GOLOMB'S PUZZLE COLUMN™

EARLY BIRD NUMBERS

Solomon W. Golomb

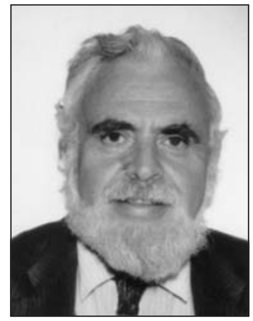
Consider the sequence (∗) consisting of the consecutive positive integers, written in decimal notation, with no intervening spaces or punctuation:

(∗) 1234567891011121314151617181920212223242 5…

It is easy to show that if this sequence is preceeded by a decimal point, the resulting real number is irrational and is "normal, to the base ten" (i.e. every sequence of k consecutive digits occurs in this sequence, asymptotically, with a frequency of 10^{-k}). It has also been shown that this real number is transcendental.

Martin Gardner has defined a positive integer to be an *early bird number* (*e.b. no.*, for short), if it can be found in the sequence (∗) earlier than its guaranteed place in the counting sequence. Thus, **12** is an e.b. no., since the sequence (∗) *begins* with 12. So too is **718**, since we find it in (∗) in the overlap of 17 and 18: (17)(18). On the other hand, the numbers from 1 to 11, inclusive, are *not* e.b. nos., nor are any two-digit numbers ending in "0". Here are some questions.

1. There are 90 two-digit integers from 10 though 99. Exactly half of these (i.e. 45) are e.b. nos. Can you describe which ones these are?
2. Suppose that n is a k-digit positive integer (k>1) such that there is a cyclic permutation n' of the digits of n, where n' begins in a digit other than 0 and ends in a digit other than 9, and $n'<n$. Prove that n must be an e.b. no.
3. Is the previous statement still true if n', the cyclic permutation of n with $n'<n$, is allowed to end with the digit 9? (Prove or disprove.)



4. a. Show that every integer from 91 to 99 (inclusive) is an e.b. no.
   b. Show that every integer from 901 to 999 is an e.b. no.
   c. Prove or disprove: "Every integer from $9{\cdot}10^d+1$ to $10^{d+1}-1$ (inclusive) is an e.b. no., for all $d \geq 1$." (If true, give a proof. If false, exhibit counter-examples.)
5. Martin Gardner observed that "31415" (the first five digits of π) is an "early" e.b. no., occurring in the sequence (∗) at (13)(14)(15). By the theorem in Problem 2, we can also get 31415 as an e.b. no. using either n'=14153 or n'=15314. (That is, n appears in the overlap of the consecutive integers (14153)(14154) and of (15314)(15315).)

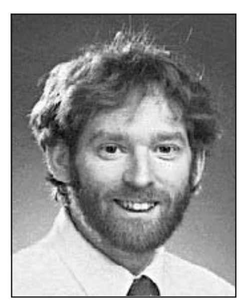
   Find a 5-digit integer that has *six* different representations as an e.b. number.
6. Asymptotically, what percentage of all positive integers are e.b. numbers?

IEEE Information Theory Society Annual Meeting
Hotel Alpha-Palmiers, Lausanne, Switzerland, Sunday June 30, 2002

Aaron Gulliver

Attendees: Erik Agrell, John Anderson, Vijay Bhargava, Daniel Costello, Tom Cover, Michelle Effros, Anthony Ephremides, Raymond Findlay, David Forney, Marc Fossorier, Thomas E. Fuja, Alex Grant, Aaron Gulliver, Joachim Hagenauer, Michael Honig, Hideki Imai, Philippe Jacques, Torleiv Kløve, Ryuji Kohno, Steven W. McLaughlin, Urbashi Mitra, Mehul Motani, Levent Onural, Lance C. Pérez, Paul H. Siegel, Wojciech Szpankowski, Alexander Vardy, Han Vinck, Raymond Yeung

1. The meeting was called to order at 1:50 PM by Society President Tom Fuja. Those present were welcomed and introduced themselves. IEEE President Raymond Findlay and Region 8 Director Levent Onural were welcomed to the meeting. The Agenda was approved as distributed.
2. The minutes of the previous meeting held in Princeton, NY on March 22, 2002, were approved as distributed.
3. Society President Tom Fuja began with a report on the ongoing financial crisis. In particular, the 'Findlay model' was discussed. This model determines how corporate infrastructure expenses will be allocated to IEEE organizational units.



