CISC 2210 Discrete Structures - Noson S. Yanofsky

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1.5

1.

$$\text{Let } f(n) = n^2 + 3 \text{ and } g(n) = 5n - 11 \text{ for } n \in \mathbb{N}$$
$$\text{Thus } f : \mathbb{N} \to \mathbb{N} \text{ and } g : \mathbb{N} \to \mathbb{Z} \text{ Calculate:}$$

(a)

$$f(1) \text{ and } g(1)$$
$$f(1) = 1^2 + 3 = 4$$
$$g(1) = 5(1) - 11 = -6$$

(b)

$$f(2) \text{ and } g(2)$$
$$f(2) = 2^2 + 3 = 7$$
$$g(2) = 5(2) - 11 = -1$$

(c)

$$f(3) \text{ and } g(3)$$
$$f(3) = 3^2 + 3 = 12$$
$$g(3) = 5(3) - 11 = 4$$

(d)

$$f(4) \text{ and } g(4)$$
$$f(4) = 4^2 + 3 = 19$$
$$g(4) = 5(4) - 11 = 9$$

(e)

$$f(5) \text{ and } g(5)$$
$$f(5) = 5^2 + 3 = 28$$
$$g(5) = 5(5) - 11 = 14$$

(f)

To think about: Is $f(n) + g(n)$ always an even number?

let's test base case of n = 0 to be sure:
$f(0)$ and $g(0)$
$f(0) = 0^2 + 3 = 3$
$g(0) = 5(0) - 11 = -11$

$f(n)$ and $g(n)$ both produce either an even or odd integer within the $\mathbb{Z}$ domain.

for $n = \{n : n \in \mathbb{N} \text{ and } n \bmod 2 = 0\}$
let z_1 and z_2 be outputs of $f(n)$ and $g(n)$ respectively,
where $z = \{z : z \in \mathbb{Z} \text{ and } z \bmod 2 = 0\}$
thus, $(z_1 + z_2) \bmod 2 = 0$

for $n + 1 = \{n : n \in \mathbb{N} \text{ and } n \bmod 2 = 1\}$
let $z_1 + 1$ and $z_2 + 1$ be outputs of $f(n)$ and $g(n)$ respectively,
where $z = \{z : z \in \mathbb{Z} \text{ and } z \bmod 2 = 0\}$
thus, $((z_1 + 1) + (z_2 + 1)) \bmod 2 = (z_1 + z_2 + 2) \bmod 2 = 0$

Yes, $f(n) + g(n)$ consistently produces an even integer within the $\mathbb{Z}$ domain.

2.

Consider the function $h : \mathbb{P} \to \mathbb{P}$ defined by $h(n) = |\{k \in \mathbb{N} : k|n\}|$
for $n \in \mathbb{P}$. In words, $h(n)$ is the number of divisors of n.
Calculate $h(n)$ for $1 \leq n \leq 10$ and for $n = 73$.

h(1) = 1
h(2) = |1,2| = 2
h(3) = |1,3| = 2
h(4) = |1,2,4| = 3
h(5) = |1,5| = 2
h(6) = |1,2,3,6| = 4
h(7) = |1,7| = 2
h(8) = |1,2,4,8| = 4
h(9) = |1,3,9| = 3
h(10) = |1,2,5,10| = 4
h(73) = |1,73| = 2

3.

Let $\sum^*$ be the language using letters from $\sum = \{a, b\}$.
We've already seen a useful function from $\sum^*$ to $\mathbb{N}$.
It is the length function, which already has a name: length. Calculate:

(a)

$$\text{length}(bab) = 3$$

(b)

$$\text{length}(aaaaaaaa) = 8$$

(c)

$$\text{length}(\lambda) = 0$$

(d)

What is the image set Im(length) for this function?

$$\mathbb{N}$$

5.

Let f be the function in example 3:
$f(m, n) = \lfloor \frac{n}{2} \rfloor - \lfloor \frac{m-1}{2} \rfloor$

(a)

Calculate $f(0, 0), f(8, 8), f(-8, -8), f(73, 73)$, and $f(-73, -73)$

$$f(0,0) = \lfloor \tfrac{0}{2} \rfloor - \lfloor \tfrac{0-1}{2} \rfloor = 0 - (-1) = 1$$
$$f(8,8) = \lfloor \tfrac{8}{2} \rfloor - \lfloor \tfrac{8-1}{2} \rfloor = 4 - 3 = 1$$
$$f(-8,-8) = \lfloor \tfrac{-8}{2} \rfloor - \lfloor \tfrac{-8-1}{2} \rfloor = -4 - (-5) = 1$$
$$f(73,73) = \lfloor \tfrac{73}{2} \rfloor - \lfloor \tfrac{73-1}{2} \rfloor = 36 - 36 = 0$$
$$f(-73,-73) = \lfloor \tfrac{-73}{2} \rfloor - \lfloor \tfrac{-73-1}{2} \rfloor = -36 - (-36) = 0$$

(b)

Find $f(n, n)$ for all (n, n) in $\mathbb{Z}x\mathbb{Z}$.
Hint: Consider the cases when n is even and when it is odd

if $n \bmod 2 = 0,\ f(n, n) = 1,$
else if $n \bmod 2 = 1,\ f(n, n) = 0$

8.

