find X. To do that, we have to solve this inequality:

$$\tan\alpha(x-b) + ab \geq x^2 + y^2 \quad (3)$$

First, let us solve it for intersection of Γ and the glass on the plane $y = 0$:

$$\tan\alpha(x-b) + b^2 \geq x^2$$

$$-x^2 + \tan\alpha(x-b) + b^2 \geq 0$$

$$(x-b)(x+b-\tan\alpha) \leq 0 \quad (4)$$

Let us notice, that $\alpha \in \left[0, \frac{\pi}{2}\right]$ then $\tan\alpha$ is a positive value but $\tan\alpha \leq 2b$. If $\tan\alpha > 2b$ then slope of Γ would be bigger than slope of glass in any point, as $g(x) = x^2 \implies g'(x) = 2x$ and as $0 \leq x \leq b$ then slope of the wall of the glass is at most $2b$. In this way we can notice that $\tan\alpha > 2b \implies \frac{V(\alpha)}{V} = 0$. Also let us notice that roots of the quadratic equation (4) are b and $\tan\alpha - b$ where $b \geq \tan\alpha - b$.

Now for given $\tan\alpha - b \leq x \leq b$ let us find y such inequality (3) is true.

$$\tan\alpha(x-b) + b^2 \geq x^2 + y^2$$
$$\sqrt{\tan\alpha(x-b) + b^2 - x^2} \geq |y|$$

As we know for given x the interval of y when plane is above plane $z = 0$ and in the glass then our integral is:

$$V(\alpha) = \int_{\tan\alpha - b}^{b} \int_{-\sqrt{\tan\alpha(x-b)+b^2-x^2}}^{\sqrt{\tan\alpha(x-b)+b^2-x^2}} \tan\alpha(x-b) + b^2 - x^2 - y^2 \,\mathrm{d}y\,\mathrm{d}x =$$
$$= \int_{\tan\alpha-b}^{b} \left[\left(\tan\alpha(x-b) + b^2 - x^2\right) y - \frac{y^3}{3}\right]_{-\sqrt{\tan\alpha(x-b)+b^2-x^2}}^{\sqrt{\tan\alpha(x-b)+b^2-x^2}} \mathrm{d}x =$$
$$= \int_{\tan\alpha-b}^{b} 2(\tan\alpha(x-b)+b^2-x^2)\sqrt{\tan\alpha(x-b)+b^2-x^2} - \frac{2(\tan\alpha(x-b)+b^2-x^2)^{\frac{3}{2}}}{3}\mathrm{d}x =$$
$$= \int_{\tan\alpha-b}^{b} \frac{4(\tan\alpha(x-b)+b^2-x^2)^{\frac{3}{2}}}{3}\mathrm{d}x = \frac{4}{3}\int_{\tan\alpha-b}^{b} (\tan\alpha(x-b)+b^2-x^2)^{\frac{3}{2}}\mathrm{d}x =$$
$$\frac{4}{3}\int_{\tan\alpha-b}^{b} (-(x-b)(x-(\tan\alpha-b)))^{\frac{3}{2}}\ \mathrm{d}x =$$
$$= \frac{4}{3}\left[-\frac{(2b-\tan\alpha)^4}{512}\left(\frac{4(\tan\alpha-2x)}{2b-\tan\alpha}\sqrt{1-\left(\frac{\tan\alpha-2x}{2b-\tan\alpha}\right)^2}\left(17-2\left(\frac{\tan\alpha-2x}{2b-\tan\alpha}\right)^2\right)\right)\right] +$$
$$+\frac{4}{3}\left[\frac{-(2b-\tan\alpha)^4}{512}\left(12\arcsin\left(\frac{\tan\alpha-2x}{2b-\tan\alpha}\right)\right)\right]_{\tan\alpha-b}^{b} =$$
$$= \frac{\pi(2b-\tan\alpha)^4}{32}$$

Now let is find V of this glass.

$$V = b^2(b)^2\pi - 2\pi\int_0^b x\cdot x^2\mathrm{d}x = b^4\pi - 2\pi\left[\frac{x^4}{4}\right]_0^b = \frac{b^4\pi}{2}$$