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In response to pressure by the organizational units to reduce costs and balance the budget, the IEEE Board of Directors has raised dues $9. Due primarily to staff

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Early Bird Numbers — Solutions

1. The 45 two-digit *early bird numbers* (e.b. nos.) can be described as follows. Let $n = ab$ (where the standard decimal notation ab stands for $10 \cdot a+b$). If $0<b<a<9$, then $ba<ab$, and the sequence (∗) of all positive integers in natural order will contain $(ba)(ba+1)$ and will exhibit ab in the overlap. (E.g. if n=53=ab, then we see n in the overlap of (35)(36).) There are $\binom{8}{2}=28$ numbers of this type.

   Next, if n=ab where b=a+1, and $0<a<9$, then we see n early in the sequence (∗) where the single-digit number a is followed by a+1. (Thus, n=23 occurs in 123456….) There are 8 such values of n. Finally, the 9 numbers from 91 to 99 appear in the overlaps of 9-10, 19-20, 29-30, …, 89-90. Altogether, this gives 28+8+9=45 two-digit e.b. nos., exactly half of the numbers from 10 through 99. (None of the others are e.b. nos.)

2. If n is a k-digit positive integer (k>1) such that there is another number n' consisting of a cyclic permutation of the digits of n, with $n'<n$, and the left-most digit of n' being from 1 to 9 inclusive, and the right-most digit of n' is other than 9, then n is an e.b. no. because it appears in the overlap of the consecutive integers n' and n'+1. (For example, if n=215, we may take n'=152, and then in (n') (n'+1) we see 152-153 with the original n in the overlap.)

3. If, in the previous problem, there is a cyclic permutation n' of the digits of n, with $n'<n$, but where the right-most digit of n' is 9, the conclusion that n is an e.b. no. is still true, but the proof is more complicated. Here are the typical situations.

   a. If n=291, we take n'=129 and see n in the overlap of n' and n'+1: (129)(130), as in the previous solution.

   b. If n=9193, we cannot use n'=1939, since the overlap of n' and n'+1, (1939)(1940), does not contain n in its overlap. However, we *can* use n'' = 3919, since now (n'')(n'' + 1) = (3919)(3920) has n in its overlap, and we still have $n''<n$.

   c. If n=919, we cannot use n' =199, since (n')(n'+1) = (199)(200) does not have n as an overlap. However, n already appears in the overlap of (91)(92).

   d. If n=9199, we cannot use n'=1999; but n already occurs in the overlap of (919)(920).

4. a. We already saw that every integer from 91 to 99 inclusive is an e.b. no. in the solution to problem 1.

   b. For the numbers from 901 to 999, problems 2 and 3 show that all are e.b. nos. with the possible exceptions of 909 and 999. (The others have cyclic permutations $n'<n$ with the required characteristics.) But 909 appears in the overlap of (90)(91); and 999 is found in the overlap of (899)(900).

   c. The generalization to all numbers from $9 \cdot 10^d + 1$ to $10^{d+1} - 1$ (inclusive) being e.b. nos. for all is false. As counter-examples, consider n=9090, n=900900, and more generally, $n=0.9(10^{2c}+10^c)$ for all $c \geq 2$. None of these is an e.b. no.

5. The 5-digit number n=11121 is an e.b. number for (at least) the following six overlap representations: a. (11)(12)(13), b. (111)(112)(113), c. (1112)(1113), d. (1211)(1212), e. (2111)(2112), f. (11112)(11113).

6. It is true that, asymptotically, 100% of all positive integers are e.b. nos. That is, if $e(x)$ denotes the number of e.b. nos. ≤ x, then $\lim_{x\to\infty} \frac{e(x)}{x} = 1$ (There are infinitely many non-e.b. nos. also, but they become increasingly infrequent.)

   To see this, observe that the "typical" positive integer has a huge number of digits. (Paradoxically, although any specific integer has a finite number of digits, the *expected* number of digits in a "random" integer is infinite!) With so many digits in the typical integer n, it is overwhelmingly likely that there is a cyclic permutation of these digits satisfying the sufficient condition in Problem 2 (or Problem 3) for n to be an e.b. no.

   An open question is to determine how many of the 10^k-10^{k-1} k-digit integers are e.b. nos., for each k. (For k=1 it is 0 for 9, and for k=2 it is 45 out of 90.) It is very likely to be easier to answer this question if we only count those e.b. nos. n that appear in the overlap of *two* consecutive integers less than n.