Let $S = \{1,2,3,4,5\}$ and consider the functions $1_s.f.g$ and h from S into S defined by $1_s(n) = n, f(n) = 6 - n, g(n) = max\{3, n\},$ and $h(n) = max\{1, n-1\}$

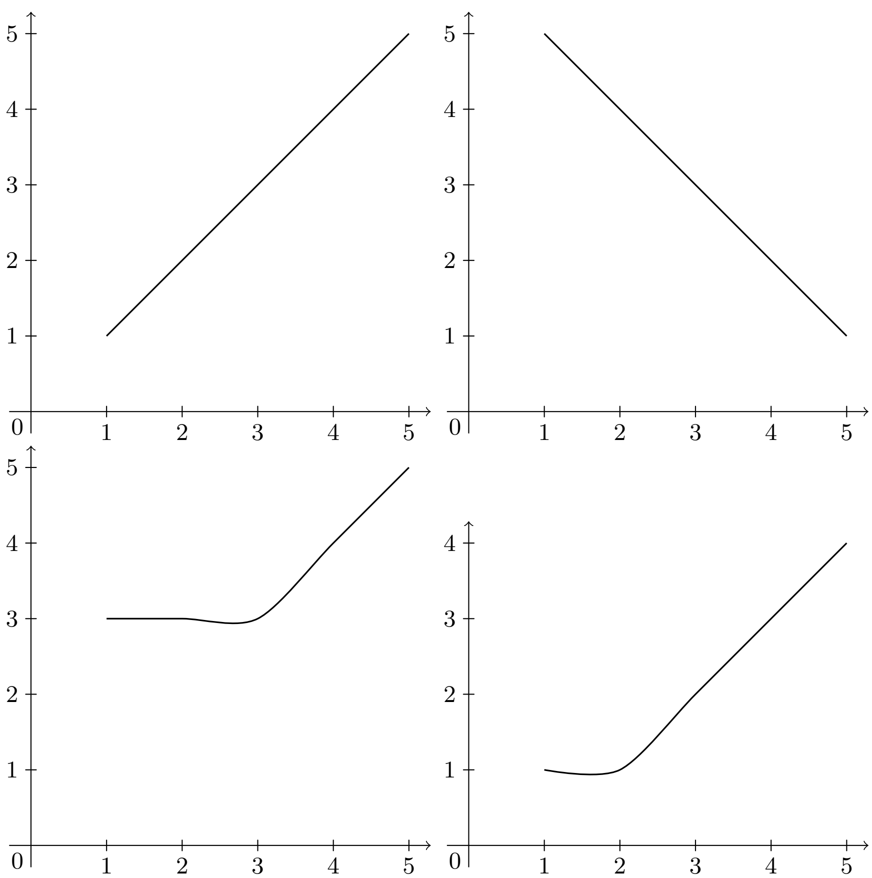
(a)

Write each of these functions as a set of ordered pairs, i.e., list the elements in their graphs:

$$
\begin{aligned}
1_s(n) &= \{(1,1),(2,2),(3,3),(4,4),(5,5)\} \\
f(n) &= \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \\
g(n) &= \{(1,3),(2,3),(3,3),(4,4),(5,5)\} \\
h(n) &= \{(1,1),(2,1),(3,2),(4,3),(5,4)\}
\end{aligned}
$$

(b)

Sketch a graph of each of these functions:



9.

For $n \in \mathbb{Z}$, let $f(n) = \frac{1}{2}[(-1)^n + 1]$.
The function f is the characteristic function for some subset of $\mathbb{Z}$.
Which subset?

{0,1}, where 1 = all even integers in $\mathbb{Z}$

10.

Consider subsets A and B of a set S

(a)

The function $X_A \cdot X_B$ is the characteristic function of some subset of S.
Which subset?

$$A \bigcap B$$

(b)

Repeat part (a) for the function $X_A + X_B - X_{A \bigcap B}$

$$A \bigcup B$$

(c)

Repeat part (a) for the function $X_A + X_B - 2 \cdot X_{A \bigcap B}$

$$A \backslash B \bigcup B \backslash A$$

13.

We define functions mapping $\mathbb{R}$ into $\mathbb{R}$ as follows:

$$f(x) = x^3 - 4x$$
$$g(x) = \frac{1}{x^2+1}$$
$$h(x) = x^4$$

(a)

Find $f \circ f$

$$f(x^3 - 4x)$$
$$= (x^3 - 4x)^3 - 4(x^3 - 4x)$$

(e)

Find $f \circ g \circ h$

$$f(g(h(x)))$$
$$= f(g(x^4))$$
$$= f(\tfrac{1}{(x^4)^2+1})$$
$$= (\tfrac{1}{(x^4)^2+1})^3 - 4(\tfrac{1}{(x^4)^2+1})$$