In this way formula we were looking for is equal to:

$$\frac{V(\alpha)}{V} = \frac{\frac{\pi(2b-\tan\alpha)^4}{32}}{\frac{b^4\pi}{2}} = \frac{(2b-\tan\alpha)^4}{16b^4} = \left(1-\frac{\tan\alpha}{2b}\right)^4 \qquad (5)$$

1. As parabolic glass was made by rotation around z-axis and it is a solid of revolution of some curve, value of $\frac{V(\alpha)}{V}$ depends only from α

2. We can use formula (5) and we get

$$\frac{V(\alpha)}{V} = \frac{\left(6 - \frac{\sqrt{3}}{3}\right)^4}{16 \cdot 81} = \frac{12313 - 2616\sqrt{3}}{11664}$$

3. In this case we have to find α such

$$\frac{1}{2} = \frac{V(\alpha)}{V} = \left(\frac{2b - \tan\alpha}{2b}\right)^4 \text{ (6)}$$

As we know that $2b \geq \tan\alpha$ then $\frac{2b-\tan\alpha}{2b} \geq 0$. Therefore (6) is equivalent to:

$$\frac{1}{\sqrt[4]{2}} = \frac{2b - \tan\alpha}{2b}$$

$$\tan\alpha = 2b\left(1 - \frac{\sqrt[4]{8}}{2}\right)$$

$$\alpha = \arctan\left(2b\left(1 - \frac{\sqrt[4]{8}}{2}\right)\right)$$

Conic glass

This is clearly the C_1-type glass where Γ has an equation $z = \tan\alpha(x-b) + ab$, $0 \leq x \leq b$ and $g(x) = ax$. Let us find X. To do that, we have to solve this inequality:

$$\tan\alpha(x-b) + ab \geq a\sqrt{x^2+y^2} \quad (7)$$

First of all let us change coordinates into polar and substitute $x = r\cos\varphi$ and $y = r\sin\varphi$ where $r \geq 0$ and solve (7) to find X. We know that:

$$\tan\alpha(r\cos\varphi - b) + ab - ar \geq 0$$

$$r\tan\alpha\cos\varphi - ar - b\tan\alpha + ab \geq 0$$

$$r(\tan\alpha\cos\varphi - a) \geq b(\tan\alpha - a)$$

Now let us notice that if slope of Γ is equal or greater than slope of wall of the glass, then glass is empty. So we can assume that $\tan\alpha < a$, and as a and $\tan\alpha$ are some positive values then $\tan\alpha\cos\varphi - a$ is a negative value. Therefore:

$$r \leq \frac{b(\tan\alpha - a)}{\tan\alpha\cos\varphi - a}$$

We can skip case when $\tan\alpha = 0$ as obviously $\frac{V(\alpha)}{V} = 1$. In this way $V(\alpha)$ is equal to:

$$V(\alpha)=\int_0^{2\pi}\int_0^{\frac{b(\tan\alpha-a)}{\tan\alpha\cos\varphi-a}} r(\tan\alpha(r\cos\varphi-b)+ab-ar)\,\mathrm{d}r\,\mathrm{d}\varphi=$$

$$\int_0^{2\pi}\left[\frac{r^3}{3}(\tan\alpha\cos\varphi-a)+\frac{r^2}{2}(ab-b\tan\alpha))\right]_0^{\frac{b(\tan\alpha-a)}{\tan\alpha\cos\varphi-a}}\,\mathrm{d}\varphi=$$

$$=\int_0^{2\pi}\frac{b^3(\tan\alpha-a)^3}{3(\tan\alpha\cos\varphi-a)^2}+\frac{b^3(\tan\alpha-a)^2(a-\tan\alpha)}{2(\tan\alpha\cos\varphi-a)^2}\,\mathrm{d}\varphi=$$

$$=\frac{b^3(a-\tan\alpha)^3}{6\tan^2\alpha}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{\left(\cos\varphi-\frac{a}{\tan\alpha}\right)^2}$$

We also know that volume is equal to:

$$V=(ab)b^2\pi-2\pi\int_0^b x\cdot ax\,\mathrm{d}x=ab^3\pi-2\pi\left[\frac{ax^3}{3}\right]_0^b=\frac{ab^3\pi}{3}$$

Hence our formula is equal to:

$$\frac{V(\alpha)}{V}=\frac{\frac{b^3(a-\tan\alpha)^3}{6\tan^2\alpha}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{\left(\cos\varphi-\frac{a}{\tan\alpha}\right)^2}}{\frac{ab^3\pi}{3}}=\frac{(a-\tan\alpha)^3}{2a\tan^2\alpha\pi}\cdot\int_0^{2\pi}\frac{\mathrm{d}\varphi}{\left(\cos\varphi-\frac{a}{\tan\alpha}\right)^2}\tag{8}$$

1. As a cone is a solid of revolution of some curve value of $\frac{V(\alpha)}{V}$ depends only from α
2. We can use (8) and we get that

$$\frac{V\left(\frac{\pi}{6}\right)}{V}=\frac{\left(1-\frac{\sqrt{3}}{3}\right)^3}{2\cdot\frac{1}{3}\pi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{\left(\cos\varphi-\sqrt{3}\right)^2}=\frac{(3-\sqrt{3})^3}{18\pi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{(\cos\varphi-\sqrt{3})^2}\tag{9}$$

   According to the computational program (https://www.wolframalpha.com/) and the online integral calculator (https://www.integral-calculator.com/) we have that:

$$\int_0^{2\pi}\frac{\mathrm{d}\varphi}{(\cos\varphi-\sqrt{3})^2}\approx 3.847649490485593$$

   and

$$\left|3.847649490485593-\sqrt{\frac{3}{2}}\pi\right|<10^{-13}$$

   So we can show that approximately value of $\frac{V\left(\frac{\pi}{6}\right)}{V}$ is:

$$\frac{V\left(\frac{\pi}{6}\right)}{V}=\frac{(3-\sqrt{3})^3}{18\pi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{(\cos\varphi-\sqrt{3})^2}\approx\frac{\pi\sqrt{3}(3-\sqrt{3})^3}{18\sqrt{2}\pi}=\frac{\sqrt{2}\cdot(3\sqrt{3}-5)}{2}$$

3. In this case solution is such α that:

$$\frac{1}{2}=\frac{V(\alpha)}{V}=\frac{(1-\tan\alpha)^3}{2\cdot\tan^2\alpha\pi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{\left(\cos\varphi-\frac{1}{\tan\alpha}\right)^2}$$

Hall of integrals

Here you can find some formulas and solutions of non-trivial integrals used here:

1.

$$\int x\sqrt{a^2-x^2}\,\mathrm{d}x = -\frac{\left(a^2-x^2\right)^{\frac{3}{2}}}{3} + C$$

Solution:

$$\int x\cdot\sqrt{a^2-x^2}\,\mathrm{d}x = \begin{vmatrix} t = a^2-x^2 \\ \frac{\mathrm{d}t}{2} = x\,\mathrm{d}x \end{vmatrix} = \frac{1}{2}\int \sqrt{t}\,\mathrm{d}t =$$
$$= \frac{1}{2}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{(a^2-x^2)^{\frac{3}{2}}}{3} + C$$

2.

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$$

Solution: Let $f(x) = \sqrt{a^2-x^2}$, $g(x) = 1$, $h(x) = x$ and use formula for integration by parts:

$$\int f(x)\left[\frac{\mathrm{d}}{\mathrm{d}x}h(x)\right]\,\mathrm{d}x = f(x)h(x) - \int\left[\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right]h(x)\,\mathrm{d}x$$
$$\int f(x)g(x)\,\mathrm{d}x = xf(x) - \int\left(\left[\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right]x\right)\,\mathrm{d}x$$

$$I = \int\sqrt{a^2-x^2}\,\mathrm{d}x = x\sqrt{a^2-x^2} - \int x\left(\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{a^2-x^2}\right)\mathrm{d}x =$$
$$= x\sqrt{a^2-x^2} - \int\frac{-x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x =$$
$$= x\sqrt{a^2-x^2} - \int\frac{a^2-x^2}{\sqrt{a^2-x^2}} - \frac{a^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x =$$
$$= x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,\mathrm{d}x + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x =$$
$$= x\sqrt{a^2-x^2} - I + a^2\arcsin\left(\frac{x}{a}\right) + C \implies$$
$$2I = x\sqrt{a^2-x^2} + a^2\arcsin\left(\frac{x}{a}\right) + C \implies$$
$$\int\sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$$

3.
$$\int (-(x-a)(x-b))^{\frac{3}{2}}\,\mathrm{d}x =$$
$$= -\frac{(a-b)^4}{512}\left(\frac{4(a+b-2x)}{a-b}\sqrt{1-\left(\frac{a+b-2x}{a-b}\right)^2}\left(17-2\left(\frac{a+b-2x}{a-b}\right)^2\right)\right)+$$
$$+\frac{-(a-b)^4}{512}\left(12\arcsin\left(\frac{a+b-2x}{a-b}\right)\right)+C$$

Solution:
$$\int (-(x-a)(x-b))^{\frac{3}{2}}\,\mathrm{d}x = \frac{1}{8}\int \left(-(a+b-2x)^2+(a-b)^2\right)^{\frac{3}{2}}\mathrm{d}x =$$
$$= \left|\begin{matrix} a+b-2x=t \\ \mathrm{d}x = -\frac{\mathrm{d}t}{2} \\ a-b=k \end{matrix}\right| = -\frac{1}{16}\int (k^2-t^2)^{\frac{3}{2}}\,\mathrm{d}t = -\frac{k^3}{16}\int\left(1-\left(\frac{t}{k}\right)^2\right)^{\frac{3}{2}}\mathrm{d}t =$$
$$= \left|\begin{matrix} \frac{t}{k}=\sin\varphi \\ \mathrm{d}t = k\cos\varphi\,\mathrm{d}\varphi \end{matrix}\right| = -\frac{k^4}{16}\int\left(1-\sin^2\varphi\right)^{\frac{3}{2}}\cos\varphi\,\mathrm{d}\varphi =$$
$$= -\frac{k^4}{16}\left(\frac{\sin(4\varphi)+8\sin(2\varphi)+12\varphi}{32}\right)+C =$$
$$= -\frac{k^4}{512}(4\sin\varphi\cos^3\varphi - 4\sin^3\varphi\cos\varphi + 16\sin\varphi\cos\varphi + 12\varphi) + C \tag{10}$$

As $\frac{t}{k}=\sin\varphi$ then $\varphi = \arcsin\left(\frac{t}{k}\right)$. In this case we have:
$$(10) = \frac{-k^4}{512}\left(4\sin\arcsin\left(\frac{t}{k}\right)\cos^3\arcsin\left(\frac{t}{k}\right) - 4\sin^3\arcsin\left(\frac{t}{k}\right)\cos\arcsin\left(\frac{t}{k}\right)\right)+$$
$$+\frac{-k^4}{512}\left(16\sin\arcsin\left(\frac{t}{k}\right)\cos\arcsin\left(\frac{t}{k}\right)+12\arcsin\left(\frac{t}{k}\right)\right)+C =$$
$$= \frac{-k^4}{512}\left(\frac{4t}{k}\cdot\left(1-\left(\frac{t}{k}\right)^2\right)^{\frac{3}{2}} - 4\left(\frac{t}{k}\right)^3\cdot\left(1-\left(\frac{t}{k}\right)^2\right)^{\frac{1}{2}}\right)+$$
$$+\frac{-k^4}{512}\left(\frac{16t}{k}\sqrt{1-\left(\frac{t}{k}\right)^2}+12\arcsin\left(\frac{t}{k}\right)\right)+C =$$
$$= \frac{-k^4}{512}\left(\frac{4t}{k}\sqrt{1-\left(\frac{t}{k}\right)^2}\left(17-2\left(\frac{t}{k}\right)^2\right)+12\arcsin\left(\frac{t}{k}\right)\right)+C =$$
$$= \frac{-(a-b)^4}{512}\left(\frac{4(a+b-2x)}{a-b}\sqrt{1-\left(\frac{a+b-2x}{a-b}\right)^2}\left(17-2\left(\frac{a+b-2x}{a-b}\right)^2\right)\right)+$$
$$+\frac{-(a-b)^4}{512}\left(12\arcsin\left(\frac{a+b-2x}{a-b}\right)\right)+C$$

References

[1] *https://en.wikipedia.org/wiki/Solid_of_revolution*

[2] *https://en.wikipedia.org/wiki/Spherical_cap.*